Leskelä–Evdoridis–Karjalainen

5 Products of sigma-algebras and measures

In this exercise you learn to work with product measures, and you also get introduced to the concept of probability kernel, an important tool in Bayesian statistics and operations research models.

5.1 *Dirac measures on* $\mathbb{R}^n$. The *Dirac measure* at a point $a \in \mathbb{R}^n$ is defined by

$$\delta_a[A] = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that δ_a is a probability measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$.

(b) Show that any Borel function $f : \mathbb{R}^n \to \mathbb{R}$ is δ_a-integrable (meaning that $\int_{\mathbb{R}^n} |f(x)| \, d\delta_a(x)$ is finite), and compute the integral $\int_{\mathbb{R}^n} f(x) \, d\delta_a(x)$.

Consider now the case $n = 1$.

(c) Does the measure δ_a have a probability density function with respect to the Lebesgue measure on $\mathbb{R}$? If yes, find out an expression for it. If not, explain why.

5.2 *Product of Borel sigma-algebras.* Let $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$ be the smallest sigma-algebra on $\mathbb{R}^3$ for which the projection maps $\mathrm{pr}_i : (x_1, x_2, x_3) \mapsto x_i$, $i = 1, 2, 3$, are measurable.

(a) Prove that $\mathcal{I} = \{A_1 \times A_2 \times A_3 : A_1, A_2, A_3 \in \mathscr{B}(\mathbb{R})\}$ is a π-system on $\mathbb{R}^3$ which generates $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$.

(b) Verify that $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \mathscr{B}(\mathbb{R}^3)$.

Hint.A set $A \subset \mathbb{R}^n$ is open if and only if it can be written as a countable union of open boxes of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ with a_i, b_i being rational numbers.

5.3 *Random vectors.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Let $X = (X_1, X_2, X_3)$ be an $\mathbb{R}^3$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that X_1, X_2, X_3 are $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) Let X_1, X_2, X_3 be $\mathbb{R}$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $X = (X_1, X_2, X_3)$ is a $\mathbb{R}^3$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

5.4 *Disintegration of independent random variables.* Let X and Y be independent real-valued random variables with laws P_X and P_Y. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be Borel function.

(a) Prove that $\mathbb{E}h(X,Y) = \int_{\mathbb{R}} \mathbb{E}\big[h(x,Y)\big]\,\mathrm{d}P_X(x) = \int_{\mathbb{R}} \mathbb{E}\big[h(X,y)\big]\,\mathrm{d}P_Y(y)$ whenever h is non-negative.

Hint. Fubini's theorem and product measure.

Let $Z = U_1 U_2$ where U_1 and U_2 be independent and uniformly distributed on $[0,1]$.

(b) Calculate the cumulative distribution function $F_Z(x) = \mathbb{P}\big[Z \le x\big]$.

(c) Does Z have a probability density function? If yes, find out an expression for it. If not, explain why.

5.5 *Probability kernels.* Denote $\mathscr{B} := \mathscr{B}(\mathbb{R})$ and let K be a probability kernel on $(\mathbb{R}, \mathscr{B})$, that is, a mapping $\mathbb{R} \times \mathscr{B} \to [0,+\infty)$ denoted by $(x,B) \mapsto K_x[B]$ such that

- for any $B \in \mathscr{B}$, the mapping $x \mapsto K_x[B]$ is Borel-measurable $\mathbb{R} \to [0,+\infty)$
- for any $x \in \mathbb{R}$, the mapping $B \mapsto K_x[B]$ is a probability measure on $(\mathbb{R}, \mathscr{B})$.

Let μ be a probability measure on $(\mathbb{R}, \mathscr{B})$.

(a) Define a set function μK by $(\mu K)[B] = \int_{\mathbb{R}} K_x[B]\,\mathrm{d}\mu(x)$, $B \in \mathscr{B}$. Show that μK is a probability measure on $(\mathbb{R}, \mathscr{B})$.

(b) Define, for Borel subsets $A \subset \mathbb{R}^2$

$$\nu[A] = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{I}_A(x_0, x_1)\,\mathrm{d}K_{x_0}(x_1) \right) \mathrm{d}\mu(x_0).$$

Show that ν is a probability measure on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$.

(c) Let $X = (X_0, X_1)$ be a random vector in $\mathbb{R}^2$ with distribution $\mathrm{Law}_X = \nu$ given in (b). Find out the distributions Law_{X_0} and Law_{X_1} of its components.

Note: *(X_0, X_1) can be viewed as the first two values of a Markov chain with initial distribution μ and transition "matrix" K. The distribution of the whole Markov chain can be defined by generalizing the above construction.*