MS-A0503 First course in probability and statistics

3B Statistical datasets

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Contents

What is statistics (as a science)?

- Applying and developing methods for studying random or uncertain phenomena in the *real world*.
  - The methods are based on the mathematical laws of probability.
  - Sources of uncertainty are many: physical randomness; unknown properties of the real world phenomenon; random sampling on purpose; measurement errors; missing data ...
  - ...generally, the same math applies.

- Roughly:
  - Probability theory tells: How a certain process produces data.
  - Statistics tells: What was the process that produced the data.

- Statistics is applicable whenever you have data; especially if there is any kind of uncertainty or randomness.
  - Most fields of engineering and business have data, so they can (and do) use statistics.

Two basic approaches of statistics

Descriptive statistics

Present and describe the data "as it is", either fully, or in a summary way.

- Tables ("the raw data")
- Graphs (visualization)
- Numerical summaries or "statistics" (e.g. average, minimum, maximum)

Statistical inference

Infer facts about the real phenomenon that lies "behind" the data.
Generalize, e.g. sample $\rightarrow$ population; or measurements $\rightarrow$ universal physical law.

- Stochastic models
- Parameter estimation
- Hypothesis testing

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Statistical data

Typically (not always), statistical data is in a table, the data frame, where

- rows correspond to units, e.g. people
- columns are the variables observed for each unit, e.g. height

Caveat: Different fields of science/engineering use different words. E.g. “units” may be “objects”, “items”, “data points” (geometrically thinking), “records” (in databases)

Depending on number of variables, we may call our data *univariate*, *bivariate*, *multivariate*.

Levels of measurement = What kind of values

- nominal scale ≈ categorical: just distinct classes
  gender: {male, female}
  piece of DNA: {**A**denine, **C**ytosine, **G**uanine, **T**hymine}
- ordinal scale: classes have meaningful order
  cloth size: { XS < S < M < L < XL }
  Likert scale: { str. disagree < disagree < neutral < agree < str. agree }
- numerical: values have arithmetic meaning
  - interval scale: differences $x - y$ are meaningful
    calendar dates, Celsius temperature
  - ratio scale: also quotients x/y are meaningful
    length, weight, distance, Kelvin temperature

Notes:

- all can be *represented* as numbers, e.g. adenine=1, cytosine=2, guanine=3, but arithmetic might not make sense.
- nominal sometimes called "qualitative", but other meanings
- this is *not* the discrete/continuous distinction. Numerical data can be well discrete; e.g. counts (frequencies)

Data set (Data frame)

- data set = sequence of elements (units) of the same type, e.g. numbers, identifiers, or lists of values (one for each variable)
- Often arranged in a table; (R terminology) "data frame"
- Order of units often not meaningful, so we could treat it as a set (or *multiset*, if many identical observations possible)

E.g. course feedback: ((12345A, 5, 1, 5), (98759K, 1, 5, 2), (33312K, 4, 4, 3), (23453B, 4, 4, 3), (21453U, 3, 3, 3))

One string variable (student id), three numerical variables (general satisfaction, workload, usefulness)

Student ID	General	Workload	Usefulness
12345A	5	1	5
98759K	1	5	2
33312K	4	4	3
23453B	4	4	3
21453U	3	3	5

5 units, 4 variables

Average and standard deviation

If we have univariate numerical data: $\vec{x} = (x_1, \ldots, x_n)$

Average (sample mean) $m(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$

Variance $\mathrm{var}(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - m(\vec{x}))^2$

Standard deviation $\mathrm{sd}(\vec{x}) = \sqrt{\mathrm{var}(\vec{x})}$

Eg. $\vec{y} = (0, 0, 1, 1, 2, 2)$

$$m(\vec{y}) \;=\; \frac{1}{6}\Big(0 + 0 + 1 + 1 + 2 + 2\Big) \;=\; 1$$

$$\mathrm{var}(\vec{y}) \;=\; \frac{1}{6}\Big((0-1)^2+(0-1)^2+(1-1)^2+(1-1)^2+(2-1)^2+(2-1)^2\Big) \;=\; \frac{2}{3}$$

$$\mathrm{sd}(\vec{y}) \;=\; \sqrt{\frac{2}{3}} \;\approx\; 0.8165$$

Caveat: Sometimes $n-1$ used as divisor for variance and sd, for technical reasons; more about this later (in parameter estimation).

Example

Calculate sample mean and standard deviation for the following data sets

$$\begin{aligned}
\vec{x} &= (1, 1, 1, 1, 1), \\
\vec{y} &= (0, 0, 1, 1, 2, 2), \\
\vec{z} &= (0, 2, 0, 2, 0, 2, 0, 2, 0, 2), \\
\vec{w} &= (\underbrace{0, 0, 0, 0, \ldots, 0, 0, 0, 0}_{\text{666666 times}}, 1000000, \underbrace{0, 0, \ldots, 0, 0}_{\text{333333 times}}).
\end{aligned}$$

Dataset	Mean	SD
$\vec{x}$	1	0.0000
$\vec{y}$	1	0.8165
$\vec{z}$	1	1.0000
$\vec{w}$	1	999.9995

Average and standard deviation are *summaries*, they do not tell everything about the data. (Just like in probability distributions.)

Computing the summary statistics *from data*

Notation	Name	R	Matlab	Excel
$m(\vec{x})$	Average	`mean()`	`mean()`	`AVERAGE()`
$\mathrm{sd}(\vec{x})$	Standard deviation	`sqrt(1-1/n)*sd()`	`std(,1)`	`STDEV.P()`
$\mathrm{sd}_s(\vec{x})$	(Corrected) std.dev.	`sd()`	`std()`	`STDEV.S()`
$\mathrm{var}(\vec{x})$	Variance	`(1-1/n)*var()`	`var(,1)`	`VAR.P()`
$\mathrm{var}_s(\vec{x})$	(Corrected) variance	`var()`	`var()`	`VAR.S()`
$\mathrm{cov}(\vec{x}, \vec{y})$	Covariance	`(1-1/n)*cov()`	`cov(,1)`	`COVARIANCE.P()`
$\mathrm{cov}_s(\vec{x}, \vec{y})$	(Corrected) covariance	`cov()`	`cov(1)`	`COVARIANCE.S()`
$\mathrm{cor}(\vec{x}, \vec{y})$	Correlation	`cor()`	`corrcoef()`	`CORREL()`
$q_{0.5}(\vec{x})$	Median	`median()`	`median()`	`MEDIAN()`
$q_{0.25}(\vec{x})$	Lower quartile	`quantile( ,.25)`	`quantile( ,.25)`	`PERCENTILE.INC( ,.25)`
$q_{0.75}(\vec{x})$	Upper quartile	`quantile( ,.75)`	`quantile( ,.75)`	`PERCENTILE.INC( ,.75)`

Caveat. Terminology and notation varies across sources. For technical reasons, many computer programs offer the so-called "unbiased" or "Bessel-corrected" *sample variance* and *sample standard deviation*, where the divisor is $n - 1$ instead of n. (Don't worry now – we will talk about this in parameter estimation.)

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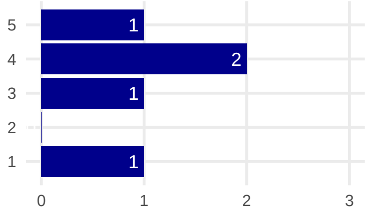
Counts = Frequencies of values

The count, or (absolute) frequency of a value x, in the univariate dataset $\vec{x}$, is

$$n_{\vec{x}}(x) \;=\; \#\{i : x_i = x\}$$

Course feedback, pick one variable "General" $\to$ univariate data $(5, 1, 4, 4, 3)$. Frequency as a table and a (horizontal) bar chart:

x	1	2	3	4	5
$n_{\vec{x}}(x)$	1	0	1	2	1



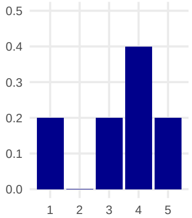
Relative frequencies

The proportion, or relative frequency of value x in dataset $\vec{x}$ is

$$f_{\vec{x}}(x) = \frac{n_{\vec{x}}(x)}{n} = \frac{\#\{i : x_i = x\}}{n}$$

Course feedback, pick "General", dataset $(5, 1, 4, 4, 3)$, relative frequencies as a table and (vertical) bar chart

x	1	2	3	4	5
$f_{\vec{x}}(x)$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$



Observation: $\sum_x f_{\vec{x}}(x) = 1$, thus $f_{\vec{x}}(x)$ is a probability distribution!
It is the empirical distribution of the dataset $\vec{x}$.

Empirical distribution

Proposition

If an element X is chosen uniformly at random, from the dataset $\vec{x} = (x_1, \ldots, x_n)$, then X is a discrete random variable, whose density corresponds to the empirical distribution: $f_X(x) = f_{\vec{x}}(x)$. Furthermore,

$$\mathbb{E}(X) = m(\vec{x}), \quad (1)$$

$$\mathrm{SD}(X) = \mathrm{sd}(\vec{x}), \quad (2)$$

$$\mathrm{Var}(X) = \mathrm{var}(\vec{x}). \quad (3)$$

Also, for any function g, we have

$$\mathbb{E}[g(X)] = \frac{1}{n}\sum_{i=1}^{n} g(x_i). \quad (4)$$

Example

For the dataset $\vec{y} = (0, 0, 1, 1, 2, 2)$, determine the empirical distribution, and its mean and standard deviation.

The relative frequencies are

y	0	1	2
$f_{\vec{y}}(y)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

If random variable Y has density $f_{\vec{y}}(y)$, then

$$\mathbb{E}(Y) \;=\; \sum_{y=0}^{2} y f_{\vec{y}}(y) \;=\; 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 2 \times \frac{1}{3} \;=\; 1,$$

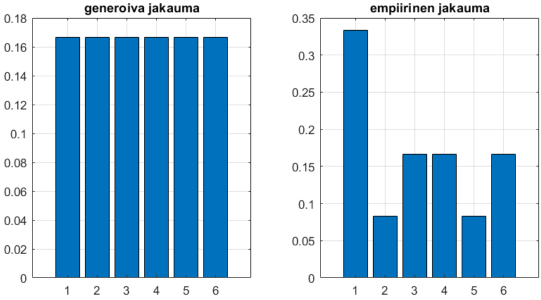
$$\text{Var}(Y) \;=\; \sum_{y=0}^{2} (y-1)^2 f_{\vec{y}}(y) \;=\; (0-1)^2 \times \frac{1}{3} + (1-1)^2 \times \frac{1}{3} + (2-1)^2 \times \frac{1}{3} \;=\; \frac{2}{3}$$

$$\implies m(\vec{y}) = \mathbb{E}(Y) = 1$$

$$\implies \text{sd}(\vec{y}) = \sqrt{\text{var}(\vec{y})} = \sqrt{\text{Var}(Y)} = \sqrt{\tfrac{2}{3}} \approx 0.8165$$

Generating vs. empirical distribution

Rolling a fair die. Generating distribution (stochastic model) versus empirical distribution from 12 rolls that were (5,1,6,4,3,1,1,6,2,4,1,3).



- The empirical distribution *is* a probability distribution and you can use all tools of probability distributions.
- But *different distribution* from the generating one.
- For large *n* they are reasonably close (by LLN).

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Binning (Grouping)

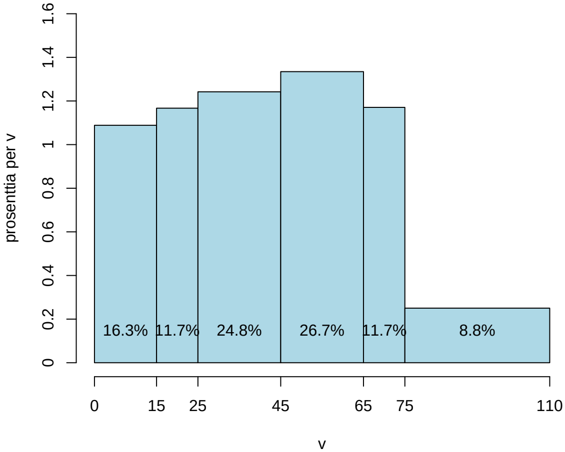
Eg. ages of all Finns 31.12.2015.
$n = 5\ 487\ 308$ units (data points)
Not probably good idea to draw as individual points (especially if ages are expressed in 1-day precision)
Let us group the data into bins at predefined boundaries.

Age (yr)	Frequency
0–14	896 023
15–24	640 387
25–44	1 363 155
45–64	1 464 640
65–74	642 428
75–	480 675

In fact we have transformed our real-valued variable (age) into a discrete one (group or bin index).

Example: Histogram with unequal widths

Finnish age distribution 31.12.2015 [Source: Tilastokeskus]



Age (yr)	Frequency
0–14	896 023
15–24	640 387
25–44	1 363 155
45–64	1 464 640
65–74	642 428
75–	480 675

The bars are an *approximation* of the true density function.

Could you find the proportion of Finns in the 1-year interval $[13, 14)$?

What about the the interval $[109, 110)$ years? Would it be accurate?

How to draw a histogram (allowing unequal bin widths)

- One bar for each bin (interval of possible values)
- Bar width = width of the interval
- Bar height = relative frequency *divided* by width

Example: Age distribution, first bar:

- Represents Finns with ages in interval $[0, 15)$
  Note: values *strictly* smaller than 15; age in whole years 0–14
- Bar width = 15 years
- Frequency = 896023, relative frequency $896023/5487308 \approx 16.3\%$
- Bar height = $16.3/15 \approx 1.09$ (unit: % per year).
- Then bar *area* is the relative frequency.

(Typically, we use equal-width intervals, but not always.)

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Bivariate data

Bivariate data = sequence (or multiset) of *pairs*

$$\vec{xy} \;=\; ((x_1, y_1), \ldots, (x_n, y_n)).$$

Alternatively, a pair $(\vec{x}, \vec{y})$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ are univariate data (note: as ordered sequences, so we know which x_i and y_i belong together).

Course feedback: Two variables "General" and "Usefulness" composed as a bivariate dataset ((5,5), (1,2), (4,3), (4,3), (3,3))

Univariate statistics $m(\vec{x}), m(\vec{y}), \mathrm{sd}(\vec{x}), \mathrm{sd}(\vec{y})$ are surely useful, but they tell nothing about the dependence between variables. Covariance and correlation tell (some aspects of) dependence.

$$\mathrm{cov}(\vec{x}, \vec{y}) \;=\; \frac{1}{n} \sum_{i=1}^{n} (x_i - m(\vec{x}))(y_i - m(\vec{y}))$$

$$\mathrm{cor}(\vec{x}, \vec{y}) \;=\; \frac{\mathrm{cov}(\vec{x}, \vec{y})}{\mathrm{sd}(\vec{x})\,\mathrm{sd}(\vec{y})}$$

Example: Heights of father-son pairs

I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P	I	P
165	152	165	171	179	174	178	178	184	187	174	168	171	174	165	178	178	184	173	165	178	170	172	176	178	181	170	161	168	170	182	173	179	179	175	199	180	168	173	173
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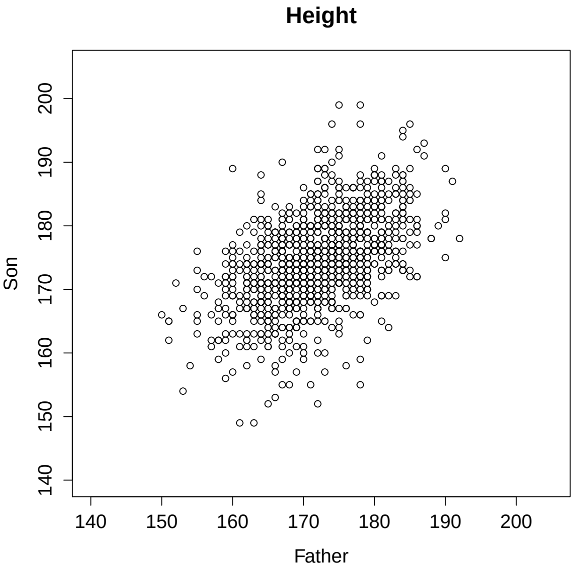
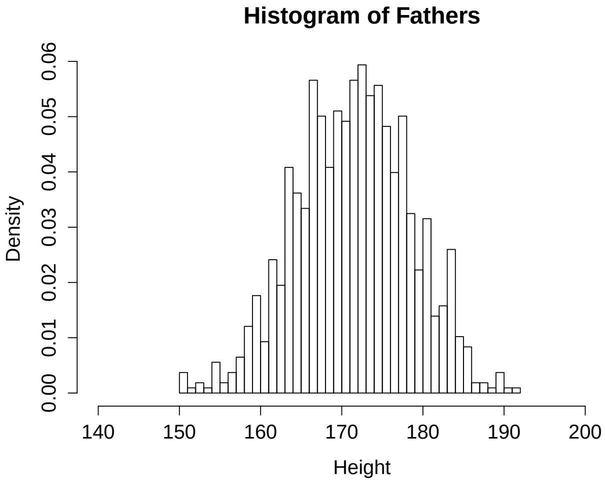
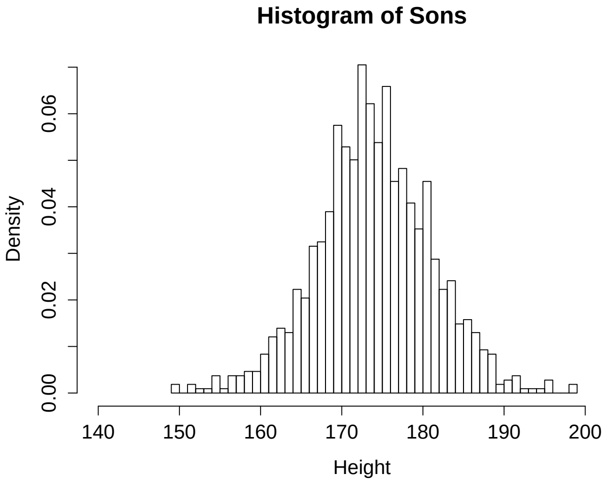
166	170	175	173	174	177	182	187	168	168	167	174	179	175	168	183	161	167	173	171	171	176	172	182	182	164	176	171	176	174	176	178	164	188	169	168	167	173	181	176
165	170	176	173	173	180	180	185	168	169	168	174	178	175	171	183	165	164	174	170	169	175	173	182	159	166	172	160	175	173	174	179	172	189	169	168	169	173	181	177
167	176	175	173	176	178	183	186	167	169	169	173	178	176	172	183	165	165	172	170	171	175	174	180	181	177	165	171	177	174	175	179	178	188	170	169	170	174	185	177
174	175	176	174	175	178	160	189	171	168	166	173	181	176	175	182	165	165	173	170	171	176	172	180	169	165	176	175	177	173	177	179	180	188	173	168	170	174	166	179
169	157	176	173	176	178	173	189	170	169	170	173	181	177	176	183	168	167	175	171	173	175	176	182	171	155	165	170	178	172	177	177	185	189	172	168	171	174	168	178
165	167	174	174	176	179	178	187	171	169	170	173	184	176	177	183	168	166	175	171	172	175	176	181	159	160	165	171	179	173	179	179	175	191	174	167	170	174	166	179
177	175	177	173	177	177	180	188	172	167	170	174	164	178	178	184	169	165	179	172	173	175	176	182	165	162	167	171	184	173	181	179	184	194	176	167	170	172	167	179

Table: 1000 pairs of heights (father, son) from Pearson's data. Note that this not just two data sets, but a set of *pairs* that belong together.

Scatterplot (scatter diagram)



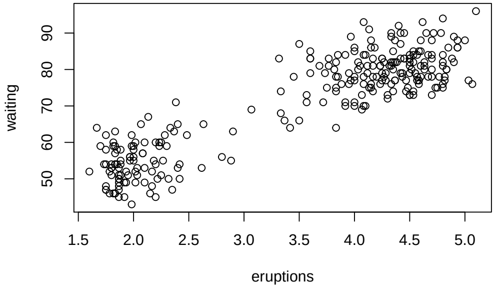

Histogram of Fathers
Density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
140
150
160
170
180
190
200
Height



Histogram of Sons
Density
0.00
0.02
0.04
0.06
140
150
160
170
180
190
200
Height

Example: Eruptions of Old Faithful geysir

Scatterplot of 272 eruptions of *Old Faithful* (Yellowstone).
Two variables: eruption length and waiting time to next eruption.



Already a visual inspection ("eyeballing") reveals interesting patterns (that are not captured by mean and standard deviation).

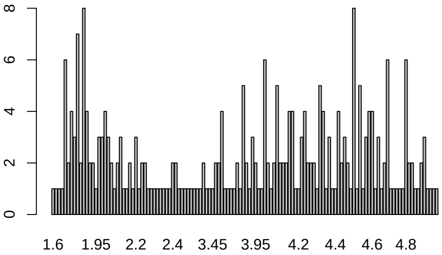
You can find the data in R, try `faithful` and `help("faithful")`

Old Faithful: bar chart of one variable. . .

We could try listing *all different values* of eruption length, and collect their frequencies (within $n = 272$)

x	1.6	1.667	1.7	1.733	1.75	...	5.1
$n_{\vec{x}}(x)$	1	1	1	1	6	...	1

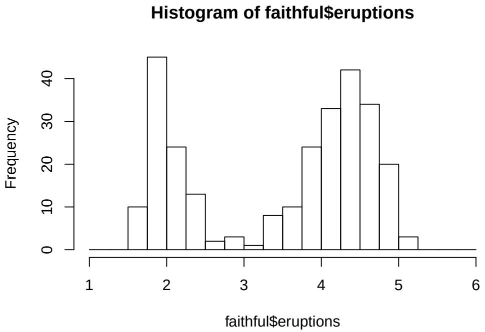
and draw a bar chart



Neither the table or the bar chart seems very informative.

Old Faithful: histogram

But if we group the data into 0.25-minute intervals such as $[2.00, 2.25)$, and plot the counts, we have a better picture of the distribution.



Try this on your own! What happens if you use more (finer) intervals? What if you use less (coarser) intervals?

Cross-tabulation (Contingency table)

The frequency of the pair (x, y), in the data set, is

$$n_{\vec{x}\vec{y}}(x, y) \; = \; \#\{i \; : \; x_i = x \;\text{ and }\; y_i = y\}$$

Course feedback: "General" and "Usefulness" as bivariate dataset ((5,5), (1,2), (4,3), (4,3), (3,3)) has this contingency table:

	y					
x	1	2	3	4	5	Sum
1	0	1	0	0	0	1
2	0	0	0	0	0	0
3	0	0	1	0	0	1
4	0	0	2	0	0	2
5	0	0	0	0	1	1
Sum	0	1	3	0	1	

Cross-tabulation of relative frequencies

The relative frequency of the pair (x, y) is

$$f_{\vec{x}\vec{y}}(x, y) \;=\; \frac{\#\{i \;:\; x_i = x \;\text{ and }\; y_i = y\}}{n}$$

	y					
x	1	2	3	4	5	Sum
1	0	$\frac{1}{5}$	0	0	0	$\frac{1}{5}$
2	0	0	0	0	0	0
3	0	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$
4	0	0	$\frac{2}{5}$	0	0	$\frac{2}{5}$
5	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$
Sum	0	$\frac{1}{5}$	$\frac{3}{5}$	0	$\frac{1}{5}$	

$\sum_{x,y} f_{\vec{x}\vec{y}}(x, y) = 1$, thus $f_{\vec{x}\vec{y}}(x, y)$ is a probability distribution.
$f_{\vec{x}\vec{y}}(x, y)$ is the empirical joint distribution of the dataset $\vec{x}\vec{y}$.

Empirical joint distribution

Proposition

If a pair (X, Y) is chosen uniformly at random, from the dataset $\vec{x}\vec{y} = ((x_1, y_1), \ldots, (x_n, y_n))$, it is a discrete random variable whose (joint) density is the empirical distribution $f_{X,Y}(x, y) = f_{\vec{x}\vec{y}}(x, y)$ and also

$$\begin{aligned} \mathbb{E}(X) &= m(\vec{x}), & \mathbb{E}(Y) &= m(\vec{y}), \\ \mathrm{SD}(X) &= \mathrm{sd}(\vec{x}), & \mathrm{SD}(Y) &= \mathrm{sd}(\vec{y}), \\ \mathrm{Var}(X) &= \mathrm{var}(\vec{x}), & \mathrm{Var}(Y) &= \mathrm{var}(\vec{y}), \end{aligned} \tag{5}$$

and

$$\mathrm{Cor}(X, Y) = \mathrm{cor}(\vec{x}, \vec{y}), \tag{6}$$

$$\mathrm{Cov}(X, Y) = \mathrm{cov}(\vec{x}, \vec{y}). \tag{7}$$

Also, for any two-argument function g, we have

$$\mathbb{E}[g(X, Y)] = \frac{1}{n} \sum_{i=1}^{n} g(x_i, y_i). \tag{8}$$

Contents

Quantiles/Percentiles of data

Suppose data can be ordered from smallest to largest.
(OK for numerical or ordinal data; not for nominal data.)

If $0 < p < 1$, then the p-quantile (or $100p$-percentile) $Q(p)$ is roughly the point x such that *proportion* p of the data is smaller than x, and $1 - p$ is greater.

- $Q(0.25)$ is lower (first) quartile; 25% of data is below
- $Q(0.5)$ is median or second quartile; 50% of data is below
- $Q(0.75)$ is upper (third) quartile; 75% of data is below

Note that half of data is between lower and upper quartiles.

R: `quantile(x,p), summary(x), median(x)`

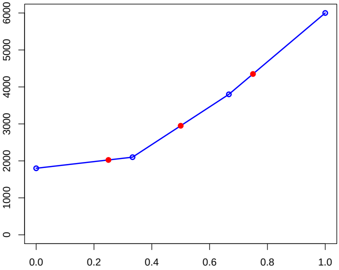
The "roughly" is because in finite data, you may not find exact quarters. There are some (varying) conventions for this.

Quantile function

One way to define the quantile function of dataset $(x_1, \ldots, x_n)$:

- Order the data as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$
- Divide the horizontal unit interval $[0, 1]$ into equal parts, at points $p_k = (k-1)/(n-1)$, $k = 1, \ldots, n$
- Plot the points $(p_k, x_{(k)})$ and connect with lines

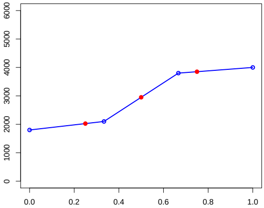
Example. Four salaries (eur/month): $3800, 1800, 2100, 6000$



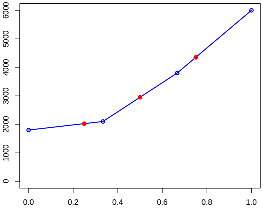
Quartiles = Evaluate the quantile function at 0.25, 0.50, 0.75

Example: Three small datasets

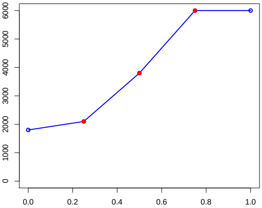
$$\vec{x} = (1800, 2100, 3800, 4000)$$
$$\vec{y} = (1800, 2100, 3800, 6000)$$
$$\vec{z} = (1800, 2100, 3800, 6000, 6000) \qquad (n \text{ not divisible by four})$$



$Q_x(0.50) = 2950$,
$m(x) = 2925$



$Q_y(0.50) = 2950$,
$m(x) = 3425$



$Q_z(0.50) = 3800$,
$m(x) = 3940$

Contents

Sample, population and “population”

The finite dataset you have, the “sample”, is often thought to “represent” qualities of a larger “population”.

sample	population
Pearson’s 1000 fathers and sons	All father-son pairs in . . . ?
1000 poll responses	Opinions of 5 million Finns now
272 eruptions of Old Faithful	All its eruptions (past? future? potential?)
Drug effect on 30 patients	Drug effect on future patients
100 rolls of a loaded die	Potential infinite sequence of rolls

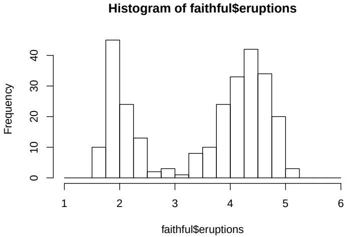
Population is statistical jargon for

- where your data came from (data-generating mechanism; data source)
- what you are trying to understand by looking at the data

Hence, terms such as *sample mean* and *population mean*.
The “population” may be quite concrete, or a figure of speech.

Old Faithful, once again

We have a *sample* of 272 eruption lengths. The real physical mechanism may be complicated, but perhaps we can think the lengths are as if they come from one particular distribution f. But what distribution?



One eruption length is a random variable X from some generating distribution or underlying distribution or true distribution. ("Population" if you want.)

Empirical distribution (shown above) approximates the generating distribution. Why? Think of the event $\{2.0 \leq X < 2.25\}$. We know by LLN that its *relative frequency* (long-run average) $\approx$ its *probability* (in generating distribution).

Next lecture is about inference.

From the data, we will estimate some parameters concerning the reality “behind” the data.