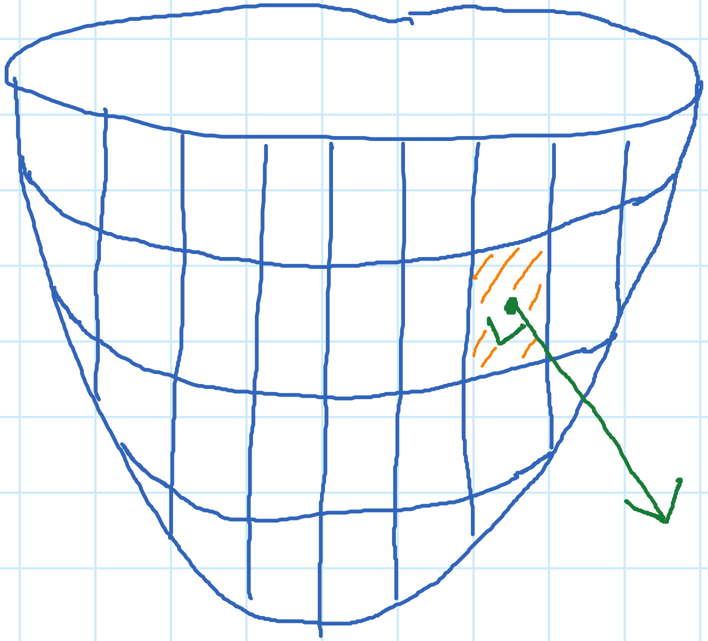
Welcome to Differential and Integral Calculus II



Welcome to Differential and Integral Calculus II

Practicalities

- MyCourses
- Open access textbooks
- Zulip chat - quick look at basics
- Grades and expectation
- How to do well in this course
- Teaching Assistants (TAs)

What's new?

- Functions of more than one variable
- 3D thinking
- Connecting geometric thinking to calculus
- Many new concepts
- Learn beautiful and useful math

What's old

- You already know how to algebra, differentiate and integrate.
- You have all the basic skills you need
- This is a very standard course so lots of material available. Many standard textbooks.

TODAY

- ✧ General definition of a parametric equation (vector valued function). Physical interpretation of position as a function of time.
- ✧ Examples of parametric equations. Line, circle, parabola.
- ✧ Parametrize the parabola in two ways and mentione that there are infinity-many ways to parametrize the same curve.
- ✧ Figure out the general formula for the parametric equation of a line in R^3
- ✧ Write the limit definition of the derivative of a vector valued function. Indicate graphically why this is the tangent vector. With a bit of algebra, show that this limit definition reduces to taking the derivative in each component separately.
- ✧ Find the tangent vector to a circle and get the result we expect.
- ✧ ~~Find the formula for arc length from first principles using the definition of the definite integral. (More details will be given in the next lecture)~~
- ✧ ~~The physical interpretation of arc length being the integral of speed makes the formula very intuitive.~~

Where to find this material (see MC for textbook links)

1. Adams and Essex 11.1 - 11.3
2. Corral, 1.8, 1.9
3. Guichard, 13.1 - 13.3
4. Active Calculus. 9.5 - 9.8

Vector-valued functions (1)

Recall

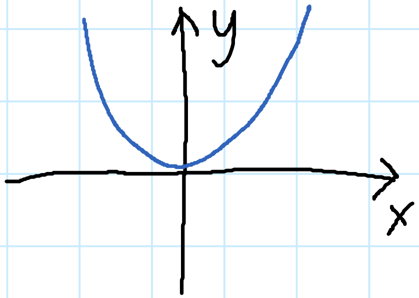
The most famaliar functions are

$$f : \underset{\uparrow \atop 1D}{\mathbb{R}} \longrightarrow \underset{\nwarrow \atop 1D}{\mathbb{R}}$$

Ex $f(x) = x^2$

The graph of a function is:

$$\{(x,y) \mid y = x^2\} = \text{subset of } \mathbb{R}^2$$



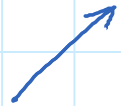
Domain = $\mathbb{R} = \{x \mid -\infty < x < \infty\}$

Range = all positive x values

$= \{x \mid x \geq 0\}$

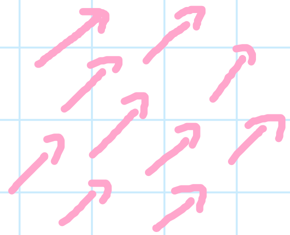
What is a vector?

magnitude + direction



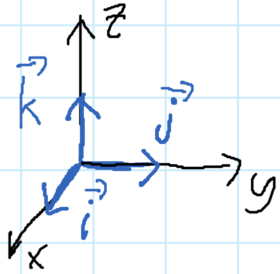
Examples: velocity, force, electric field

ATTENTION!! Vectors do $\boxed{\text{not}}$ have a base point



These are all the $\boxed{\text{same}}$ vector

Notation



$\vec{i} = \langle 1,0,0\rangle$

$\vec{j} = \langle 0,1,0\rangle$

$\vec{k} = \langle 0,0,1\rangle$

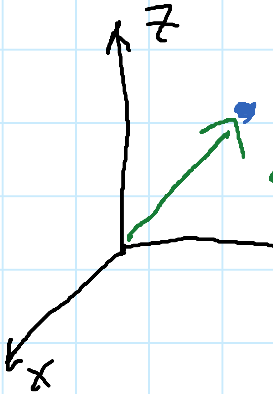
$$\vec{V} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \langle a,b,c\rangle = a\vec{i} + b\vec{j} + c\vec{k}$$

↳ Matrix Algebra course

$$\|\vec{V}\| = \sqrt{a^2+b^2+c^2}$$

Vector-valued functions (2)

Warning! We use $\mathbb{R}^3$ for both the set of points and the set of vectors



(a,b,c) - Point

$\langle a,b,c \rangle$ - called the position vector

A function

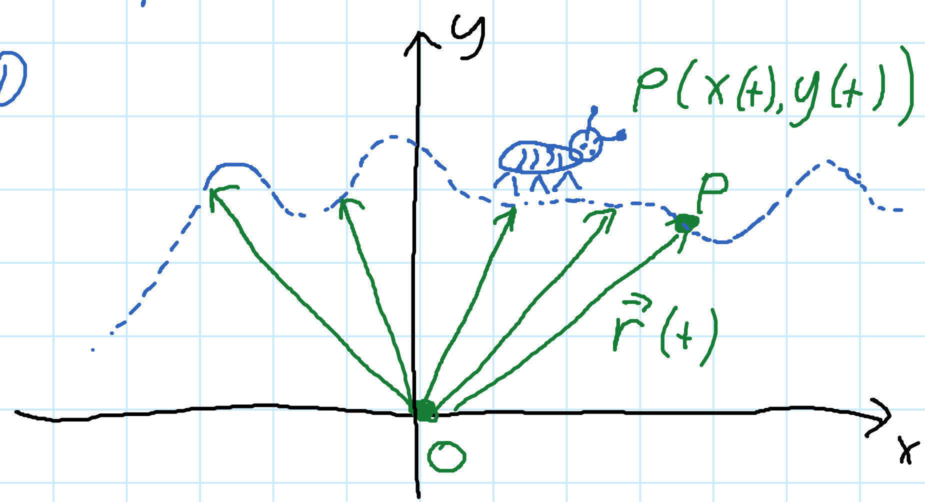
$$f: \mathbb{R} \longrightarrow \mathbb{R}^n \text{ is}$$

called a <u>vector-valued</u> function

Note: In this course we mostly deal with R^2 and R^3.
Most of the new ideas already appear in these dimensions.
Going to higher dimensions (n>3) is not conceptully different

Examples

①



$$\vec{r}(t) = \vec{OP}(t) = \langle x(t), y(t) \rangle$$
$$= x(t)\,\vec{i} + y(t)\,\vec{j}$$
$$= \text{Position as a function of time}$$

② Similarly,

$\vec{v}(t) =$ velocity

$\vec{a}(t) =$ acceleration

Vector-valued functions (3)

Exmaples: (1) Line, (2) Circle, (3) Parabola

(1) Line $y = f(x) = 3x + 2$

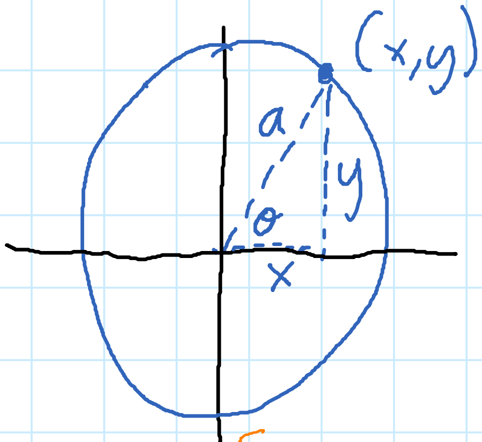
Let $x = t$

Then $y = 3t+2$

$\vec{r}(t) = \langle t, 3t+2 \rangle$

(This works for any function $f(x)$)

(2) Circle of radius 'a' centered at (0,0)



$x = a\cos\theta$, $y = a\sin\theta$

Using t instead of θ,

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$= a\langle \cos(t), \sin(t) \rangle$

For 1 revolution $0 \leq t \leq 2\pi$

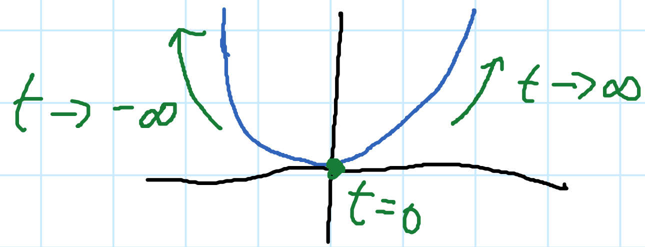
For continual revolution $-\infty < t < \infty$

Note: $x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$

which is the familiar equation of the circle

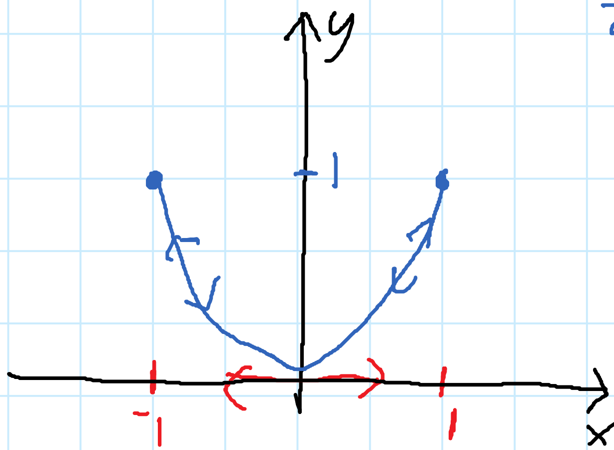
(3) $y = x^2$

(a) $x = t$, $y = t^2$, $-\infty < t < \infty$



(b) $x = \sin(t)$, $y = \sin^2(t)$

$-\infty < t < \infty$



x oscillates between -1 and 1

Sub $x = \sin(t)$ into y, get $y = x^2$

Lines in 3D

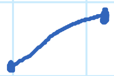
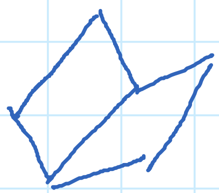
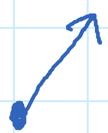
Lines in 2D

- y = mx + b
- 2 variable (R^2) and 1 linear equation
- Solution set is 1 dimensional (affine) space.
- If b= 0 then the solution set is a subspace (matrix alg couse)

But in 3D

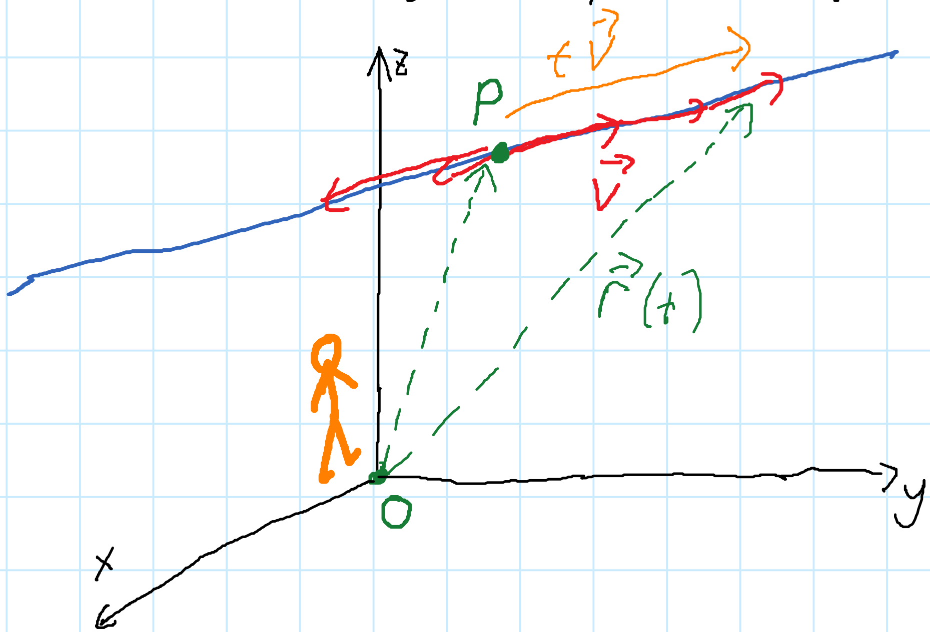
- The analogue of the 2D equation is naively
  z = mx + ny + b
- 3 variable (R^3) and 1 linear equation
- The solution set is a 2 dimensional space
- This is a plane
- To get a line we need 2 equations

What data spceifies a line (see zulip poll)?

- 2 points 
- 1 point + 2 slopes
- Int of 2 planes 
- 1 point + 1 vector 

Parametric description of a line in 3D

Given data: $P(x_0, y_0, z_0)$, $\vec{v} = \langle a, b, c \rangle$



(1) Go to P. Vector $\vec{OP}$

(2) move in the direction of v

Parametric equation of the line

$$\vec{r}(t) = \vec{OP} + t\vec{v}$$
$$= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

OR

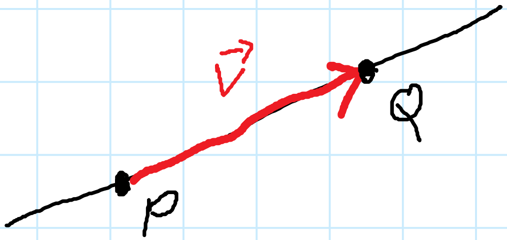
$$\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}$$

Line is 3D (2)

Example: Find the equation of the line passing through the points P(2, 3, -1) and Q(5, 4, -2).

Need: 1 point ✓ (P or Q)

1 vector ?



$$\vec{v} = \vec{PQ} = \langle 5-2,\ 4-3,\ -2-(-1) \rangle$$
$$= \langle 3,\ 1,\ -1 \rangle$$

Using P as the point, we have

$$\vec{r}(t) = \langle 2,3,-1 \rangle + t\langle 3,1,-1 \rangle$$

(Parametric equation - vector form)

OR

Solve for t: $t = \dfrac{x-2}{3}$

$$x(t) = 2+3t$$
$$y(t) = 3+t$$
$$z(t) = -1-t$$

(Parametric equations - component form)

OR

$$t = \frac{x-2}{3} = y-3 = \frac{z+1}{-1}$$

(Symmetric equation)

Derivatives of vector valued functons

Recall: (Homework #1)

The definition of the derivative of a function of one variable is:

$$\frac{df}{dx}(a) = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Let's just write down the analogue for a vector valued function.

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$

Definition: $$\vec{r}\,'(a) = \lim_{h\to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$$

Does this make sense?

(1) $\vec{r}(a+h) - \vec{r}(a)$ is ok because we can add and subtract vectors

(2) $\frac{\text{" "}}{h}$ makes sense as it is just scalar multiplication

$$\frac{1}{h}\left(\vec{r}(a+h) - \vec{r}(a)\right)$$

OK, so fine we have a definition that makes mathematical sence. But (A) how can we compute it? And (B), what does it measure?

(A) Let's just write it out in components.

$$\vec{r}\,'(a) = \lim_{h\to 0} \frac{1}{h}\Big(\langle x(a+h), y(a+h) \rangle - \langle x(a), y(a) \rangle \Big)$$

$$= \lim_{h\to 0} \frac{1}{h} \langle x(a+h) - x(a), y(a+h) - y(a) \rangle$$

$$= \lim_{h\to 0} \left\langle \frac{x(a+h) - x(a)}{h}, \frac{y(a+h) - y(a)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(a+h) - x(a)}{h}, \lim_{h\to 0} \frac{y(a+h) - y(a)}{h} \right\rangle$$

$$= \langle x'(a), y'(a) \rangle$$

= Woohoo! Just the regular derivative in each component

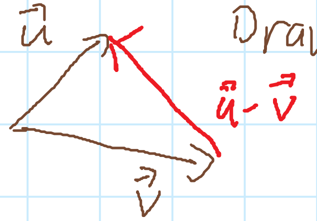
Ex $$\frac{d}{dt} \langle t^2, \sin(t) + t \rangle$$

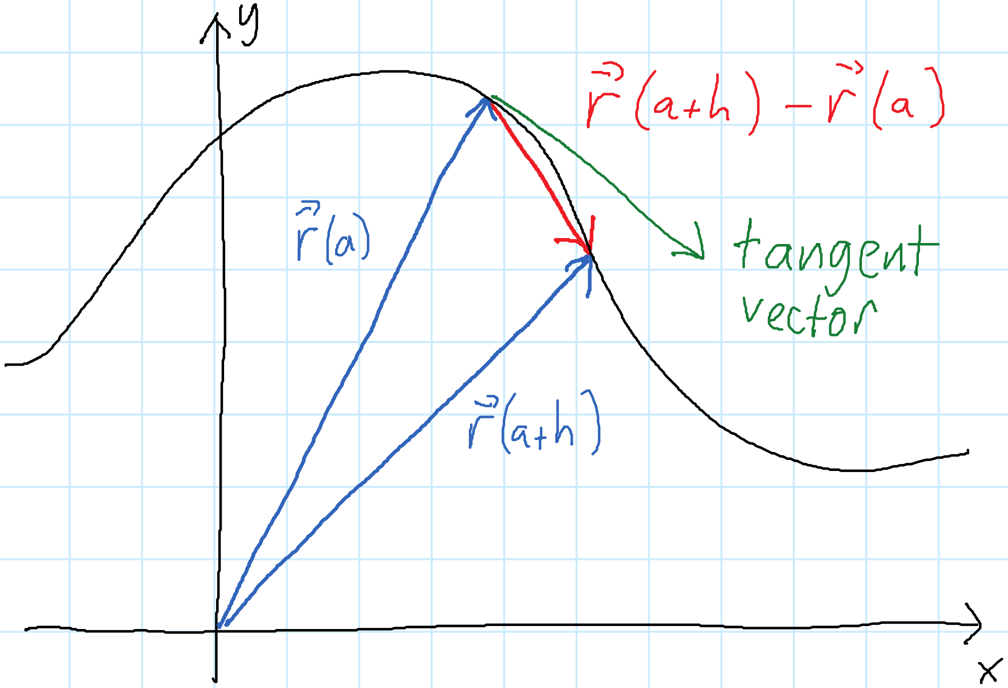
$$= \langle 2t, \cos(t) + 1 \rangle$$

Derivatives of vector valued functons (2)

(B) What does r'(t) measure? Let's draw a picture.

ASIDE: $\vec{u}$ Draw $\vec{u}-\vec{v}$





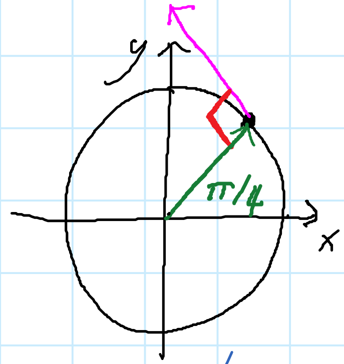
We see that as $h \to 0$ the difference $\vec{r}(a+h)-\vec{r}(a)$ becomes parallel to the tangent vector

Conclusion:

r'(a) is the tangent vector to the curve r'(t) at t=a

Example Find the tangent vector to the circle of radius 1 centered at (0,0) at the point $(1/\sqrt{2}, 1/\sqrt{2})$

Solution



$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

at $t = \pi/4$

$\vec{r}(\pi/4) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

The tangent vector is

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

Note $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$. So the tangent vector is perpendicular to the radial vector as we know from basic school geometry

The required tangent is $\vec{r}\,'(\frac{\pi}{4}) = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$