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3 Stochastic dependence and independence

In this exercise you learn techniques for investigating when given random variables are mutually independent. You also get to analyse events concerning infinite random sequences.

3.1 *Independent pairs.* Let X and Y be $\mathbb{R}$-valued random variables defined on a common probability space $(\Omega, \mathscr{F}, \mathsf{P})$.

(a) Show that if X and Y are independent, then $f(X)$ and $g(Y)$ are independent for any Borel-measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

(b) Show that if $f(X)$ and $g(Y)$ are independent for all Borel-measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$, then X and Y are independent.

(c) Assume that $\mathsf{P}\big[X + Y = 42\big] = 1$. Is it possible that X and Y are independent?

3.2 *Independence of three sigma-algebras.* Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space and let $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3 \subset \mathscr{F}$ be sigma-algebras on Ω. Assume that $\mathscr{G}_k$ is generated by a π-system $\mathscr{I}_k$ which contains Ω.

(a) Show that $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3$ are independent if and only if

$$\mathsf{P}\big[I_1 \cap I_2 \cap I_3\big] = \mathsf{P}[I_1]\,\mathsf{P}[I_2]\,\mathsf{P}[I_3]$$

for all $I_1 \in \mathscr{I}_1$, $I_2 \in \mathscr{I}_2$, $I_3 \in \mathscr{I}_3$.

(b) Why did we require that $\mathscr{I}_k$ contains Ω?

3.3 *Product of independent random signs.* Let $(X_0, X_1, X_2, \dots)$ be a sequence of independent random integers such that $\mathsf{P}\big[X_n = -1\big] = \mathsf{P}\big[X_n = +1\big] = \frac{1}{2}$ for all n, and let

$$Z_n = X_0 X_1 \cdots X_n.$$

(a) Are Z_0 and Z_2 independent?

(b) Is the collection $\{Z_0, Z_1, Z_2\}$ independent?

(c) Is the collection $\{Z_0, Z_1, Z_2, \dots\}$ independent?

Justify your answers carefully.

3.4 *Decimal digits of a uniform random number.* Let P be the uniform probability measure on the unit interval $[0,1]$ (see Example II.12). Let $D_k(\omega)$ be the k-th digit in the decimal representation of a number $\omega \in [0,1]$, so that

$$\omega = \sum_{k=1}^{\infty} D_k(\omega)\, 10^{-k},$$

and denote the run length of zeroes starting from the k-th digit by

$$Z_k = \begin{cases} 0, & \text{if } D_k \neq 0, \\ m, & \text{if } D_k = D_{k+1} = \cdots = D_{k+m-1} = 0 \text{ and } D_{k+m} \neq 0. \end{cases}$$

For this exercise you may assume it known that $D_1, D_2, \ldots$ are mutually independent random variables (on the sample space $\Omega = [0,1]$).

(a) Show that $\mathsf{P}\big[Z_k = m\big] = \frac{9}{10^{m+1}}$ for all integers $m \geq 0$ and $k \geq 1$.

(b) Show that $\mathsf{P}\big[Z_k = 42 \text{ for infinitely many } k\big] = 1$.

(c) Show that $\mathsf{P}\big[Z_k = k \text{ for infinitely many } k\big] = 0$.

Hint. Borel–Cantelli lemmas.

3.5 *Infinite-horizon events.* Let $X_1, X_2, X_3, \ldots$ be a sequence of random numbers ($\mathbb{R}$-valued random variables) defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Which of the following events belong to the tail sigma-algebra $\mathcal{T}_\infty = \bigcap_{n\geq 1} \sigma(X_{n+1}, X_{n+2}, \ldots)$? Justify your answers carefully.

(a): $\Big\{\omega \in \Omega \,\Big|\, \lim_{n\to\infty} X_n(\omega) \text{ exists}\Big\}$

(b): $\Big\{\omega \in \Omega \,\Big|\, \sum_{n=1}^{\infty} X_n(\omega) \leq 42\Big\}$

(c): $\Big\{\omega \in \Omega \,\Big|\, \forall\, \ell \in \mathbb{N}\ \exists n \in \mathbb{N} \text{ such that } X_n(\omega) = X_{n+1}(\omega) = \cdots = X_{n+\ell}(\omega)\Big\}$
$= \{\text{there exists arbitrarily long repetitions in the sequence } X_1, X_2, \ldots\}$