Computational Algebraic Geometry

Geometry, Algebra and Algorithms

Kaie Kubjas

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January 11, 2021

Organization

Schedule:

- lectures Mo and We 14.15-16.00 (Kaie Kubjas)
- exercises Fr 12.15-14.00 (Muhammad Ardiyansyah)

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Lecture materials:

- “Ideals, Varieties and Algorithms” by Cox, Little and O’Shea
- “Numerically solving polynomial systems with Bertini” by Bates, Hauenstein, Sommese and Wampler
- Further reading: “Nonlinear algebra” by Michalek and Sturmfels

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Grade:

- five weekly homework assignments (50% of the grade)
- homework is handed out on Tuesday and the deadline is one week later on Wednesday
- it is encouraged to discuss homework in small groups (2-3 persons), but everyone has to write down their solutions
- a final exam at the end of the course (50% of the grade)
- correcting mistakes gives 0.5 points

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Optional extra homework:

- You can submit any exercise from “Nonlinear algebra” by Michalek and Sturmfels as extra homework.
- Each exercise gives 3 points.
- Sections 1-4 are most related to this course.

Exam

Suggestion: February 22 (Monday), 13:00-17:00

- the exam will be an open book exam
- if this time doesn't work for you, let me know before the lecture on Wednesday

What is this course about?

- An important goal is to learn basic theory and tools for investigating systems of polynomial equations, e.g.

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  - If the solution set is infinite, then we can aim to describe each irreducible component.
- To be able to solve simple problems computationally.
- To learn to recognize polynomial systems in applications.

Content

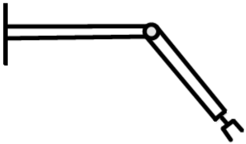
We will cover chapters 1-4 and 6 from “Ideals, Varieties and Algorithms”:

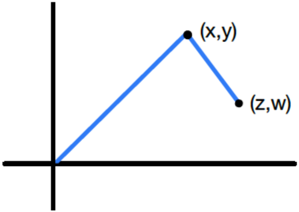
- Chapter 1: Geometry, Algebra and Algorithms (1 week)
- Chapter 2: Groebner Bases (1.5 weeks)
- Chapter 3: Elimination Theory (1 week)
- Chapter 4: The Algebra-Geometry Dictionary (1.5 weeks)
- Chapter 6: Robotics (1 lecture)
- Additional topic: Numerical algebraic geometry (1 lecture)

Most results will be presented together with proofs.

Robotics

- suppose we have a robot arm in the plane consisting of two linked rods of lengths 1 and 2, with the longer rod anchored at the origin





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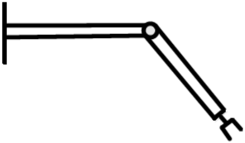

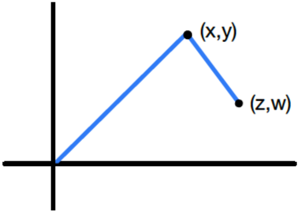

- the "state" of the arm is completely described by the coordinates (x, y) and (z, w) indicated in the figure

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- not all 4-tuples can occur as states of the arm
- the subset of possible states is the affine variety in $\mathbb{R}^4$ defined by the equations

$$\begin{aligned} x^2 + y^2 &= 4 \\ (x - z)^2 + (y - w)^2 &= 1. \end{aligned}$$

Today

- Polynomials and affine space
- Affine varieties
- Parametrizations of affine varieties

Today's lecture is based on Chapters 1.1-1.3 in "Ideals, Varieties and Algorithms".

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Example

The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are fields, but integers $\mathbb{Z}$ is not a field.

Fields

Fields are important: linear algebra works over any field!

We will employ different fields for different purposes:

- The rational numbers $\mathbb{Q}$ for doing computations.
- The real numbers $\mathbb{R}$ for drawing pictures.
- The complex numbers $\mathbb{C}$ for proving theorems.

Monomials

Definition

A **monomial** in $x_1, \ldots, x_n$ is a product of the form

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative integers. The **total degree** of this monomial is $\alpha_1 + \ldots + \alpha_n$.

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- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ n-tuple of nonnegative integers

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

- the total degree $|\alpha| = \alpha_1 + \ldots + \alpha_n$

Polynomials

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A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in k is a finite linear combination (with coefficients in k) of monomials. We will write a polynomial f in the form

$$f = \sum_{\alpha} a_\alpha x^\alpha, a_\alpha \in k.$$

The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in k is denoted by $k[x_1, \ldots, x_n]$.

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Definition

- We call a_α the coefficient of the monomial x^α.
- If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a term of f.
- The total degree of f, denoted $\deg(f)$, is the maximum $|\alpha|$ such that the coefficient a_α is nonzero.

Polynomials

Quiz

Let $f = 2x^3y^2z + \frac{3}{2}y^3z^3 - 3xyz + y^2$*.*

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- we refer to $k[x_1, \ldots, x_n]$ as a polynomial ring

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Given a field k and a positive integer n, we define the n-dimensional **affine space** over k to be the set

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- this makes it possible to link algebra and geometry

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These two statements are not equivalent in general.

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Let $k = \mathbb{F}_2$ and $f = x^2 - x \in \mathbb{F}_2[x]$. It gives the zero function, but not the zero polynomial.

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Proposition

Let k be an infinite field, and let $f \in k[x_1, \ldots, x_n]$. Then $f = 0$ in $k[x_1, \ldots, x_n]$ if and only if $f : k^n \to k$ is the zero function.

Proof: The zero polynomial clearly gives the zero function.

<u>Other direction</u>: WNTS if $f(a_1,\dots,a_n)=0$ for all $(a_1,\dots,a_n)\in k^n$, then f is the zero polynomial. We will use induction.

$\boxed{n=1:}$ A nonzero polynomial in $k[x]$ of degree m has at most m roots (we will prove this next time). We assume $f(a)=0$ for all $a\in k$. Since k is infinite, this means that f has infinitely many roots, hence f must be the zero polynomial.

<u>Induction step</u>: Assume that the statement holds for $n-1$. We can write f in the form

$$f=\sum_{i=0}^{N} g_i(x_1,\dots,x_{n-1})\,x_n^i,$$

where $g_i\in k[x_1,\dots,x_{n-1}]$. We will show that each g_i is the zero polynomial in $n-1$ variables.

Let us fix $(a_1,\dots,a_{n-1})\in k^{n-1}$. We get the pl.

$f(a_1, \dots, a_{n-1}, x_n) \in k[x_n]$. It follows from the case $n=1$ that $f(a_1, \dots, a_{n-1}, x) \in k[x_n]$ is the zero polynomial. Hence $g_i(a_1, \dots, a_{n-1}) = 0$ for all i. Since $(a_1, \dots, a_{n-1}) \in k^{n-1}$ was chosen arbitrarily, g_i is the zero polynomial for all i. Hence f is the zero polynomial. $\blacksquare$

Polynomial vs function

Corollary

Let k be an infinite field, and let $f, g \in k[x_1, \ldots, x_n]$. Then $f = g$ in $k[x_1, \ldots, x_n]$ if and only if $f : k^n \to k$ and $g : k^n \to k$ are the same function.

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Every nonconstant polynomial $f \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

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We say that a field k is algebraically closed if every nonconstant polynomial in $k[x]$ has a root in k.

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Example

Thus $\mathbb{R}$ is not algebraically closed ($x^2 + 1$ has no roots over $\mathbb{R}$), whereas by the previous theorem $\mathbb{C}$ is algebraically closed.

Affine varieties

Definition

Let k be a field, and let $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$. Then we set

$$\mathbb{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0 \quad \text{for all } 1 \leq i \leq s\}.$$

We call $\mathbb{V}(f_1, \ldots, f_s)$ the **affine variety** defined by $f_1, \ldots, f_s$.

Examples

What is the variety $\mathbb{V}(x^2 + y^2 - 1)$ in the plane $\mathbb{R}^2$?

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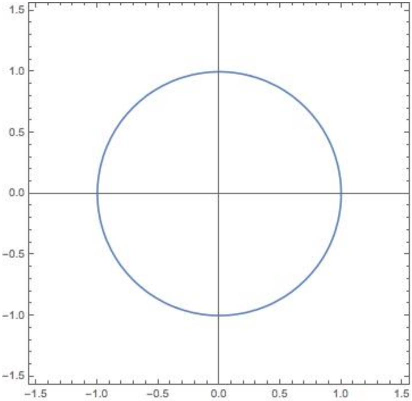
What is the variety $\mathbb{V}(x^2 + y^2 - 1)$ in the plane $\mathbb{R}^2$? It is the circle of radius 1 centered at the origin:



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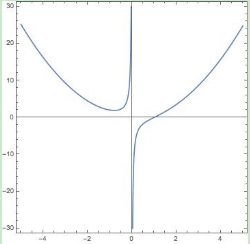
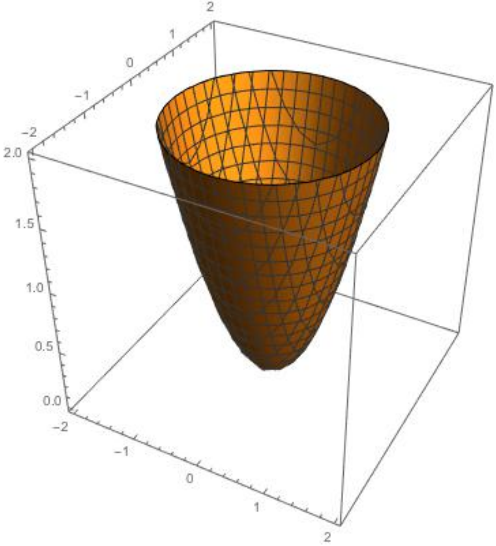
Example

The graph of $y = \frac{x^3-1}{x}$ gives the affine variety $\mathbb{V}(xy - x^3 + 1)$.



Examples

Paraboloid of revolution $\mathbb{V}(z - x^2 - y^2)$:



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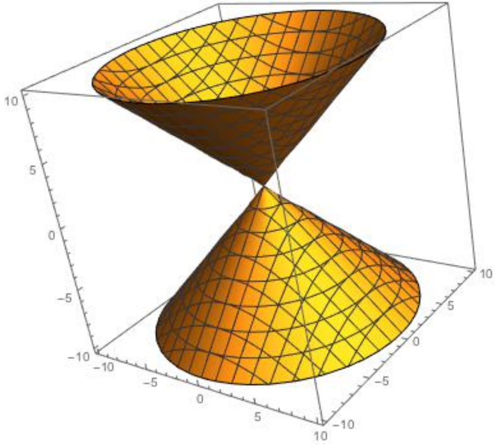
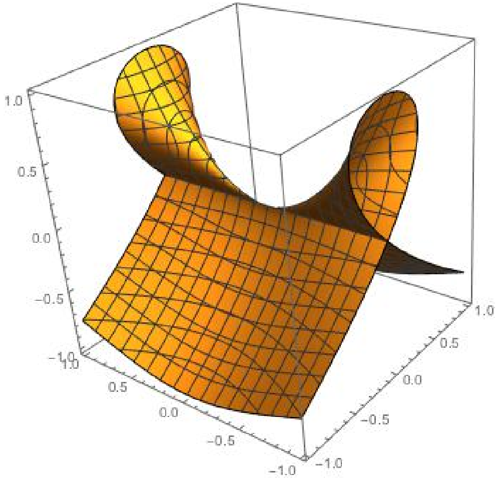


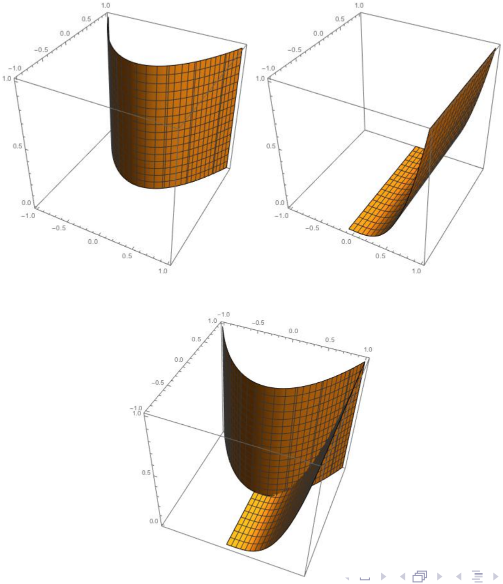
Examples

Much more complicated surface is $\mathbb{V}(x^2 - y^2z^2 + z^3)$:



Examples

Twisted cubic $\mathbb{V}(y - x^2, z - x^3)$:



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- however, the notion of dimension is more subtle than indicated by the above examples

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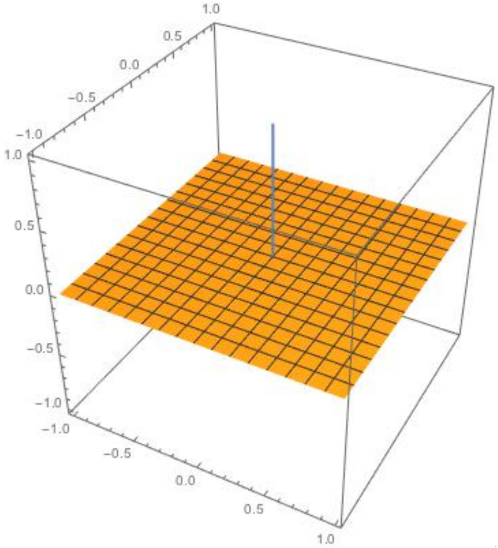

Linear varieties

Consider a system of m linear equations in n unknowns $x_1, \ldots, x_n$ with coefficients in k:

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- the dimension is determined by the number of independent equations

Lagrange multipliers

- we want to find the minimum and maximum values of $f(x, y, z) = x^3 + 2xyz - z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1$ being zero

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- these equations define an affine variety in $\mathbb{R}^4$
- intuition suggests it consists of finitely many points

Properties

Lemma

If $V, W \subset k^n$ are affine varieties, then so are $V \cup W$ and $V \cap W$. Suppose $V = \mathbb{V}(f_1, \ldots, f_s)$ and $W = \mathbb{V}(g_1, \ldots, g_t)$. Then

$$V \cap W = \mathbb{V}(f_1, \ldots, f_s, g_1, \ldots, g_t)$$
$$V \cup W = \mathbb{V}(f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t)$$

Proof: "$V \cup W \subseteq \mathbb{V}(f_i \cdot g_j)$"

If $(a_1, \dots, a_n) \in V$, then $f_i(a_1, \dots, a_n) = 0$ $\forall i$. Hence $(f_i g_j)(a_1, \dots, a_n) = 0$ $\forall i, j$. Hence $(a_1, \dots, a_n) \in \mathbb{V}(f_i \cdot g_j)$.

"$\mathbb{V}(f_i g_j) \subseteq V \cup W$"

Let $(a_1, \dots, a_n) \in \mathbb{V}(f_i g_j)$. If $(a_1, \dots, a_n) \in V$, then we are done. If $(a_1, \dots, a_n) \notin V$, then there exists f_i s.t. $f_i(a_1, \dots, a_n) \neq 0$. Since $(f_i g_j)(a_1, \dots, a_n) = 0$ $\forall j$, hence it must be that $g_j(a_1, \dots, a_n) = 0$ $\forall j$. Hence $(a_1, \dots, a_n) \in W$. $\blacksquare$

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- (Dimension) Can we determine the "dimension" of $\mathbb{V}(f_1, \ldots, f_s)$?

Parametrizations of affine varieties

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Example

Let $k = \mathbb{R}$ and consider the system of equations

$$\begin{aligned} x + y + z &= 1 \\ x + 2y - z &= 3. \end{aligned}$$

We use row operations to obtain the equivalent equations

$$\begin{aligned} x + 3z &= -1 \\ y - 2z &= 2. \end{aligned}$$

Letting $z = t$, this implies that all solutions are given by

$$\begin{aligned} x &= -1 - 3t, \\ y &= 2 + 2t, \\ z &= t \end{aligned}$$

as t varies over $\mathbb{R}$.

Parametrizations of the unit circle

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This parametrization does not describe the whole circle: the point $(-1, 0)$ is not covered.

Rational parametrizations

Definition

A **rational function** in $t_1, \ldots, t_m$ with coefficients in k is a quotient f/g of two polynomials $f, g \in k[t_1, \ldots, t_m]$, where g is not the zero polynomial. Two rational functions f/g and h/k are equal, provided that $kf = gh$ in $k[t_1, \ldots, t_m]$. The set of all rational functions is denoted $k(t_1, \ldots, t_m)$.

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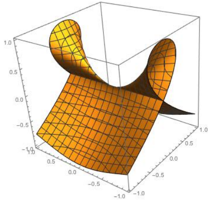

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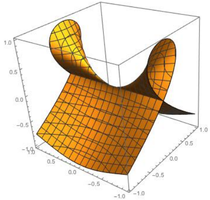

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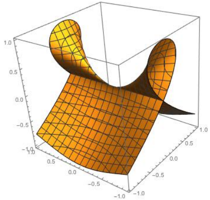

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- if we want to know whether the point $(1, 2, -1)$ is on the above surface, then implicit presentation is useful: $1^2 - 2^2(-1)^2 + (-1)^3 = 1 - 4 - 1 = -4$

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- We will learn in two weeks that the answer to the second question is always yes.

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Consider the parametric representation

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Hence the parametric equations define the affine variety $\mathbb{V}(y - x^2 + 2x - 2)$.

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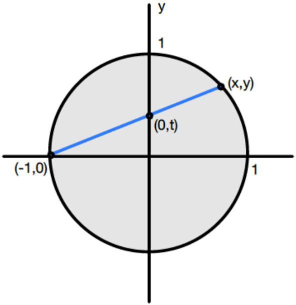

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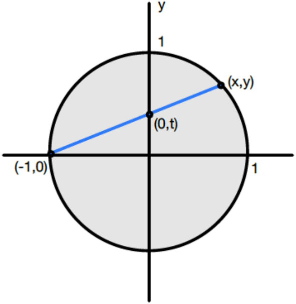

- geometric parametrization: given t, draw the line connecting $(-1, 0)$ to $(0, t)$ and let (x, y) be the point where the line meets $x^2 + y^2 = 1$

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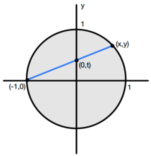

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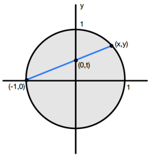

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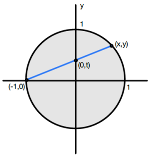

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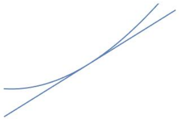

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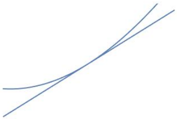
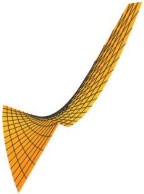


- taking tangent lines for all points gives the tangent surface of the twisted cubic



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$$x = t + u, y = t^2 + 2tu, z = t^3 + 3t^2 u$$

- t tells where we are on the curve and u tells where we are on the tangent line

Parametrization example 2

- the twisted cubic has parametrization

$$x = t, y = t^2, z = t^3$$

- the tangent vector to the curve at a point is $(1, 2t, 3t^2)$
- the tangent line is parametrized

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- t tells where we are on the curve and u tells where we are on the tangent line
- in the next weeks we will learn that the implicit representation is

$$-4x^3 z + 3x^2 y^2 - 4y^3 + 6xyz - z^2 = 0$$

Bezier cubic

- engineers need curves and surfaces that are varied in shape, easy to describe, quick to draw

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- Bezier cubic (introduced by Renault auto designer P. Bezier) is given parametrically by the equations

$$x = (1-t)^3 x_0 + 3t(1-t)^2 x_1 + 3t^2(1-t)x_2 + t^3 x_3,$$
$$y = (1-t)^3 y_0 + 3t(1-t)^2 y_1 + 3t^2(1-t)y_2 + t^3 y_3$$

for $0 \leq t \leq 1$ where x, y are constants specified by the design engineer

Bezier cubic

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Bezier cubic

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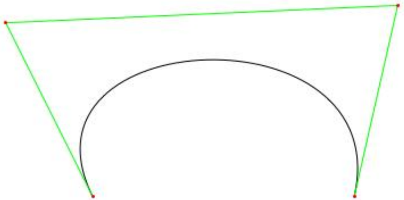
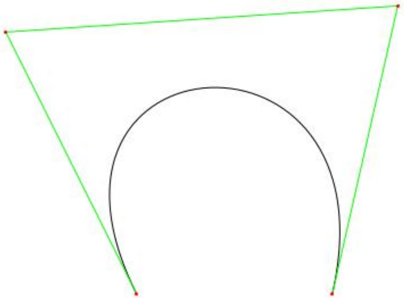
- the placement of (x_2, y_2) controls the tangent direction at the end of the curve

Bezier cubic





Conclusion and next time

Today:

- monomials and polynomials
- polynomials as functions - link between algebra and geometry
- affine varieties
- rational parametric description and implicit representation

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- ideals
- polynomials in one variable