Aalto University
School of Electrical
Engineering

Brief Recap of Chapter 4 of Brown et al. (2014)

Simo Särkkä

Contents of Chapter 4 ”Magnetization, Relaxation, and the Bloch Equation”

Introduction

- **The spins interact with their surroundings.**
- **The effect can be modeled by the phenomenological Bloch equation.**
- **Formulated in terms of the average magnetic dipole moment density M.**
- **The relaxation decay times are T1, T2, T2', and T2*.**

Magnetization Vector

- **Material consists of a huge number of protons.**
- **Magnetization is the sum of the individual magnetic moments per volume:**

$$\vec{M} = \frac{1}{V} \sum_{i = \text{protons in V}} \vec{\mu}_i$$

- **The set of same-phase spins in voxel V is called a spin 'isochromat'**
- **If we neglect interactions of spins, we have**

$$\frac{1}{V} \sum_i \frac{d\vec{\mu}_i}{dt} = \frac{\gamma}{V} \sum_i \vec{\mu}_i \times \vec{B}_{ext}$$

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}_{ext} \qquad \text{(non-interacting protons)}$$

Magnetization Vector

- **Assume that the field is static in z-direction:** $\vec{B}_{ext} = B_0\hat{z}.$
- **Longitudinal and transverse components of magnetization are then:**

$$M_{||} = M_z$$

$$\vec{M}_{\perp} = M_x\hat{x} + M_y\hat{y}$$

- **The differential equations for them are:**

$$\frac{dM_z}{dt} = 0 \qquad \text{(non-interacting protons)}$$

$$\frac{d\vec{M}_{\perp}}{dt} = \gamma\vec{M}_{\perp} \times \vec{B}_{ext} \qquad \text{(non-interacting protons)}$$

- **The components ‘relax’ differently due to spin interactions**

Spin-Lattice Interaction and Regrowth Solution: T_1

- **The equilibrium value of magnetization** $M = M_0\, z$.
- $M(t)$ **approaches the equilibrium value due to spin-lattice interactions.**
- **The differential equation for the longitudal** z**-component is**

$$\frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z) \qquad (\vec{B}_{ext} \parallel \hat{z})$$

- T_1 **is the experimental 'spin-lattice relaxation time'**
- **The solution to the differential equation is**

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1})$$

- **Typical** T_1**s are given on the right:**

Tissue	T_1 (ms)	T_2 (ms)
gray matter (GM)	950	100
white matter (WM)	600	80
muscle	900	50
cerebrospinal fluid (CSF)	4500	2200
fat	250	60
blood[3]	1200	100–200[4]

Problem 1

Problem 4.1

The key equation (4.12) can be used to investigate general questions. If unmagnetized material is placed in a region with a finite static field at $t = 0$ ($M_z(0) = 0$):

> Find the time it takes, in units of T_1, for the longitudinal magnetization to reach 85% of M_0.

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \qquad (\vec{B}_{ext} \parallel \hat{z}) \qquad (4.12)$$

Spin-Spin Interaction and Transverse Decay: T_2

- **Due to spin-spin interactions, the individual spins 'fan out' or 'dephase'.**
- **The transverse relaxation is modeled by differential equation**

$$\frac{d\vec{M}_\perp}{dt} = \gamma \vec{M}_\perp \times \vec{B}_{ext} - \frac{1}{T_2}\vec{M}_\perp$$

- **Here T_2 is the 'spin-spin' relaxation time.**
- **In rotating frame of reference this is**

$$\left(\frac{d\vec{M}_\perp}{dt}\right)' = -\frac{1}{T_2}\vec{M}_\perp \qquad \text{(rotating frame)}$$

with the solution

$$\vec{M}_\perp(t) = \vec{M}_\perp(0)e^{-t/T_2} \qquad \text{(rotating frame)}$$

- **In practice, $T_1 > T_2$**

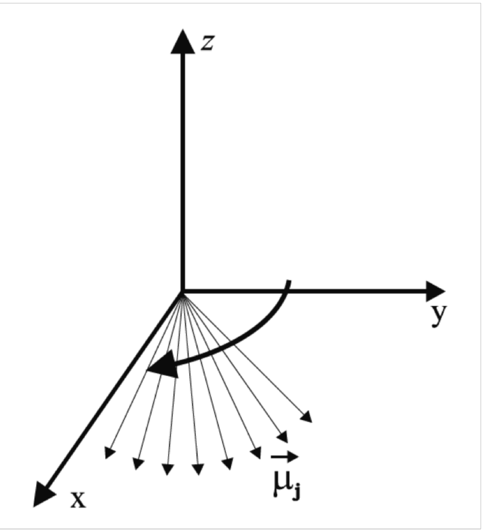

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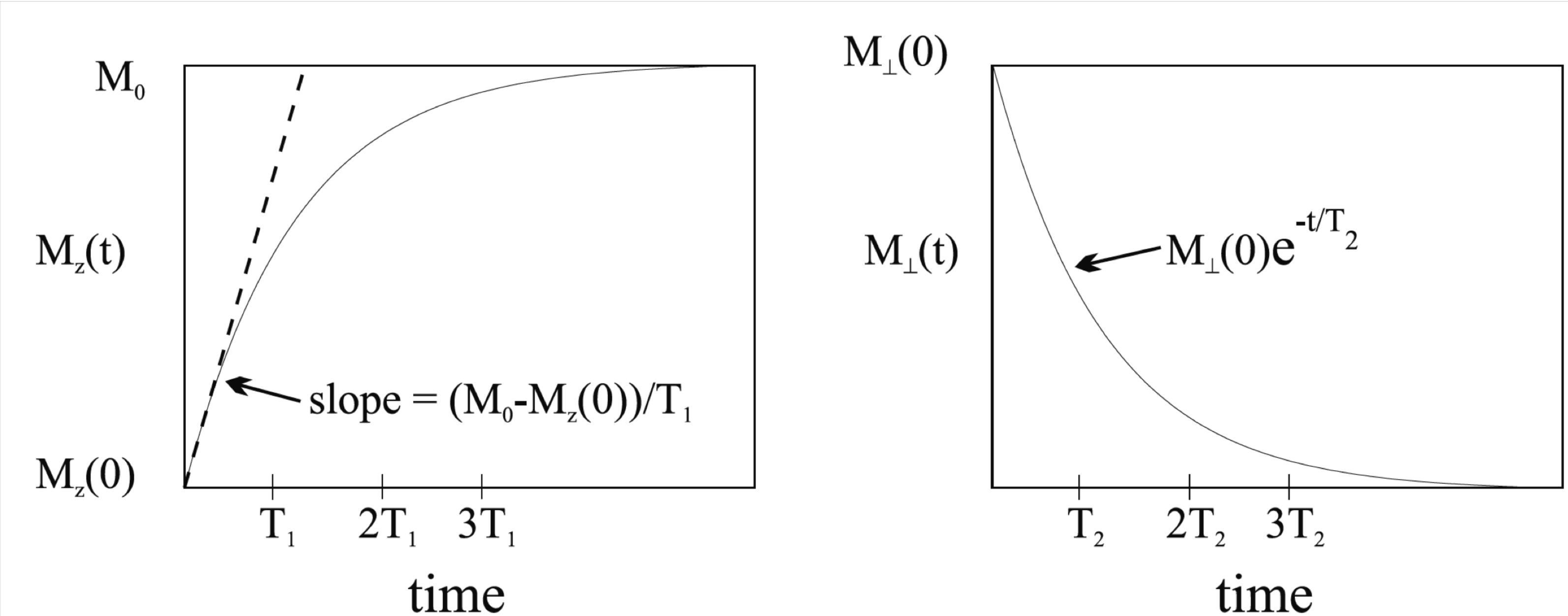
Illustration of T_1 and T_2



$$\frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z)$$

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1})$$

$$\left(\frac{d\vec{M}_\perp}{dt}\right)' = -\frac{1}{T_2}\vec{M}_\perp$$

$$\vec{M}_\perp(t) = \vec{M}_\perp(0)e^{-t/T_2}$$

Introduction of T_2' and T_2*

- **Define the relaxation rates by $R_1 = 1/T_1$ and $R_2 = 1/T_2$**
- **Additional dephasing results from magnetic field inhomogeneities which introduces rate $R_2' = 1/T_2'$**
- **The total relaxation rate due to external relaxation**

$$R_2^* = R_2 + R_2'$$

- **In terms of relaxation times ($R_2^* = 1/T_2^*$)**

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

- **Loss of transverse magnetization due to T_2' is <u>recoverable</u>.**
- **The intrinsic T_2 losses are <u>not recoverable</u>**
- **Related to "echoes" – we come back to this in later chapters**

Bloch Equation and Static-Field Solutions

- **The Bloch equation:**

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}_{ext} + \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}\vec{M}_\perp$$

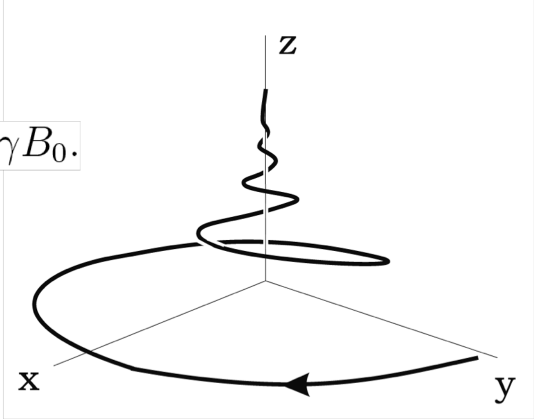
- **Then the component-wise equations are when** $\vec{B}_{ext} = B_0\hat{z}.$

$$\begin{aligned} \frac{dM_z}{dt} &= \frac{M_0 - M_z}{T_1} \\ \frac{dM_x}{dt} &= \omega_0 M_y - \frac{M_x}{T_2} \\ \frac{dM_y}{dt} &= -\omega_0 M_x - \frac{M_y}{T_2} \end{aligned}$$

$\omega_0 \equiv \gamma B_0.$

- **The solutions are**

$$\begin{aligned} M_x(t) &= e^{-t/T_2}\left(M_x(0)\cos\omega_0 t + M_y(0)\sin\omega_0 t\right) \\ M_y(t) &= e^{-t/T_2}\left(M_y(0)\cos\omega_0 t - M_x(0)\sin\omega_0 t\right) \\ M_z(t) &= M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \end{aligned}$$



Problem 2

A direct derivation of the steady-state solution, when it exists, of a system of differential equations can often be found by the following procedure. Assuming that the system evolves to constant value for large times, all time derivatives can be set to zero. The problem reduces to a system that can often be solved analytically. Show that the steady-state solution of the Bloch equations (4.37)–(4.39) is

$$M_{x'}^{ss} = M_0 \frac{\Delta\omega T_2}{D} \omega_1 T_2,$$

$$M_{y'}^{ss} = M_0 \frac{1}{D} \omega_1 T_2,$$

$$M_z^{ss} = M_0 \frac{1 + (\Delta\omega T_2)^2}{D},$$

where

$$D = 1 + (\Delta\omega T_2)^2 + \omega_1^2 T_1 T_2.$$

Problem (cont.)

$$\left(\frac{dM_z}{dt}\right)' = -\omega_1 M_{y'} + \frac{M_0 - M_z}{T_1} \qquad (4.37)$$

$$\left(\frac{dM_{x'}}{dt}\right)' = \Delta\omega\ M_{y'} - \frac{M_{x'}}{T_2} \qquad (4.38)$$

$$\left(\frac{dM_{y'}}{dt}\right)' = -\Delta\omega\ M_{x'} + \omega_1 M_z - \frac{M_{y'}}{T_2} \qquad (4.39)$$

with

$$\Delta\omega \equiv \omega_0 - \omega \qquad (4.40)$$

Complex representation of transverse magnetization

- **We can also denote**

$$M_+(t) \equiv M_x(t) + iM_y(t)$$

- **The solution in static field case**

$$M_+(t) = e^{-i\omega_0 t - t/T_2} M_+(0)$$

- **Alternatively we can write**

$$M_+(t) = |M_+(t)|e^{i\phi(t)} = M_\perp(t)e^{i\phi(t)}$$

$$M_\perp(t) = e^{-t/T_2} M_\perp(0)$$

$$\phi(t) = -\omega_0 t + \phi(0)$$

The Combination of Static and RF Fields

- **Let us add left-circularly polarized rf field B1:**

$$\vec{B}_{ext} = B_0\hat{z} + B_1\hat{x}'$$

- **The effective field in that frame is**

$$\vec{B}_{eff} = (B_0 - \frac{\omega}{\gamma})\hat{z} + B_1\hat{x}'$$

- **The Bloch equations in rotating frame:**

$$\left(\frac{dM_z}{dt}\right)' = -\omega_1 M_{y'} + \frac{M_0 - M_z}{T_1}$$

$$\left(\frac{dM_{x'}}{dt}\right)' = \Delta\omega\ M_{y'} - \frac{M_{x'}}{T_2}$$

$$\left(\frac{dM_{y'}}{dt}\right)' = -\Delta\omega\ M_{x'} + \omega_1 M_z - \frac{M_{y'}}{T_2}$$

$$\omega_1 = \gamma B_1$$

$$\Delta\omega \equiv \omega_0 - \omega$$

Short-Lived and Long-Lived RF Pulses

- **For short RF pulses, we can ignore relaxations**
  - Thus we get the flip equation as before
- **After the short pulse we can use the Bloch equations with the T_1 and T_2 relaxations**
- **For long pulses, the system saturates and is described with steady-state solutions**