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1 Event spaces and probability distributions

This exercise gets you introduced to the notion of a sigma-algebra, the mathematical model of an event space used in probability and statistics. In addition, you start to treat probability distributions as measures defined on a sigma-algebra.

1.1 *Properties of sigma-algebras.* Let $\mathcal{F}$ be a sigma-algebra on a set Ω. Prove the following statements using the basic axioms of a sigma-algebra.

Hint. de Morgan's laws may be helpful.

(a) $\mathcal{F}$ contains the empty set $\emptyset$.

(b) $\mathcal{F}$ is closed under symmetric set differences: $A \Delta B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Hint. de Morgan's laws.

(c) $\mathcal{F}$ is closed under countable intersections: $\cap_{i \geq 1} A_i \in \mathcal{F}$ whenever $A_1, A_2, \ldots \in \mathcal{F}$.

1.2 *Sigma-algebras on small finite sets.* Let a, b, c be three distinct elements.

(a) Write down all sigma-algebras on $\Omega = \{a, b\}$.

(b) Write down all sigma-algebras on $\Omega' = \{a, b, c\}$.

(c) Give an explicit counterexample which shows that the union of two sigma-algebras is not necessarily a sigma-algebra.

1.3 *Probability distributions on countable spaces.* Let S be a finite set, and denote by $\mathcal{P}(S)$ the collection of all subsets of S. A function $p : S \to \mathbb{R}$ is called a *probability mass function (pmf)* if $p(s) \geq 0$ for all s and $\sum_{s \in S} p(s) = 1$.

(a) Show that if p is a pmf on S, then the set function $\mu(F) = \sum_{s \in F} p(s)$ is a probability measure on $(S, \mathcal{P}(S))$.

(b) Show that if μ is a probability measure on $(S, \mathcal{P}(S))$, then the function $p(s) = \mu(\{s\})$ is a pmf on S.

1.4 *Monotone continuity of probability measures.* Let $(S, \mathcal{F}, \mu)$ be a probability space, that is, μ is a probability measure on $(S, \mathcal{F})$. For real numbers $x_1, x_2, \dots$ we write

- $x_n \uparrow x$ if $x_1 \le x_2 \le \cdots$ and $x = \lim_{n\to\infty} x_n$,
- $x_n \downarrow x$ if $x_1 \ge x_2 \ge \cdots$ and $x = \lim_{n\to\infty} x_n$.

For events $F_1, F_2, \dots$ we write

- $F_n \uparrow F$ if $F_1 \subset F_2 \subset \cdots$ and $\cup_{n\ge 1} F_n = F$.
- $F_n \downarrow F$ if $F_1 \supset F_2 \supset \cdots$ and $\cap_{n\ge 1} F_n = F$.

(a) Prove that $F_n \uparrow F \implies \mu(F_n) \uparrow \mu(F)$.

(b) Prove that $F_n \downarrow F \implies \mu(F_n) \downarrow \mu(F)$.

(c) Are the statements (a) and (b) true also in the case when μ is just assumed to be a measure on $(S, \mathcal{F})$, not necessarily a probability measure?

1.5 *Borel sets of the two-dimensional Euclidean space.* The Borel sigma-algebra $\mathcal{B}(\mathbb{R}^2)$ is defined as the smallest sigma-algebra on $\mathbb{R}^2$ which contains all open sets in $\mathbb{R}^2$. Denote the collection of closed south-west quadrants of $\mathbb{R}^2$ by

$$\pi(\mathbb{R}^2) \;=\; \Big\{(-\infty, x] \;\times (-\infty, y] : x, y \in \mathbb{R}\Big\}.$$

Prove that this collection generates $\mathcal{B}(\mathbb{R}^2)$, that is, $\mathcal{B}(\mathbb{R}^2) = \sigma(\pi(\mathbb{R}^2))$.

Hint. The same line of proof as in the one-dimensional case ([Kyt19, Prop I.11] or [JP04, Thm 2.1] or [Wil91, 1.2.a])works here. You may use the fact that every open set in $\mathbb{R}^2$ can be written as a countable union $\bigcup_{n=1}^{\infty} R_n$ of open rectangles of the form $R_n = (a_n, b_n) \times (a'_n, b'_n)$.

References

[JP04] Jean Jacod and Philip Protter. *Probability Essentials.* Springer, second edition, 2004.

[Kyt19] Kalle Kytölä. Probability theory. Lecture notes, 2019.

[Wil91] David Williams. *Probability with Martingales.* Cambridge University Press, 1991.