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# 4 Integrations and expectations

In this exercise you learn to apply Lebesgue's abstract integration theory to derive useful formulas related to probability distributions and expectations of real-valued random variables, both discrete (countable range) and continuous (having a Lebesgue-integrable probability density function).

**4.1** *The absolute value of a random number.* Let $Y = |X|$ where $X$ is a real-valued random variable with distribution $P_X[B] = \mathsf{P}\big[X \in B\big]$.

(a) Prove that $Y$ is a random variable.

(b) Assume that we know the cumulative distribution function $F_X(x) = \mathsf{P}[X \le x]$ of $X$. What is the cumulative distribution function of $Y$?

(c) Assume that $X$ has a continuous law with a density function $f_X(x)$. Does $Y$ also have a continuous law in this case? If yes, write down an expression for a density function of $Y$ in terms of $f_X$. If not, explain why not.

**4.2** *Discrete random numbers.* A random variable is *discrete* if its range $A = X(\Omega)$ is finite or countably infinite. The probability mass function of a discrete random variable $X$ is defined by $p_X(x) = \mathsf{P}\big[X = x\big]$ for $x \in A$. Prove that any discrete real-valued random variable satisfies:

(a) $\mathbb{E}\big[h(X)\big] = \sum_{x \in A} h(x)\, p_X(x)$ for all Borel functions $h : \mathbb{R} \to [0, \infty)$.

(b) $h(X) \in \mathcal{L}^1(\Omega, \mathscr{F}, \mathsf{P})$ if and only if $\sum_{x \in A} |h(x)|\, p_X(x) < \infty$.

(c) Explain why the formula in (a) is true for all $h \in \mathcal{L}^1(\mathbb{R}, \mathscr{B}, P_X)$.

**Hint.**Recall the argument leading to the equation (see lectures or Williams, Sec. 6.12)

$$\mathbb{E}\big[h(X)\big] := \int_\Omega h(X(\omega))\, \mathrm{d}\mathsf{P}(\omega) = \int_{\mathbb{R}} h(x)\, \mathrm{d}P_X(x).$$

**4.3** *Pointwise convergence and expectation.* Write down an example of a random sequence $X_1, X_2, \dots$ such that each $X_n$ is integrable and $X_n \to X$ pointwise, but $\mathbb{E}[X_n] \not\to \mathbb{E}[X]$.

**4.4** *Random numbers with continuous laws.* Assume that $X$ has a continuous law with a density function $f_X$.

(a) Show that $X$ is integrable if and only if $\int_{\mathbb{R}} |x|\, f_X(x)\, \mathrm{d}x < \infty$.

(b) If $X$ is integrable, show that the expectation of $X$ is given by

$$\mathbb{E}\big[X\big] = \int_{\mathbb{R}} x\, f_X(x)\, \mathrm{d}x.$$

(c) If $X$ is integrable with expected value $\mu = \mathbb{E}[X]$, show that the variance of $X$ can be computed as

$$\mathrm{var}(X) := \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \int_{\mathbb{R}} (x-\mu)^2 f_X(x)\,\mathrm{d}x.$$

(d) Can a random variable with continuous law have more than one density function?

**Hint.**Remember the hint for Problem 4.2, and seek a unified argument leading to (a) and (b) — perhaps even (c).

**4.5** *Jensen's inequality.* Let $I \subset \mathbb{R}$ be an interval. Let $\phi \colon I \to \mathbb{R}$ be a convex function, i.e., a function such that for any $x, y \in I$ and any $r \in (0,1)$ we have

$$\phi\Big((1-r)x + ry\Big) \;\le\; (1-r)\,\phi(x) + r\,\phi(y).$$

(a) For any $a, b \in I$ such that $a < b$, form the difference quotient

$$Q(a,b) := \frac{\phi(b) - \phi(a)}{b - a}.$$

Show that for a fixed $a \in I$, the function $b \mapsto Q(a,b)$ is increasing.

**Hint.** Choose $x = a$, $y = b_2$, $r = \frac{b_1 - a}{b_2 - a}$ for $a < b_1 < b_2$ in the definition of convexity.

(b) Similarly show that for a fixed $b \in I$, the function $a \mapsto Q(a,b)$ is increasing.

(c) Show that for any interior point $z$ of the interval $I$ the limits

$$D_z^+ := \lim_{h \downarrow 0} \frac{\phi(z+h) - \phi(z)}{h} \qquad \text{and} \qquad D_z^- := \lim_{h \downarrow 0} \frac{\phi(z) - \phi(z-h)}{h}$$

exist and $D_z^- \le D_z^+$ and both limits are finite.

(d) Let $z$ be any interior point of the interval $I$ and let $D_z$ be a number such that $D_z^- \le D_z \le D_z^+$. Show that for all $x \in I$ we have $\phi(x) \ge \phi(z) + (x - z)D_z$.

(e) Let $X$ be a random variable with values in the interval $I \subset \mathbb{R}$, and assume that both $X$ and $\phi(X)$ are integrable random variables. Prove Jensen's inequality:

$$\mathbb{E}\big[\phi(X)\big] \;\ge\; \phi\big(\mathbb{E}[X]\big).$$

**Hint.**Take $z = \mathbb{E}[X]$ in the previous part, and use monotonicity and linearity.