A somewhat rigorous mathematical note on the arc length formula

(This is beyond the requirements and expectations of the course. It may be of interest to those intending to pursue more advanced mathematics.)

Let $\mathbf{r}(t) : [a, b] \to \mathbb{R}^n$, be a smooth (has a continuous derivative) curve which we denote by C. For a positive integer N let $\Delta t = (b-a)/N$. For $i = 0, \ldots, N$, let $t_i = a + i\Delta t$ and for $i = 1, \ldots, N$ let $\boldsymbol{\Delta\ell}_i = \mathbf{r}'(t_i) - \mathbf{r}'(t_{i-1})$. We define

$$\text{arc length}(C) = \lim_{N\to\infty} \sum_{i=1}^{N} \|\boldsymbol{\Delta\ell}_i\|$$

Claim:

$$\text{arc length}(C) = \int_a^b \|\mathbf{r}'(t)\|\, dt$$

Proof: The integral above is defined and finite since $\|\mathbf{r}'(t)\|$ is continuous on the close and bounded interval $[a, b]$ (by a standard calculus theorem). By the definiton of the definite integral,

$$\int_a^b \|\mathbf{r}'(t)\|\, dt = \lim_{N\to\infty} \sum_{i=1}^{N} \|\mathbf{r}'(t_i)\|\, \Delta t$$

We need to show

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\|\boldsymbol{\Delta\ell}_i\| - \|\mathbf{r}'(t_i)\|\, \Delta t\right) = 0 \tag{1}$$

That is, by the definition of the limit, given any $\epsilon_1 > 0$ we need to find an $N > 0$ such that for all $K \geq N$,

$$\left|\sum_{i=1}^{K} \left(\|\boldsymbol{\Delta\ell}_i\| - \|\mathbf{r}'(t_i)\|\, \Delta t\right)\right| < \epsilon_1. \tag{2}$$

Now, from the definition of the derivative,

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

and using the formal definition of the limit we have that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta$ implies

$$\left\|\mathbf{r}'(t) - \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}\right\| < \epsilon.$$

Since by assumption $\mathbf{r}'(t)$ is continuous on the closed and bounded interval $[a, b]$, it is actually *uniformly continuous*. This implies that given an $\epsilon > 0$ there exists a single $\delta > 0$ such that the previous inequality holds with for each $t = t_i, i = 1, \ldots, N$. Thus, if $|h| < \delta$ then for $i = 1, \ldots, N$

$$\left\|\mathbf{r}'(t_i) - \frac{\boldsymbol{\Delta\ell}_i}{h}\right\| < \epsilon.$$

Starting with the left-hand-side of (2) and using the *triangle inequality* (for real numbers) and the *reverse triangle inequality* we get that for $\Delta t < \delta$,

$$\begin{aligned}
\left|\sum_{i=1}^{K} \left(\|\boldsymbol{\Delta\ell}_i\| - \|\mathbf{r}'(t_i)\|\, \Delta t\right)\right| &= \left|\sum_{i=1}^{K} \left(\left\|\frac{\boldsymbol{\Delta\ell}_i}{\Delta t}\right\| - \|\mathbf{r}'(t_i)\|\right)\right| \Delta t \\
&\leq \sum_{i=1}^{K} \left|\left(\left\|\frac{\boldsymbol{\Delta\ell}_i}{\Delta t}\right\| - \|\mathbf{r}'(t_i)\|\right)\right| \Delta t \\
&\leq \sum_{i=1}^{K} \left\|\frac{\boldsymbol{\Delta\ell}_i}{\Delta t} - \mathbf{r}'(t_i)\right\| \Delta t \\
&< N\epsilon\Delta t \\
&= \epsilon(b-a).
\end{aligned}$$

Note that since $\Delta t = (b-a)/N$, $\Delta t < \delta \iff N > (b-a)/\delta$. Now choosing $\epsilon < \epsilon_1/(b-a)$ and $N > (b-a)/\delta$ we obtain (2) as desired.