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6 Convergence of random sequences

In this exercise you study which random sequences converge almost surely or converge in probability.

6.1 *Variance.* Denote the variance of a random variable $X \in \mathcal{L}^2(\mathsf{P})$ with expected value $m_X = \mathbb{E}[X]$ by $\operatorname{Var}(X) = \mathbb{E}\big[(X - m_X)^2\big]$.

(a) Prove that $\operatorname{Var}(X + a) = \operatorname{Var}(X)$ and $\operatorname{Var}(aX) = a^2 \operatorname{Var}(X)$ for all $a \in \mathbb{R}$.

(b) If $X_1, \ldots, X_n \in \mathcal{L}^2(\mathsf{P})$ are independent, show that $\operatorname{Var}(\sum_{k=1}^n X_k) = \sum_{k=1}^n \operatorname{Var}(X_k)$.

(c) Does the result of (b) remain true for dependent random variables? Prove the claim or give a counterexample

6.2 *Some characteristic functions.*

(a) Let $p \in [0, 1]$. Calculate the characteristic function $\varphi_B(\theta) = \mathbb{E}[e^{\mathrm{i}\,\theta B}]$ of a random variable B such that $\mathsf{P}[B = 1] = p$ and $\mathsf{P}[B = 0] = 1 - p$ (we denote $B \sim \text{Bernoulli}(p)$).

(b) Let $p \in [0, 1]$ and $n \in \mathbb{N}$. Calculate the characteristic function $\varphi_Z(\theta) = \mathbb{E}[e^{\mathrm{i}\,\theta Z}]$ of a random variable Z such that $\mathsf{P}[Z = k] = \binom{n}{k} p^k (1 - p)^{n-k}$ for all $k \in \{0, 1, 2, \ldots, n\}$ (we denote $Z \sim \text{Bin}(n, p)$).

(c) Show that if X and Y are independent, with characteristic functions φ_X and φ_Y, then the characteristic function of $X + Y$ is $\varphi_{X+Y}(\theta) = \varphi_X(\theta)\, \varphi_Y(\theta)$.

(d) Let $B_1, \ldots, B_n$ be independent and identically distributed, with $\mathsf{P}[B_j = 1] = p$ and $\mathsf{P}[B_j = 0] = 1 - p$, for all j. Compute the characteristic function of $S = B_1 + \cdots + B_n$ using parts (a) and (c). Compare with the result of part (b), and conclude that $S \sim \text{Bin}(n, p)$.

6.3 *Convergence in probability and convergence almost surely along a subsequence.* Assume that $X_1, X_2, \ldots$ are real-valued random variables and $X_n \xrightarrow{\mathsf{P}} X$. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be two sequences of positive real numbers such that $a_k \downarrow 0$ and $\sum_{k=1}^{\infty} b_k < +\infty$ — for example $a_k = \frac{1}{k}$ and $b_k = 2^{-k}$.

(a) Show that there exist positive integers $n_1 < n_2 < \cdots$ such that $\mathsf{P}\big[|X_{n_k} - X| \geq a_k\big] \leq b_k$.

(b) With the sequence $(n_k)_{k \in \mathbb{N}}$ chosen as in part (a), show that

$$\mathsf{P}\Big[|X_{n_k} - X| \geq a_k \text{ for infinitely many } k\Big] = 0.$$

(c) With the sequence $(n_k)_{k \in \mathbb{N}}$ as in part (a), show that $X_{n_k} \xrightarrow{\text{a.s.}} X$ as $k \to \infty$.

Hint. Recall a suitable Borel-Cantelli lemma.

6.4 *Continuity and convergence in probability.* A function $h\colon \mathbb{R} \to \mathbb{R}$ is called *uniformly continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$ satisfying $|x-y| < \delta$ we have $|h(x) - h(y)| < \varepsilon$. Let $X, X_1, X_2, \ldots$ be real-valued random variables defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Prove the following statements:

(a) For any uniformly continuous $h\colon \mathbb{R} \to \mathbb{R}$,

$$X_n \xrightarrow{\mathbb{P}} X \qquad \Longrightarrow \qquad h(X_n) \xrightarrow{\mathbb{P}} h(X).$$

Remark: The conclusion is actually valid for any continuous $h\colon \mathbb{R} \to \mathbb{R}$. You don't have to prove this, if you are busy with other things.

(b) For any bounded uniformly continuous $h\colon \mathbb{R} \to \mathbb{R}$,

$$X_n \xrightarrow{\mathbb{P}} X \qquad \Longrightarrow \qquad \begin{cases} \mathbb{E}\big[|h(X_n) - h(X)|\big] \to 0, \\ \mathbb{E}\big[h(X_n)\big] \to \mathbb{E}\big[h(X)\big]. \end{cases}$$

Remark: The conclusions are not valid without the assumption of boundedness. Can you give a counterexample in that case?

6.5 Let $X_3, X_4, \ldots$ be independent random variables such that for $k = 3, 4, \ldots$ we have

$$\mathsf{P}\big[X_k = 0\big] = 1 - \frac{1}{k\,\log(k)} \qquad \text{and} \qquad \mathsf{P}\big[X_k = +k\big] = \frac{1}{2k\,\log(k)} = \mathsf{P}\big[X_k = -k\big].$$

(a) Calculate the expected value and variance of X_k.

(b) Show that we have

$$\sum_{j=3}^{\infty} \frac{1}{j\,\log(j)} = \infty \qquad \text{and} \qquad \frac{1}{n^2} \sum_{j=3}^{n} \frac{j}{\log(j)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Hint. Recall that the integral function of $x \mapsto \frac{1}{x\,\log(x)}$ is $x \mapsto \log(\log(x))$.

For $n \geq 3$, define the average

$$A_n = \frac{1}{n-2} \sum_{k=3}^{n} X_k.$$

(a) Does the sequence $(A_n)_{n=3,4,\ldots}$ of averages converge almost surely?

(b) Does the sequence $(A_n)_{n=3,4,\ldots}$ of averages converge in probability?