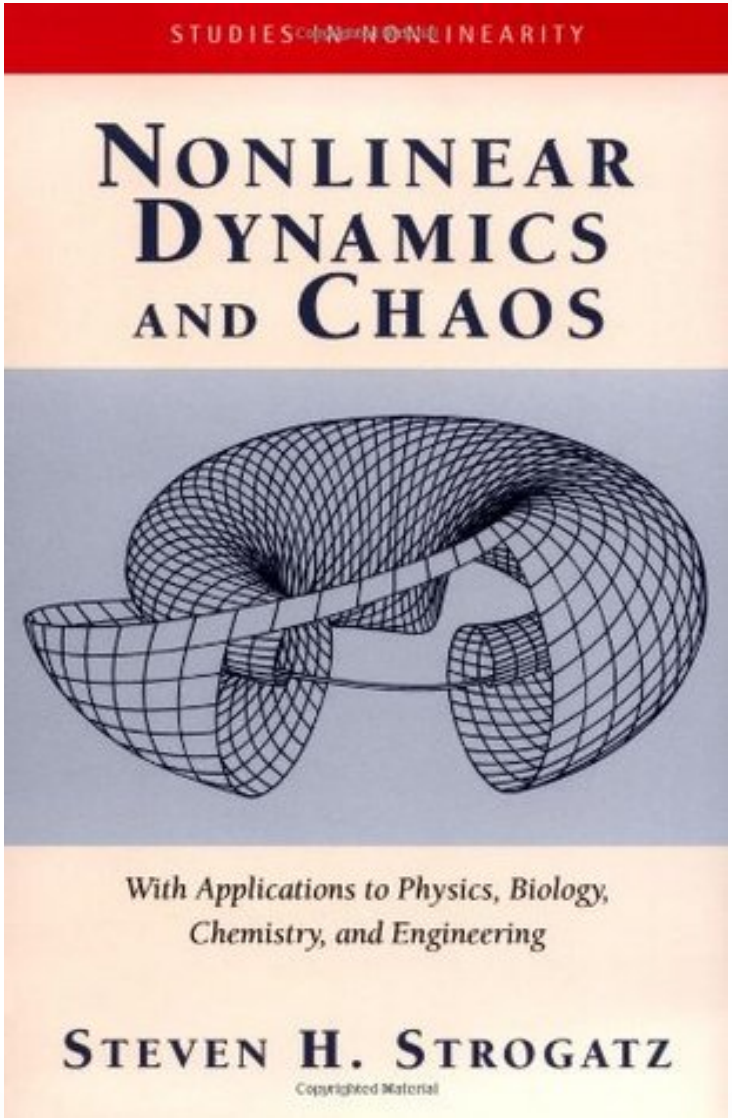
Nonlinear dynamics & chaos

Lecture I

The Book



… is on the course page (MyCourses/Materials/).

Outline

I One-Dimensional Flows

1) Flows on the line
2) Bifurcations
3) Flows on the Circle

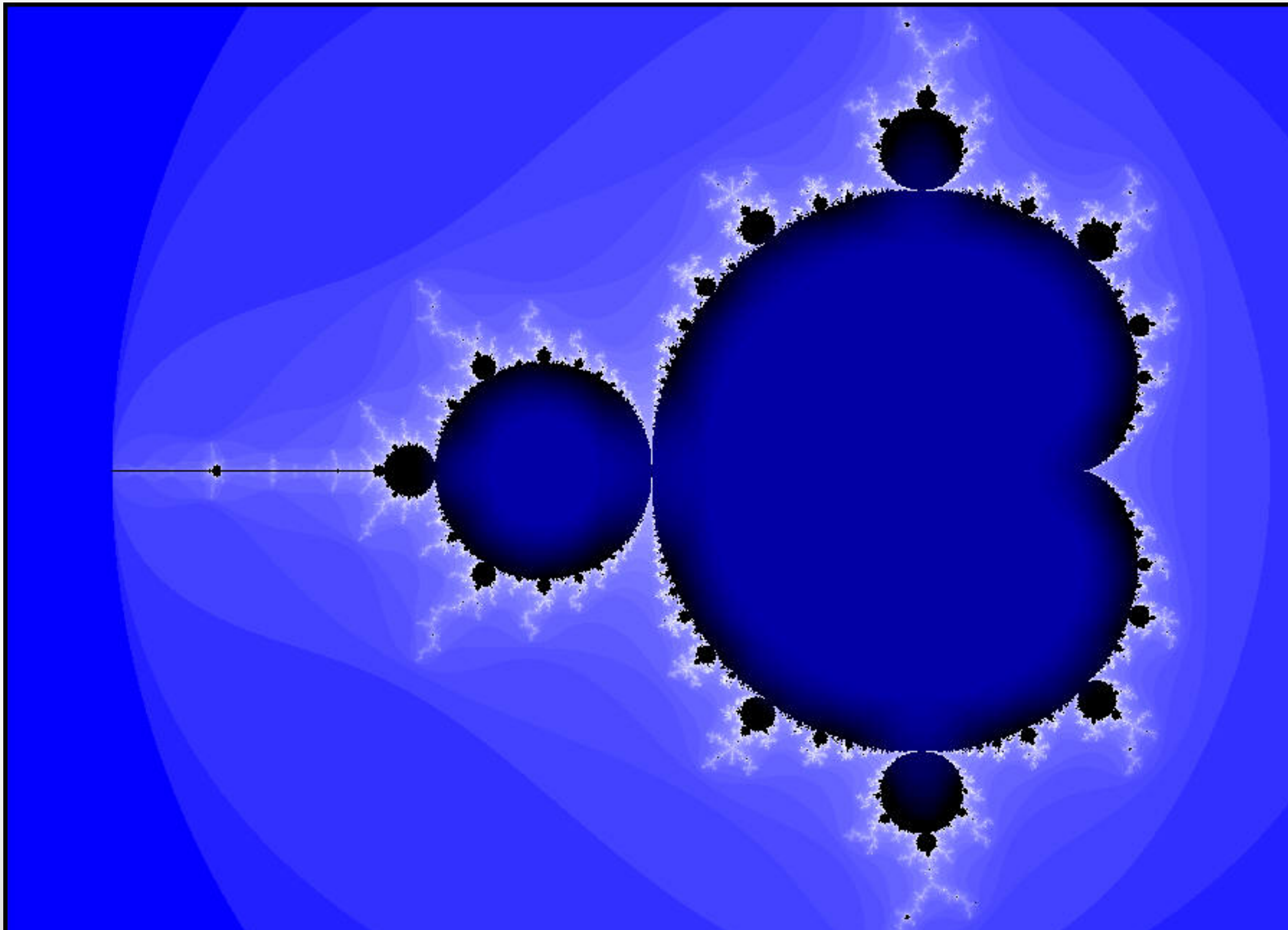
II Two-Dimensional Flows

1) Linear systems
2) Phase plane
3) Limit cycles
4) Bifurcations in Two Dimensions

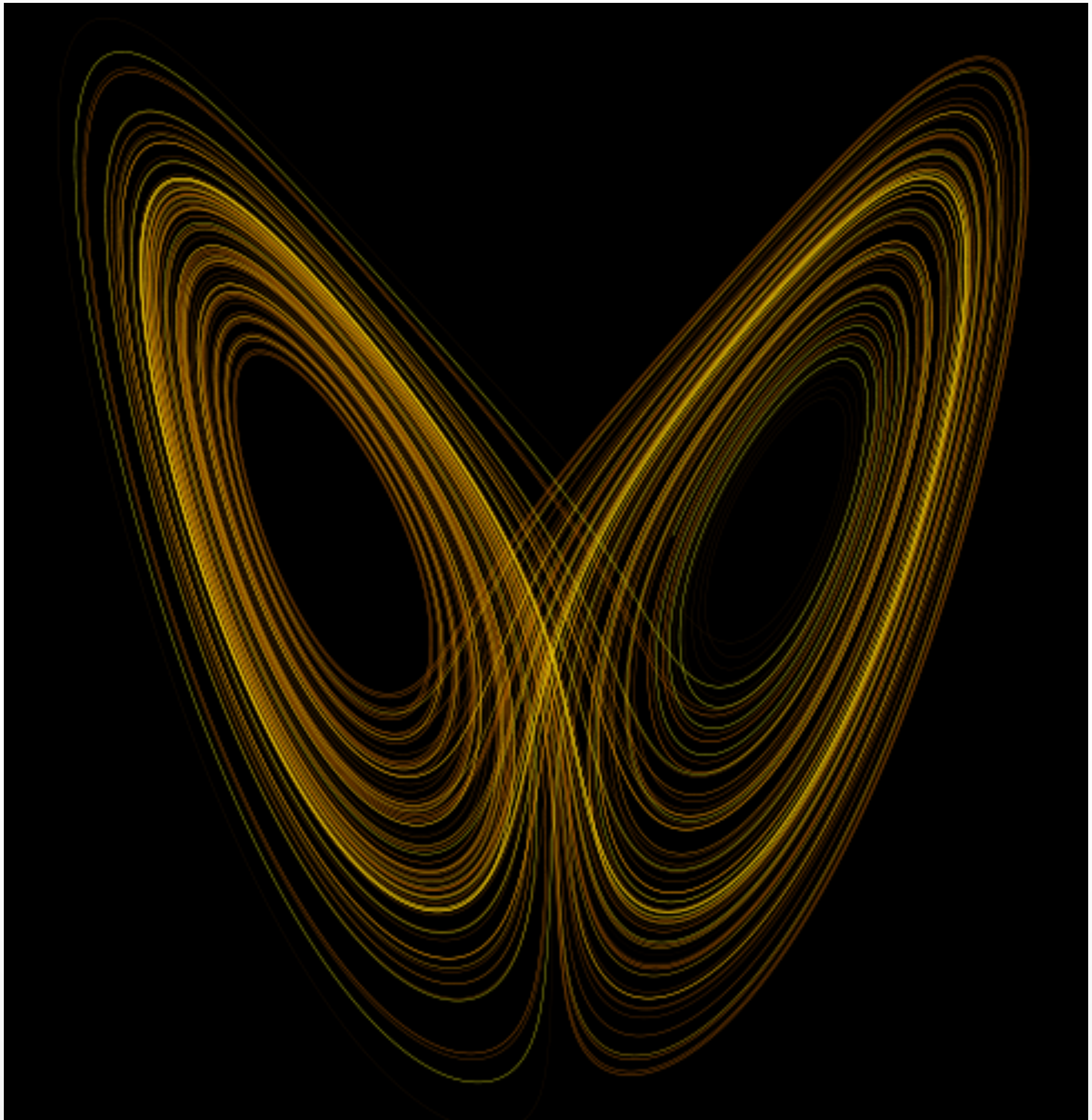
III Chaos

1) Lorentz Equations
2) One-Dimensional Maps

Fractals; self-similarity



Introduction: chaos



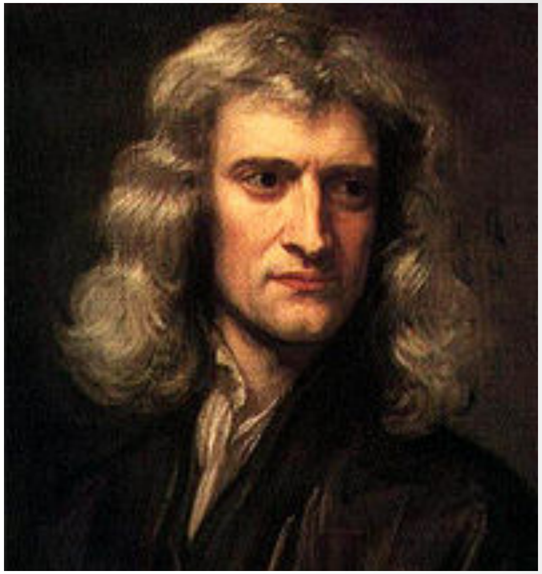
Introduction: dynamics

Fractals and chaos are part of dynamics, i.e. the subject that deals with systems evolving in time

A dynamic system may:
1) Reach a steady state (equilibrium)
2) Reach a periodic orbit (limit cycle)
3) Do otherwise (e.g. follow chaotic orbits)

Dynamic systems occur in a wide variety of fields:
1) Classical mechanics
2) Chemical kinetics
3) Population biology
4) Etc.

History of Dynamics



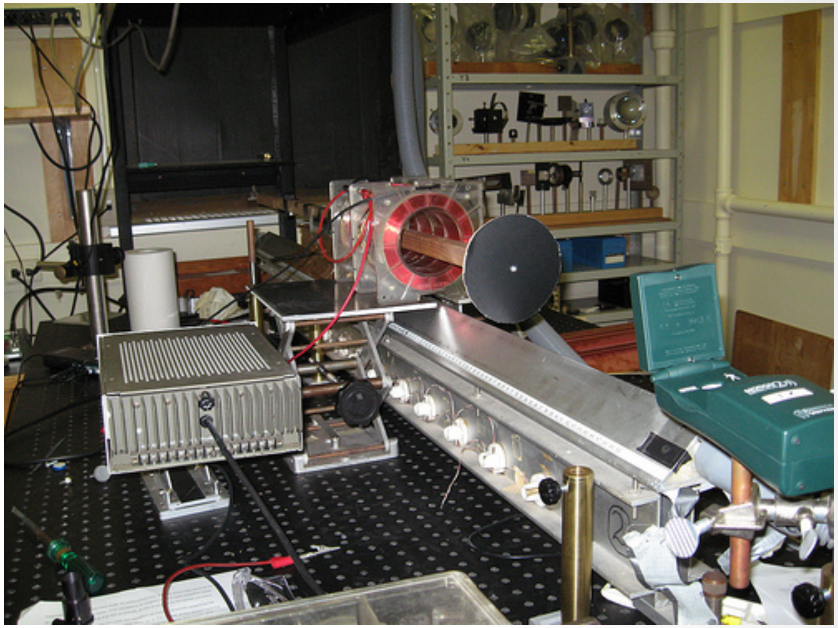
Birth (mid-1600s): Newton invented differential calculus and discovered laws of motion.
He solved two-body problem: motion of the earth around the sun and the inverse-square law of gravitational attraction.
Subsequent generations failed in the attempt to extend Newton's analytical methods to three bodies. Three-body motion is analytically unsolvable, no explicit formulas can be found!

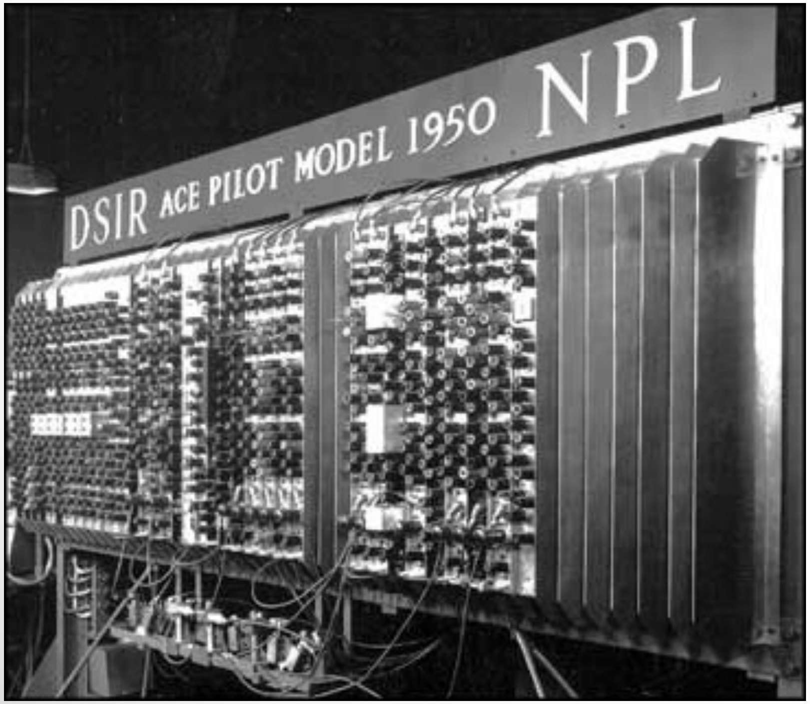
History of Dynamics

Breakthrough by Poincaré (late 1800s): development of **geometric approach** to analyze qualitative questions of e.g. stability. This approach has evolved into the modern science of dynamics. Poincaré was the first to conceive the idea of chaos, where a deterministic system exhibits aperiodic behaviour that sensitively depends on the initial conditions.

History of Dynamics

First half of the 20th century: nonlinear oscillators.
Applications in radio, radar, phase-locked loops, laser …

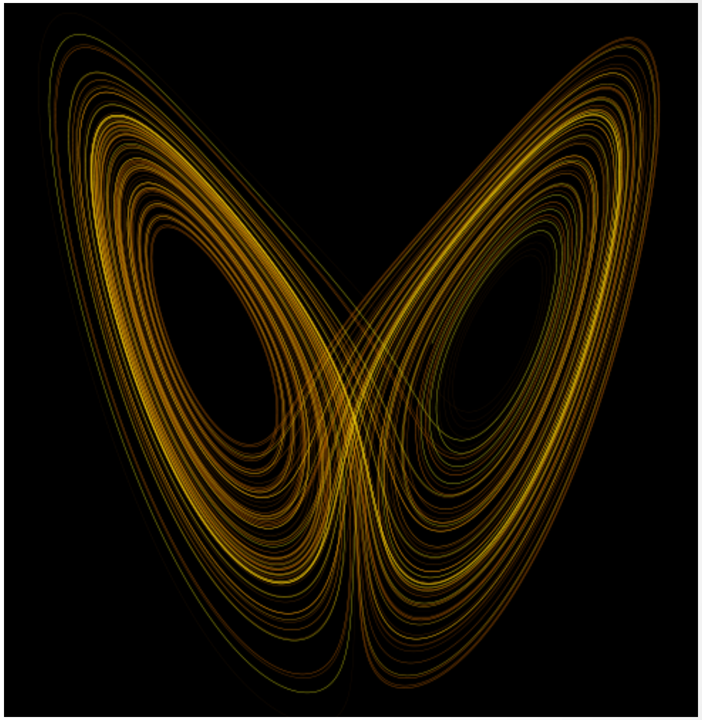




New mathematical techniques and extension of Poincare's geometric methods in classical mechanics (Kolmogorov).

History of Dynamics

"High-speed" computers in the 50's allowed for solving dynamic equations numerically ➔



Birth of chaos: Lorenz, 1963

Studies of a simplified model of convection rolls in the atmosphere for weather forecast ➔ Discovery of chaotic motion on a strange attractor (Lorenz attractor).

Dependence on initial conditions: the distance of two particles starting from slightly different points grows exponentially in time!

Lorenz attractor is "an infinite complex of surfaces": fractal.

History of Dynamics

1970s: the golden age of chaos

1971: new theory of turbulence by Ruelle and Takens using strange attractors

1976: May introduces the logistic map

1978: Feigenbaum discovers universality in one-dimensional maps; different systems may go chaotic in the same way → link between chaos and critical phenomena

1980s: experimental verification of chaotic behavior on fluids, chemical reactions, electronic circuits, mechanical oscillators, semiconductors (Gollub, Libchaber, Swinney, Linsay, Moon, Westervelt)

History of Dynamics

Dynamics - A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

Dynamics

Two types:

1) Differential equations: evolution in continuous time
2) Iterated maps (difference equations): evolution in discrete time; iterated maps are useful in chaotic dynamics

An example of a differential equation: damped harmonic oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Ordinary equation: one independent variable (time t)

Dynamics

Another example: heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2},$$

where (in physics) u is the temparature and κ is the diffusivity. This is a **partial** differential equation: two independent variables (space x, time t).

Our concern is purely temporal behaviour: exclusively ordinary differential equations.

Dynamics

General framework for ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned}$$

$$\left(\dot{x}_i = \frac{dx_i}{dt} \right)$$

x_1, …, x_n might represent concentrations of chemicals, populations of different species, or the positions and velocities of the planets. (Or: signals on different EEG/MEG sensors.)

Dynamics

High-order differential equations can be rewritten as a system of first-order equations.

Example: damped harmonic oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Trick: $x_1 = x; \;\; x_2 = \dot{x}_1$

$$\begin{aligned} \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

Dynamics

$$\begin{aligned} \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

The system is linear, there are only first powers of the variables!

Dynamics

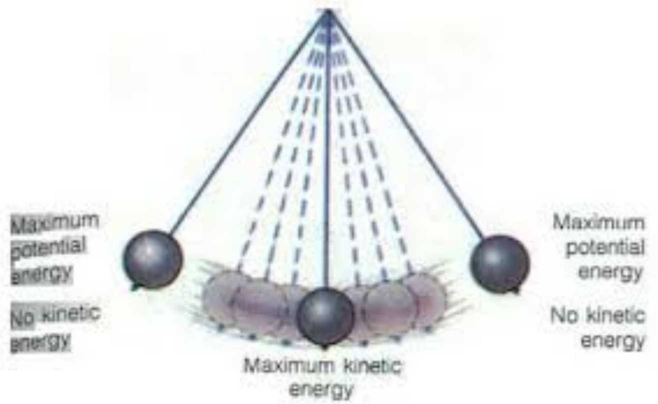
Example of a nonlinear equation: swinging pendulum!

$$\ddot{x} + \frac{g}{L} \sin x = 0$$

$x =$ angle of pendulum from vertical

$g =$ acceleration due to gravity

$L =$ length of the pendulum



Equivalent nonlinear first-order system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{L} \sin x_1 \end{aligned}$$

Dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1 \end{aligned}$$

Analytical solution is very difficult due to the nonlinear term (sin x).
(Standard) trick: Linearisation for small-angle oscillations
$\rightarrow$

$$\sin x \sim x \quad \text{for } x \ll 1$$

Problem with linearisation: no way to know what happens when the pendulum swirls over the top

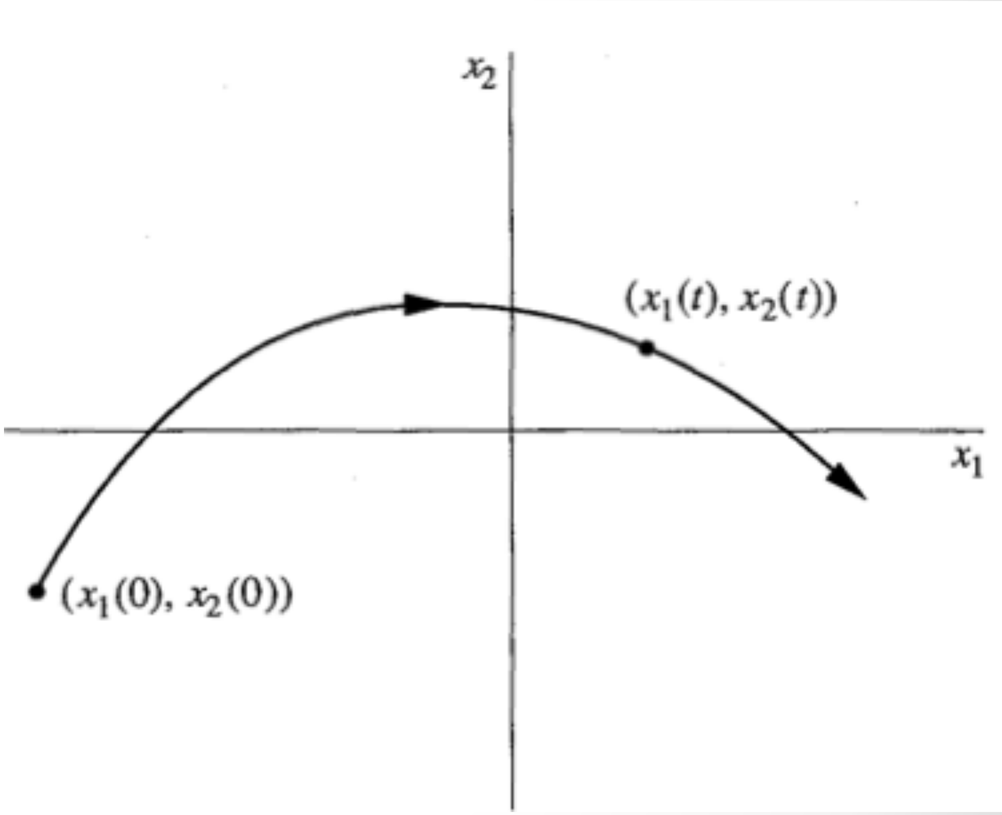
Scope of the course: to understand the features of evolution using geometric methods, without explicitly solving the equation of motion

Dynamics

Phase space: set spanned by all possible trajectories of a system

Trajectory: evolution of position(s) and velocity(ies)

Example: for one-dimensional motion phase space is two-dimensional

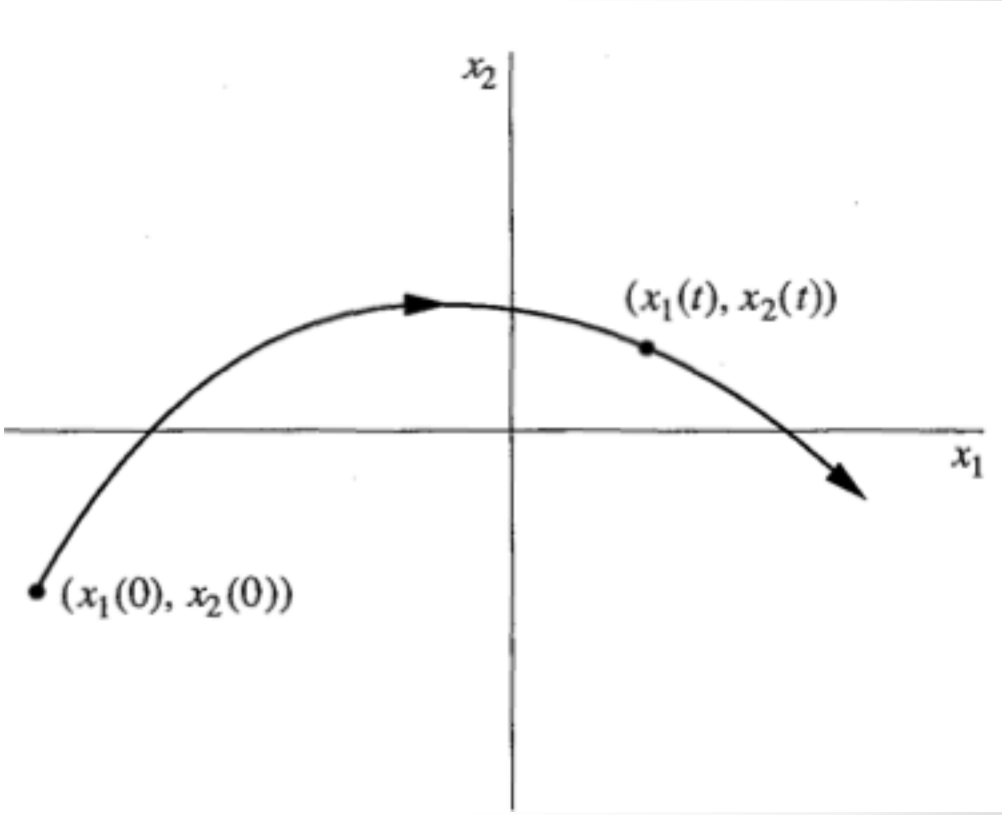


A system whose phase space is n-dimensional = an nth-order system.

Dynamics

Phase space is completely filled with trajectories as each point can be used as initial condition for the motion

Our goal: given the system, draw the trajectories without solving the equations! (geometric reasoning)



Dynamics

Can we handle equations with explicit time-dependence (nonautonomous equations)?

Example: forced harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos t$$

In addition to $x_1 = x$ and $x_2 = \dot{x}$, introduce $x_3 = t$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(-kx_1 - bx_2 + F\cos x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

Three-dimensional system with the explicit time dependence removed. → View *frozen* trajectories in 3-D phase space.

Dynamics

Rule: a time-dependent n-th order equation can be turned into an $(n+1)$-dimensional system without explicit time dependence.

'Physical' reason: For the motion to fully unfold, that is, to predict a future state from the present, we need three numbers: position, velocity, and time. Hence the 3-dimensional phase space.

This leads to nontraditional terminology. The forced harmonic oscillator, normally regarded as a second-order linear equation, will be in our vocabulary a third-order nonlinear (because of the cosine term) system.

Dynamics

Question: why are nonlinear systems so hard to solve?

Answer: linear systems can be decomposed in parts, which can be solved separately and then recombined to get the answer. Nonlinear systems cannot be decomposed.

Linear systems are the sum of their parts, nonlinear systems are not!

Nonlinearity is everywhere: weather, fluid dynamics, population and social dynamics, economics and finance, neurons and brain, lasers, Josephson junctions, etc.

Dynamics

Number of variables →

Nonlinearity ↓

	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
				The frontier	
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

One-Dimensional Flows

Flows on the line

One-dimensional (first-order) systems

$$\dot{x} = f(x)$$

$x(t)$ real-valued function of time

$f(x)$ smooth real-valued function of x, not explicitly depending on time. In case there were an explicit time dependence it would be regarded as a two-dimensional system

One-dimensional systems

Example: $$\dot{x} = \sin x$$

Exact solution: $$dt = \frac{dx}{\sin x}$$

$$t - t_0 = \int_{x_0}^{x} \frac{1}{\sin x'} dx' \qquad [x(t_0) = x_0]$$

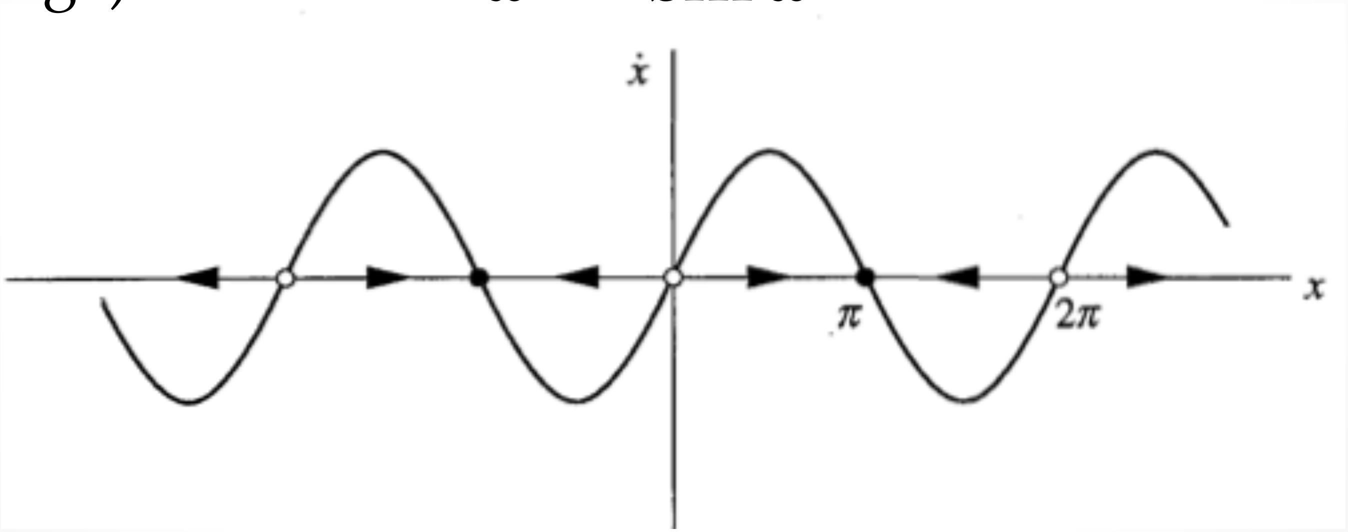
$$t - t_0 = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

Exact solution not transparent: what happens when $t \to \infty$?

Geometric approach: vector fields

Interpret a differential equation as a vector field: how the velocity of the particle depends on its position (mechanical analogs are used – "velocities" are just rates of change).

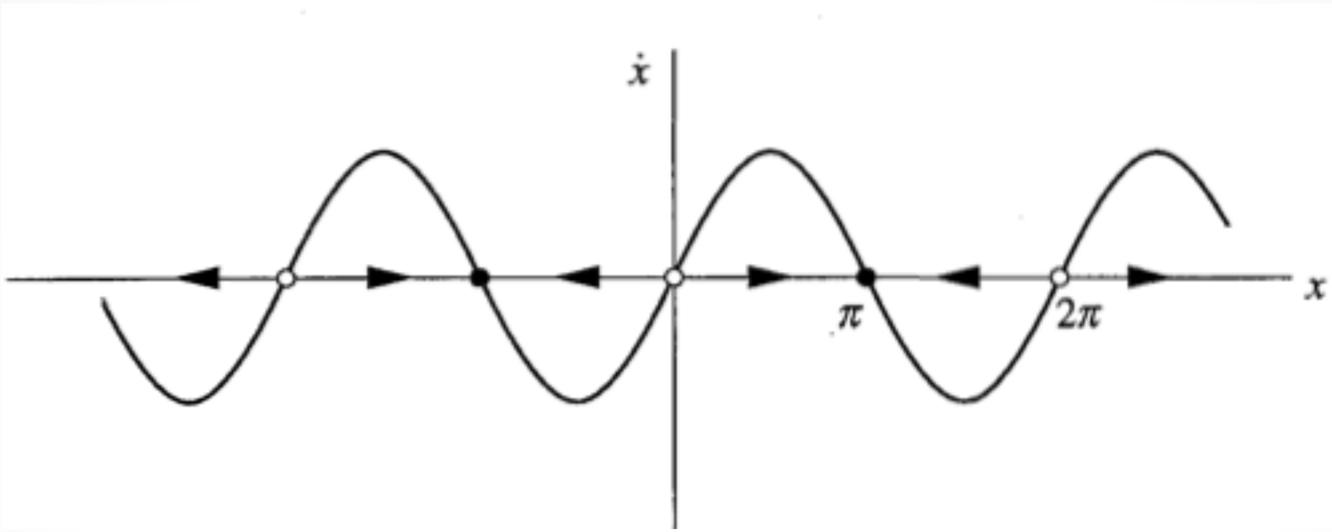
$$\dot{x} = \sin x$$



$$\dot{x} > 0 \quad \rightarrow \quad \text{particle/flow moves to the right}$$

$$\dot{x} < 0 \quad \rightarrow \quad \text{particle/flow moves to the left}$$

Fixed points

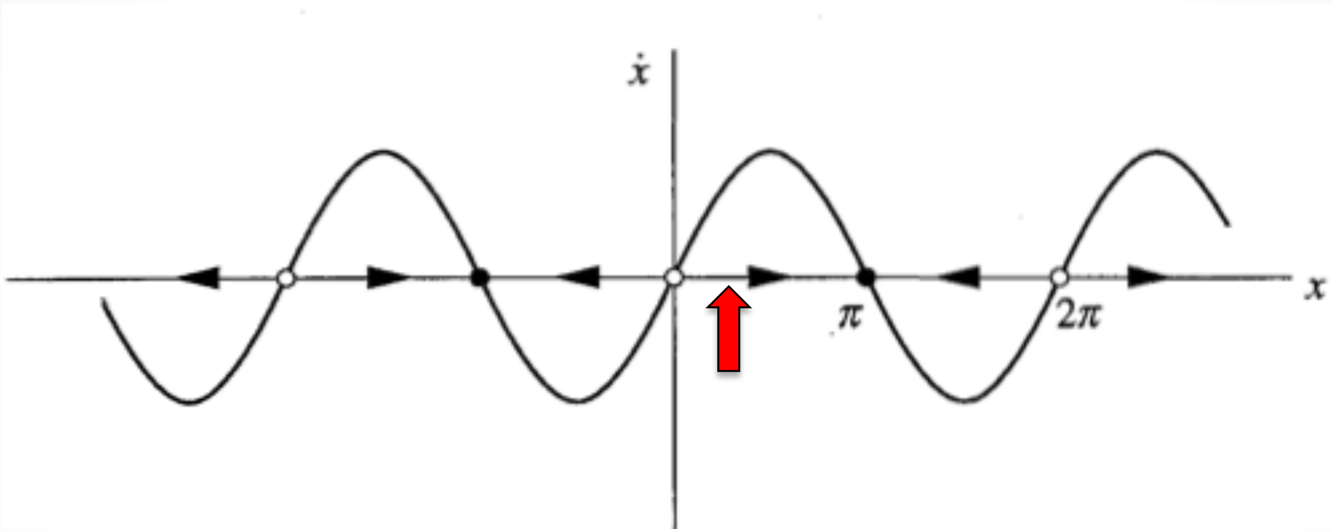


At points $\dot{x} = 0$ there is no flow: fixed points.

Two types of FPs:

1) Stable fixed points (attractors, sinks): flow converges towards them
2) Unstable fixed points (repellers, sources): flow goes away from them

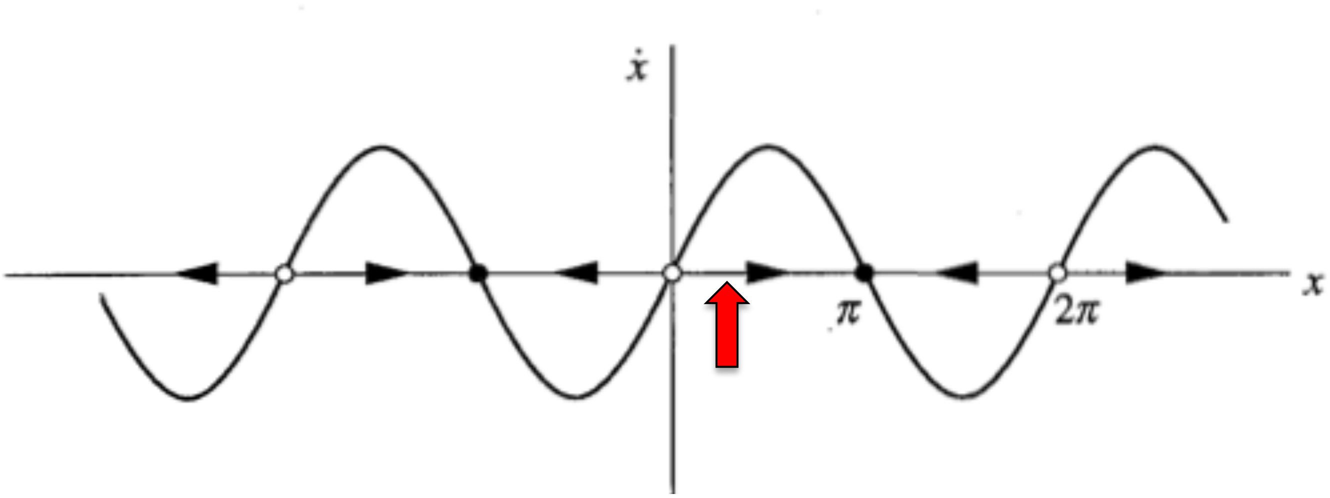
Fixed points

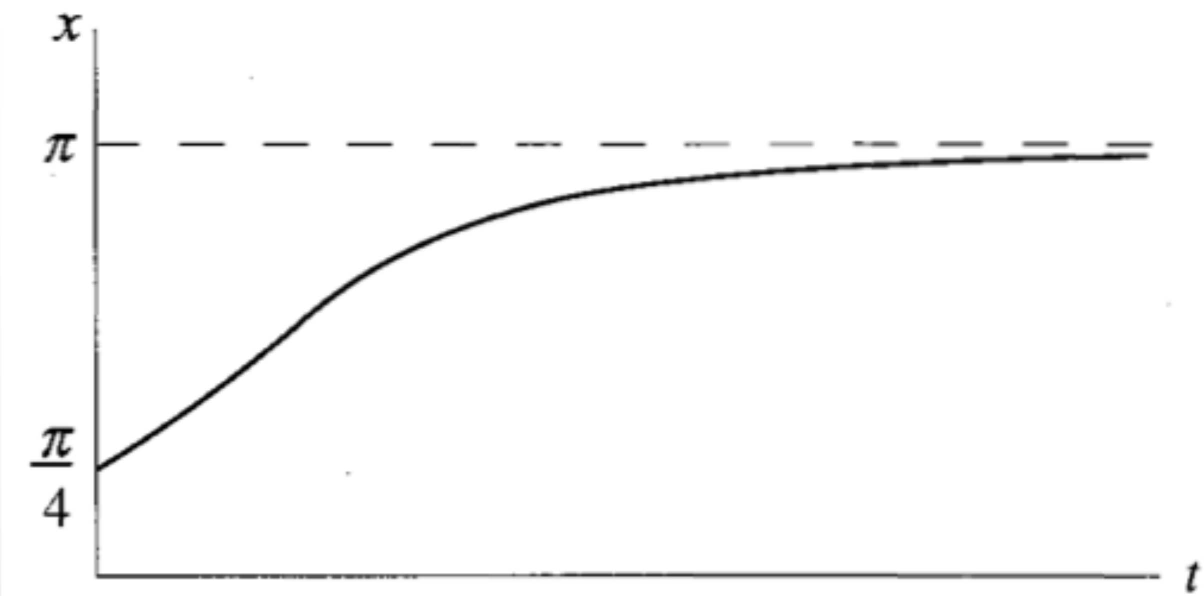


Example: initial condition $x_0 = \dfrac{\pi}{4}$

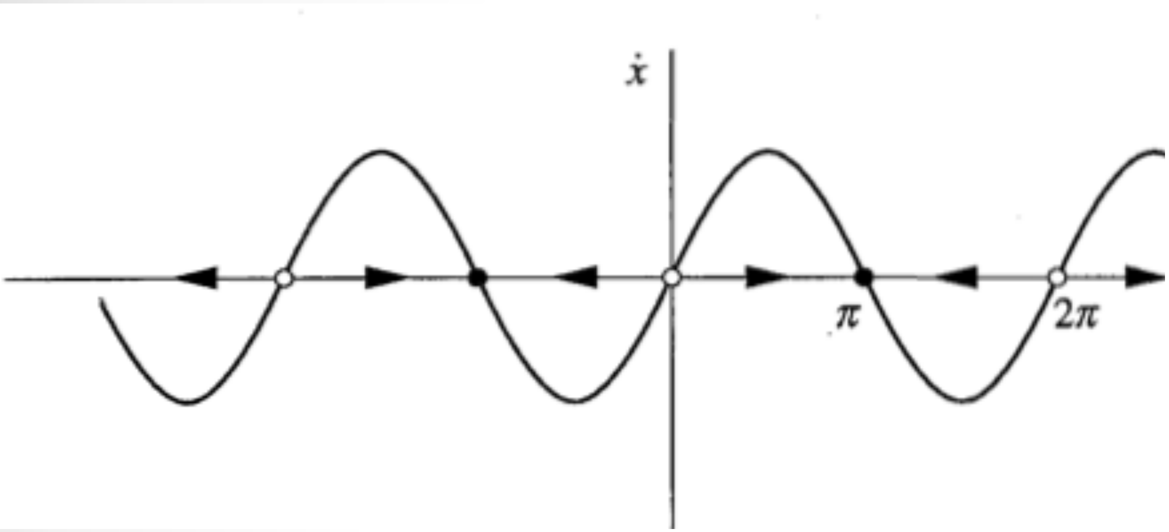
Solution: particle starting at x = π/4 moves to the right with increasing velocity, then slows down after x = π/2 until it reaches asymptotically the stable fixed point x = π.

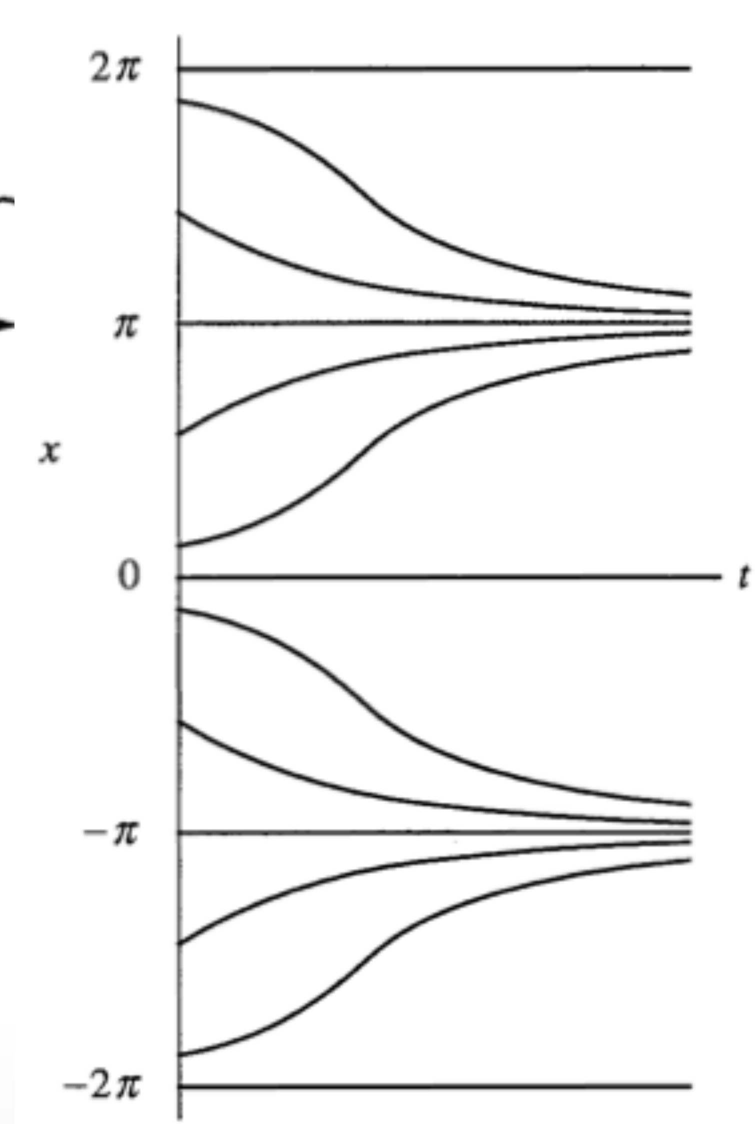
Fixed points





Fixed points





If the particle starts at a point where

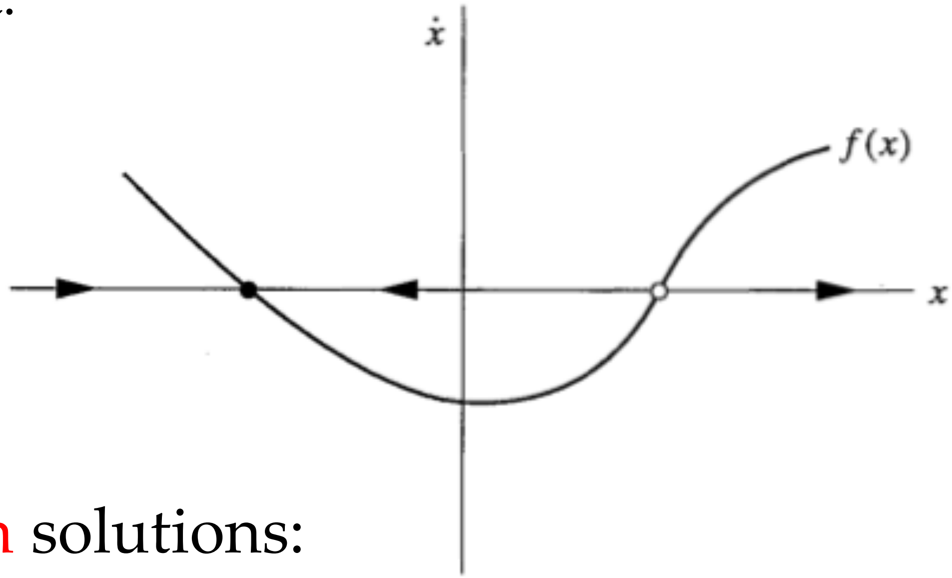
$$\dot{x} = 0$$

it will stay there forever

Fixed points & stability

$$\dot{x} = f(x)$$

Imaginary particle, **phase point**, is carried by the flow along the **trajectory** $x(t)$. The diagram is called a **phase portrait**.



General procedure:

1) Draw the graph of $f(x)$
2) Identify fixed points (intersections with x-axis)
3) Classify fixed points

Fixed points x^* are equilibrium solutions:

1) Stable equilibrium: the effect of small perturbations vanishes in time
2) Unstable equilibrium: the effect of small perturbations grow in time

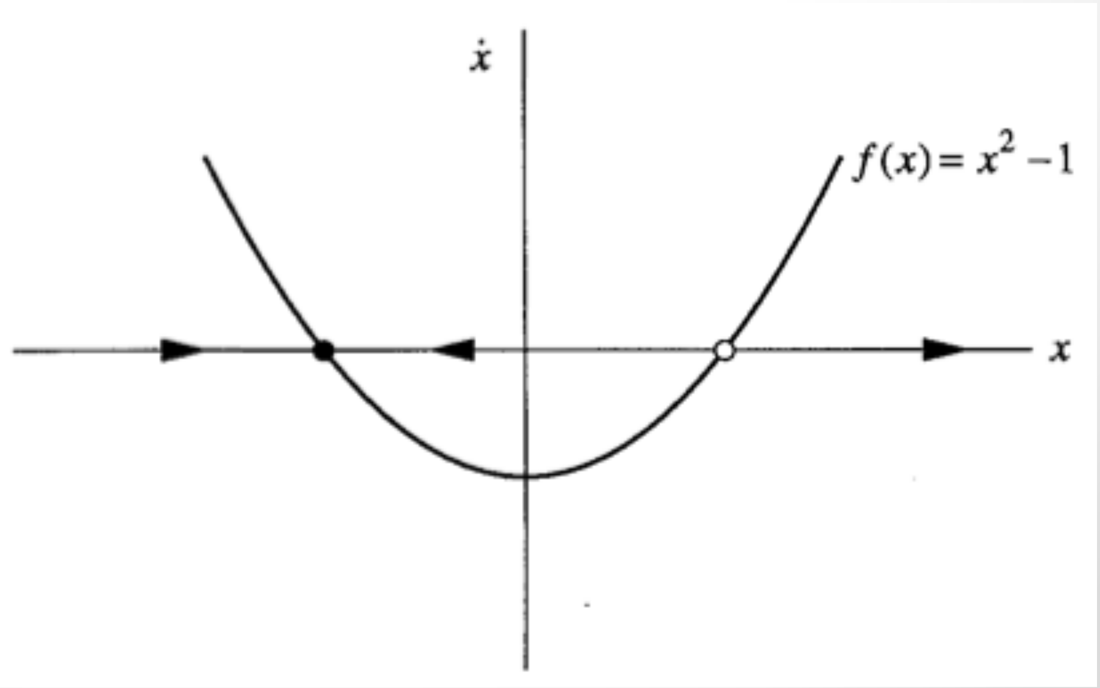
Example I

$$\dot{x} = x^2 - 1$$

Fixed points:

$$f(x^*) = 0 \rightarrow x^{*2} - 1 = 0 \rightarrow x^* = \pm 1$$

Note: Stability of a FP is determined by *small* perturbations → here FPs are stable or unstable *locally* – not globally.



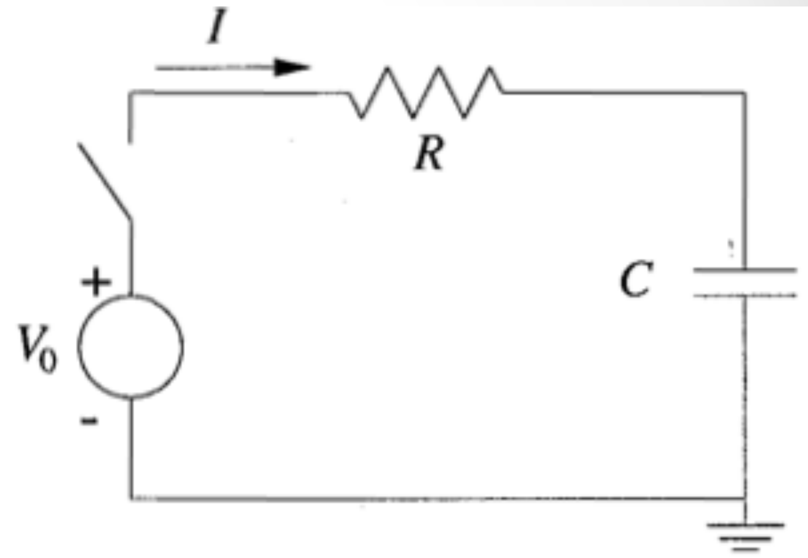
Example II: electric circuit

$Q(t)$ = charge of capacitor at time t

R = resistance

C = capacity

I = current flowing through the resistor



Circuit equation: total voltage drop of system must be zero

$$-V_0 + RI + \frac{Q}{C} = 0$$

Example II: electric circuit

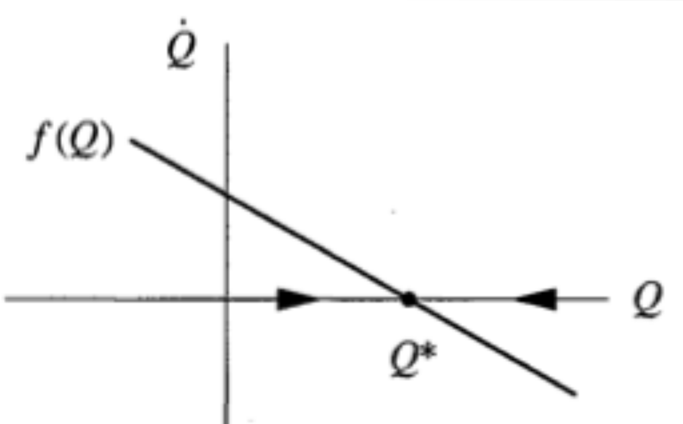
$$I = \dot{Q} \quad \rightarrow \quad -V_0 + R\dot{Q} + \frac{Q}{C} = 0$$

$$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}$$

Fixed points:

$$f(Q^*) = 0 \quad \rightarrow \quad \frac{V_0}{R} - \frac{Q^*}{RC} = 0$$

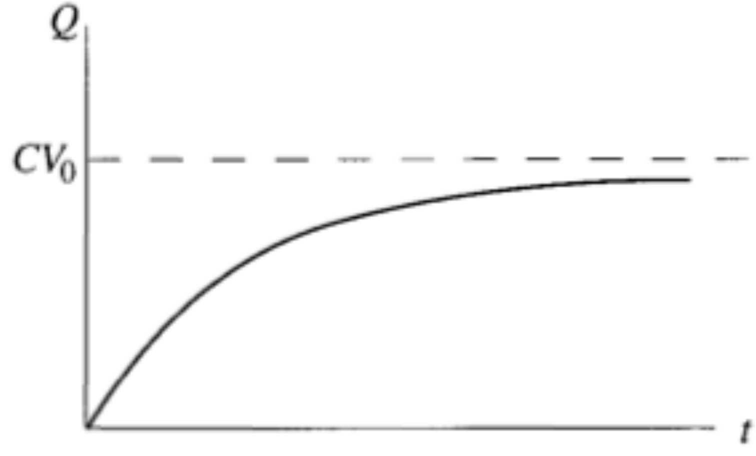
$$Q^* = CV_0$$



The fixed point is globally stable, i.e. it is approached by all initial conditions; in other words, even large perturbations/disturbances decay.

Example II: electric circuit

For the initial condition at the origin:



Example III

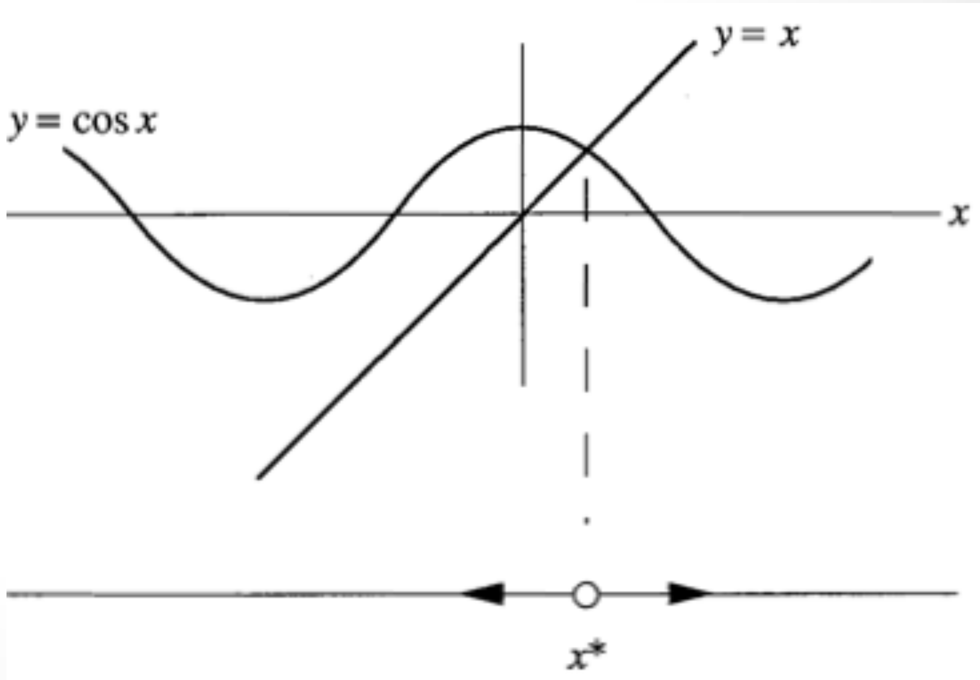
$$\dot{x} = x - \cos x$$

Fixed points:

$$f(x^*) = 0 \quad \rightarrow \quad x^* = \cos x^*$$

Solution: either plot $x - \cos x$ directly or separately plot

$$\begin{cases} y = x \\ y = \cos x \end{cases}$$



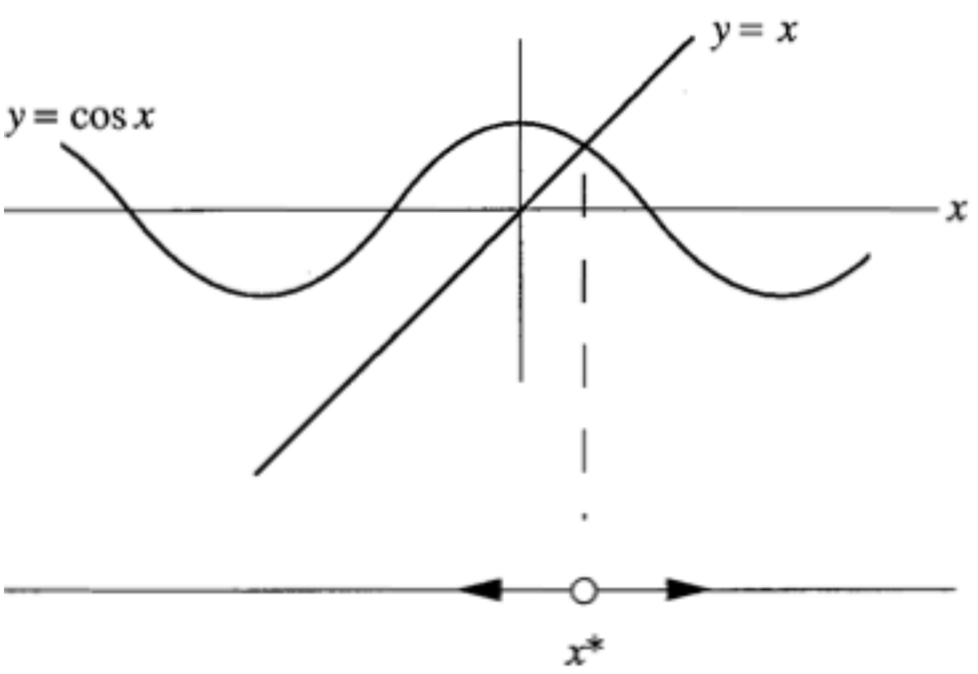
Only one fixed point!

Example III

$$\dot{x} = x - \cos x$$

To the right of x^*, $x > \cos x$
$\rightarrow x - \cos x > 0$: velocity is positive.

To the left of x^*, $x < \cos x$
$\rightarrow x - \cos x < 0$: velocity is negative.



The fixed point $x^* = \cos x^*$ is unstable!

Example IV: population growth

Simplest model:

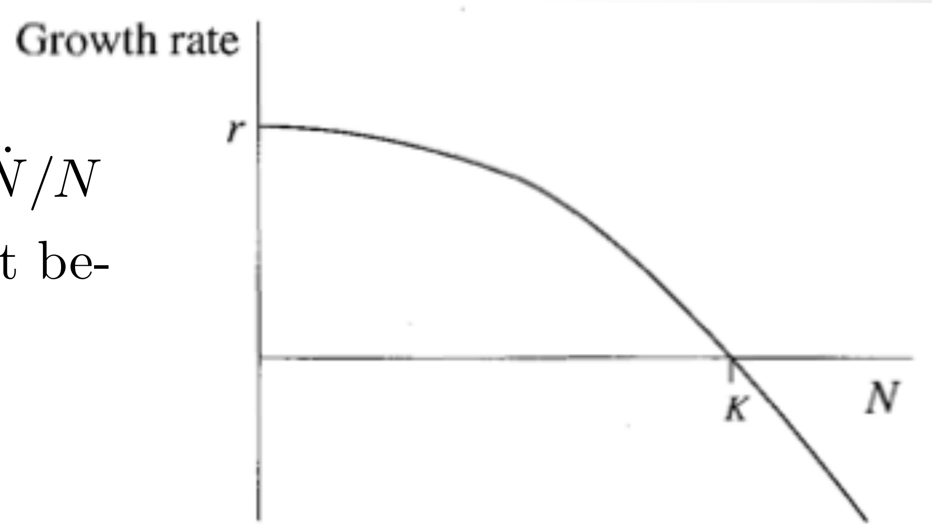
$$\dot{N} = rN$$

Consequence: exponential growth

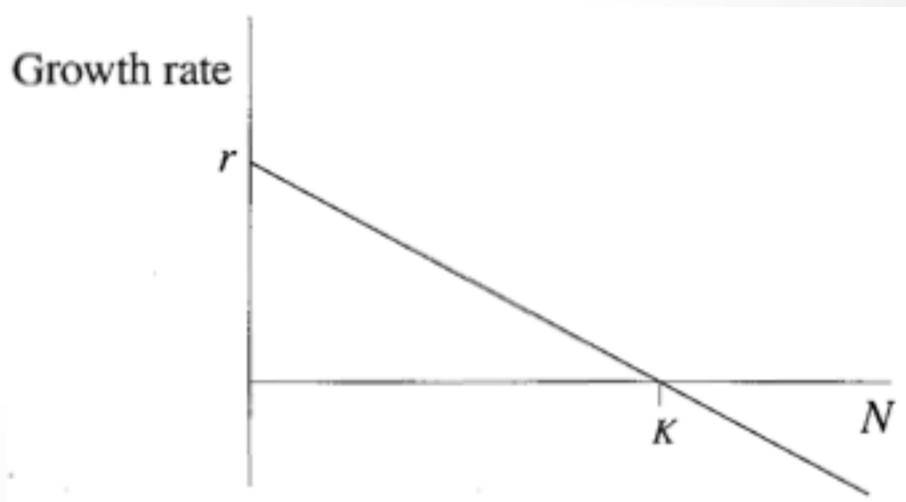
$$N(t) = N_0\, e^{rt}$$

Exponential growth cannot last forever.

Example IV: population growth



Hypothesis: per capita growth rate $\dot{N}/N$ decreases with N until, for $N > K$, it becomes negative



Simple hypothesis: linear decrease

$\rightarrow$ logistic equation.

Example IV: population growth

Logistic equation (Verhulst, 1838)

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Analytically solvable. Here we use geometric approach.

Fixed points:

$$N^*\left(1 - \frac{N^*}{K}\right) = 0 \quad \rightarrow \quad N_1^* = 0, N_2^* = K$$

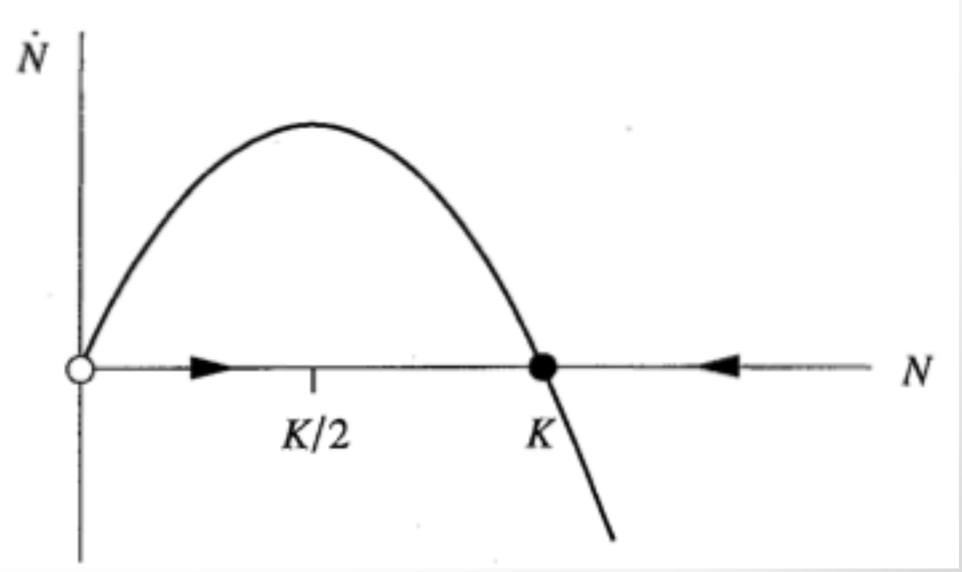
Example IV: population growth

Logistic equation (Verhulst, 1838)

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

$N_1^* = 0$ is unstable

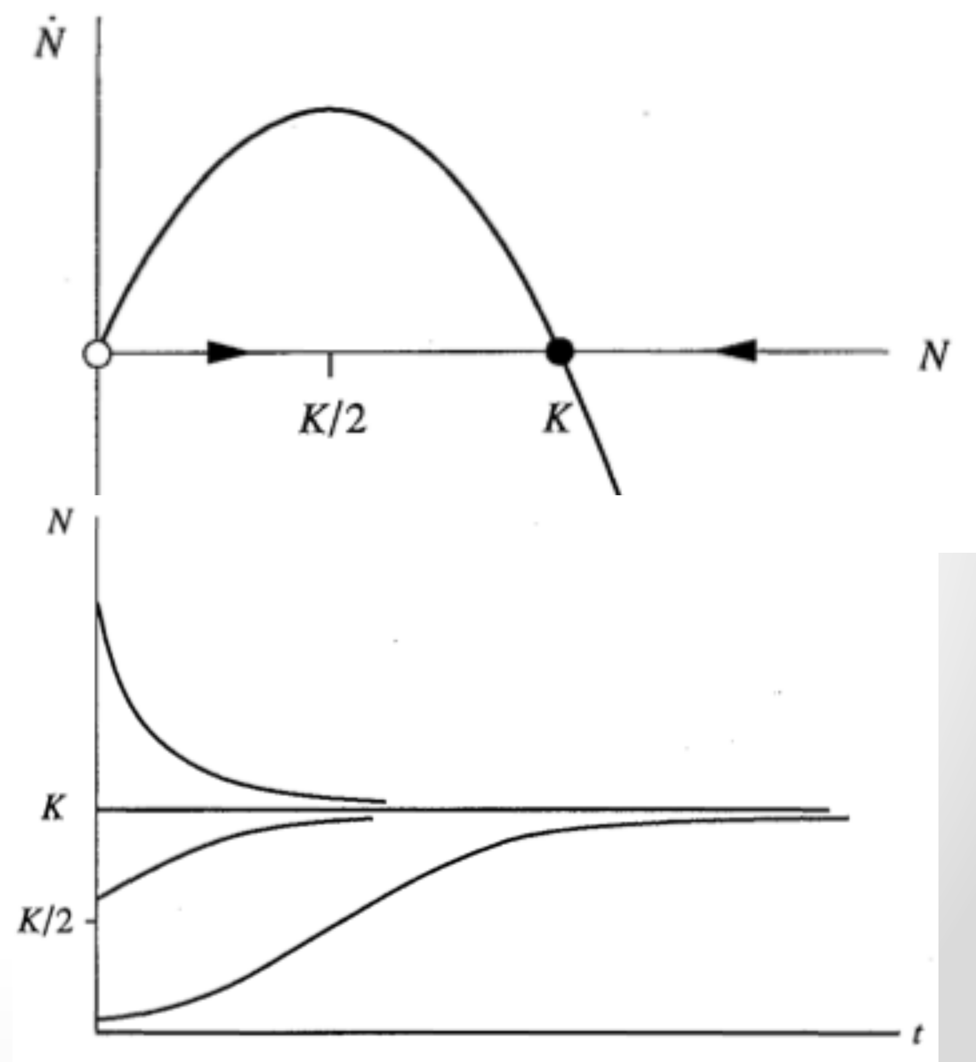
$N_2^* = K$ is stable



Example IV: population growth

For $N_0 \sim 0$ there is a rapid growth until the growth rate peaks ($N = K/2$), then the population grows slower and slower until it reaches the stationary value K (carrying capacity).

For $N_0 > K$ the growth rate is negative and the population decreases until it reaches K.



Linear stability analysis

Let x^* be a fixed point and $\eta(t) = x(t) - x^*$ a small perturbation away from the fixed point.

Question: How does the perturbation grow or decay with time?

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x} = f(x) = f(x^* + \eta)$$

Taylor's expansion:

$$f(x^* + \eta) = \underbrace{f(x^*)}_{0 \text{ (fixed point)}} + \eta f'(x^*) + O(\eta^2)$$

$$\dot{\eta} = f(x^* + \eta) = \eta f'(x^*) + O(\eta^2)$$

Linear stability analysis

If $f'(x^*) \neq 0 \quad \rightarrow \quad \dot{\eta} \sim \eta f'(x^*)$

$\boxed{\text{Linearisation about } x^*}$

if $f'(x^*) > 0$ the perturbation grows exponentially in time

if $f'(x^*) < 0$ the perturbation decays exponentially in time

if $f'(x^*) = 0$ $O(\eta^2)$ terms are non-negligible: nonlinear analysis is needed

Key point: the slope of $f(x)$ at a fixed point determines the stability of the fixed point:

1) If the slope is positive, the fixed point is unstable
2) If the slope is negative, the fixed point is stable

$1/|f\,'(x^*)|$ is a characteristic time scale:
it determines the time required for $x(t)$ to vary significantly near x^*; $\eta = \eta_0 \exp(t/\tau)$.

Example I

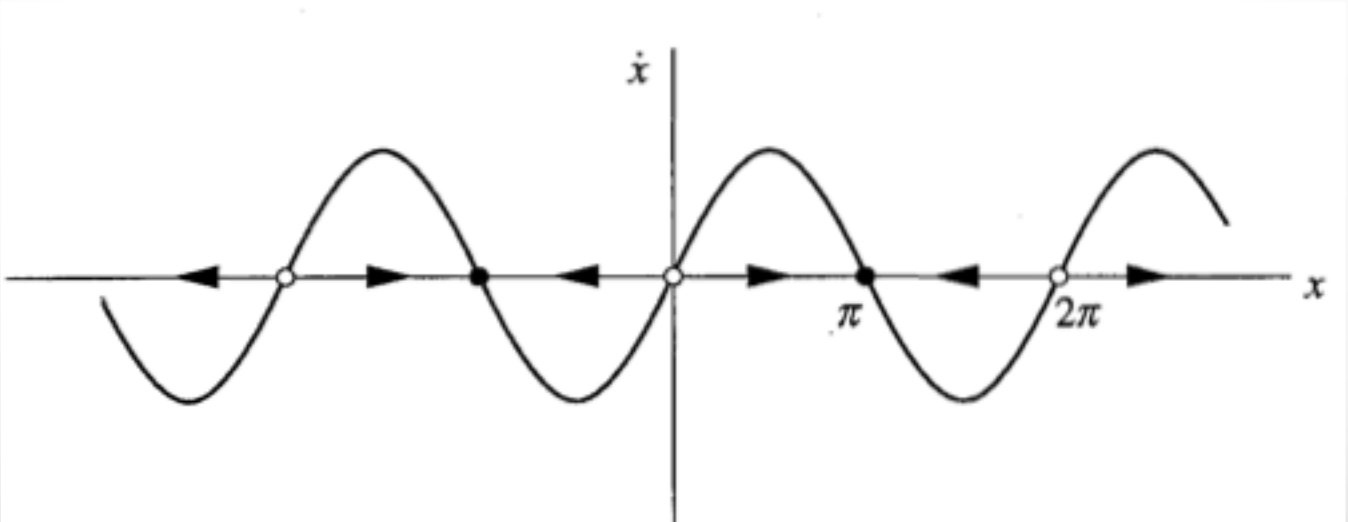
$$\dot{x} = \sin x$$

Fixed points:

$$f(x^*) = 0 \quad \rightarrow \quad \sin x^* = 0 \quad \rightarrow \quad x^* = h\pi, \ \ h = 0, \pm 1, \pm 2, \ldots$$

$$f'(x^*) = \cos h\pi = \begin{cases} 1 & \text{for } h \text{ even} \\ -1 & \text{for } h \text{ odd} \end{cases}$$

x^* is unstable if h is even, stable if h is odd



Example II

For logistic equation $\dot{N} = rN(1 - \frac{N}{K})$

Fixed points:

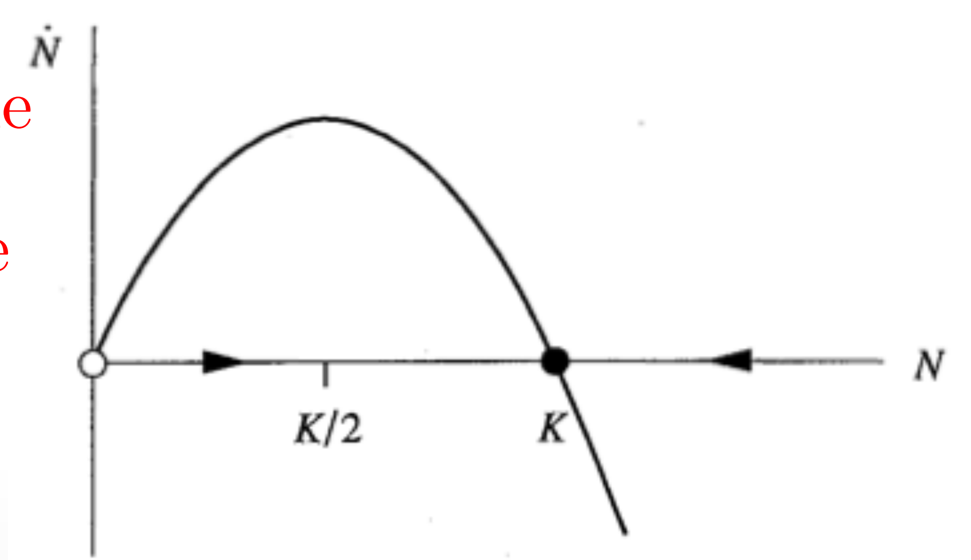
$$f(N^*) = 0 \quad \rightarrow \quad rN^*(1 - N^*/K) = 0 \quad \rightarrow \quad N_1^* = 0, N_2^* = K$$

$$f'(N^*) = r - \frac{2rN^*}{K}$$

$$f'(N_1^*) = r > 0 \rightarrow \text{unstable}$$

$$f'(N_2^*) = -r < 0 \rightarrow \text{stable}$$

Characteristic time scale for both
$N < K$ and $N > K$: $1/|f'(x)| = 1/r$



Example III

When $f'(x^*) = 0$, dynamics cannot be linearized even close to x^*.
The stability of a fixed point $f'(x^*) = 0$ depends on $f(x)$.

Examples:

$$(a)\ \dot{x} = -x^3 \quad (b)\ \dot{x} = x^3 \quad (c)\ \dot{x} = x^2 \quad (d)\ \dot{x} = 0$$

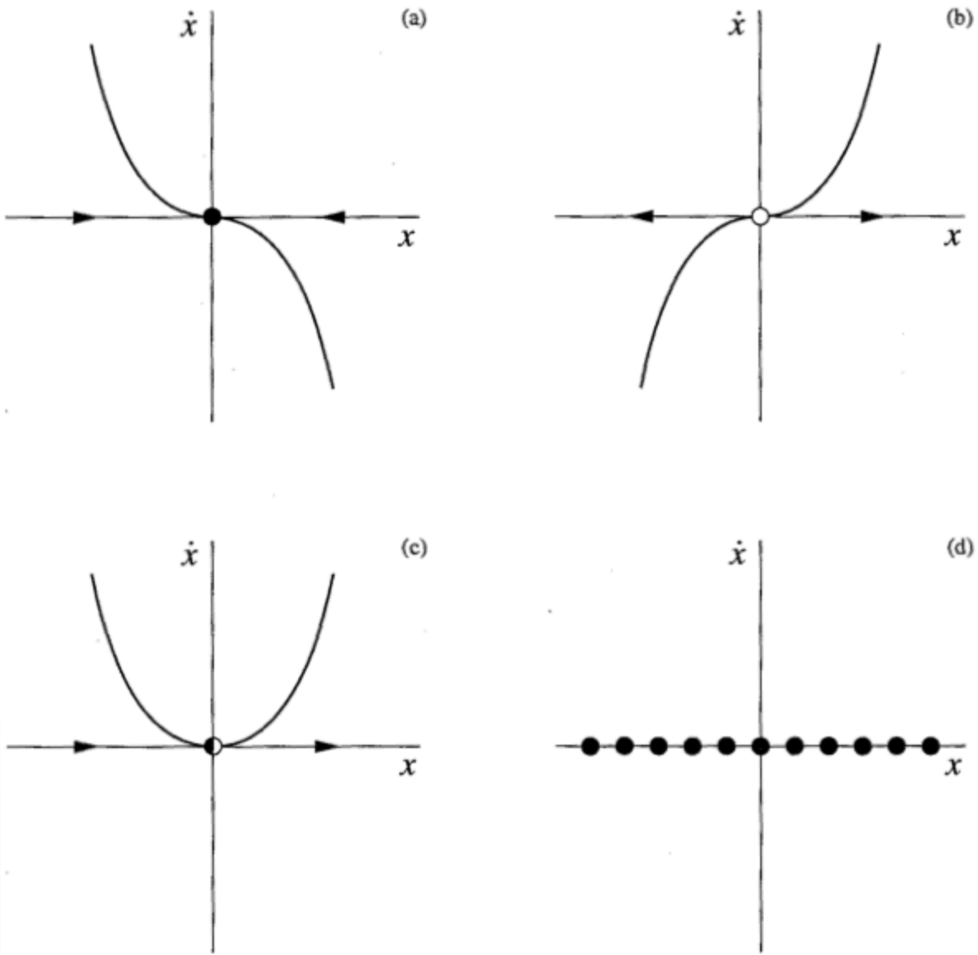
Fixed points: $x^* = 0$ in (a), (b), (c); the whole x-axis for (d)

Example III

$$(a)\ \dot{x} = -x^3 \quad (b)\ \dot{x} = x^3 \quad (c)\ \dot{x} = x^2 \quad (d)\ \dot{x} = 0$$

Stable

Unstable



Half-Stable

Line of fixed points: Perturbations neither grow nor decay.

Existence and uniqueness

Are we sure that $\dot{x} = f(x)$ always has a solution and, in that case, that it is unique?

Example:

$$\dot{x} = x^{1/3}, \quad \text{for } x_0 = 0$$

Trivial solution:

$$x(t) = 0 \quad \forall t$$

Other solution: (imposing initial condition $x(0) = 0$)

$$\int_{x_0=0}^{x} x'^{\,-1/3} dx' = \int_{t_0=0}^{t} dt' \Leftrightarrow \frac{3}{2} x^{2/3} = t \Leftrightarrow x(t) = \left(\frac{2}{3} t\right)^{3/2}$$

Existence and uniqueness

$$\dot{x} = x^{1/3}, \quad \text{for } x_0 = 0$$

There are actually infinitely many solutions along

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}.$$

Without uniqueness, geometric approach fails.

Where does non-uniqueness come from?

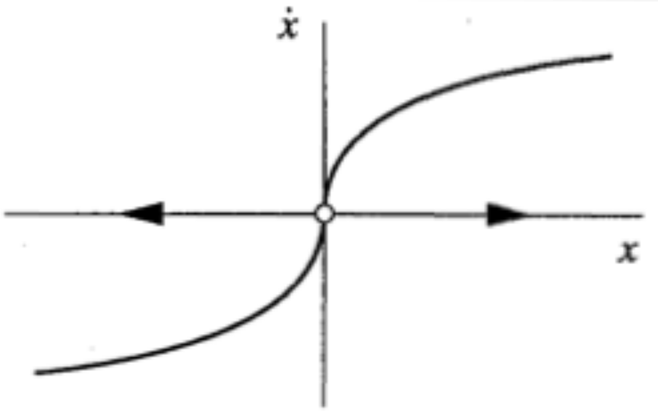
Existence and uniqueness

$$\dot{x} = x^{1/3}, \quad \text{for } x_0 = 0$$

Fixed points

$$x^* = 0 \quad \rightarrow \quad f'(x^*) = \frac{1}{3} x^{*\,-2/3} = \infty$$

Fixed point has vertical slope, so it is *extremely* unstable!



Existence & uniqueness

In fact, there are infinitely many solutions to

$\dot{x} = x^{1/3}, \quad \text{for } x_0 = 0$

$x(t) = \left(\frac{2}{3}t\right)^{3/2}$ is a solution.

We can construct solutions such as $x(t) = \left[\frac{2}{3}(t - t_0)\right]^{3/2}$.

Now $x(t) = 0$ is the only solution for $t < t_0$. For $t > t_0$,

x moves away from 0 following $x(t) = \left[\frac{2}{3}(t - t_0)\right]^{3/2}$

t_0 is arbitrary, so there are infinitely many solutions.

Existence and uniqueness

Existence and Uniqueness Theorem

Consider the initial value problem:

$$\dot{x} = f(x), \quad x(0) = x_0$$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis and that x_0 is a point in R. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

For the layman: *If $f(x)$ is smooth, solutions exist and are unique.*

Existence and uniqueness

Solutions do not necessarily exist forever! The theorem guarantees a solution only in a time interval around $t = 0$.

Example:

$$\dot{x} = 1 + x^2, \qquad x(0) = x_0$$

$f(x) = 1 + x^2$ is continuous and has continuous derivative for all x → there is a unique solution for any initial condition $x(0)$.

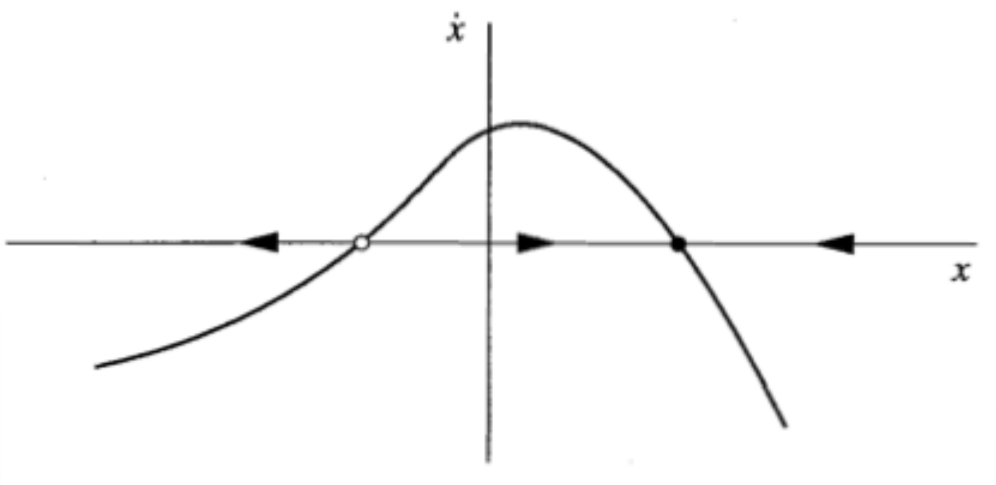
$$x(0) = 0 \quad \rightarrow \quad \int_{x(0)=0}^{x} \frac{dx'}{1 + x'^2} = \int_{t=0}^{t} dt' \quad \rightarrow \quad \tan^{-1} x = t \quad \rightarrow \quad x(t) = \tan t$$

However, the solution exists only for $-\pi/2 < t < \pi/2$, outside this interval there is no solution, since $x(t) \rightarrow \pm\infty$ as $t \rightarrow \pm\pi/2$.

Blow-up: solutions reach infinity in finite time.

Impossibility of oscillations

In a vector field on the real line particles either approach a fixed point or diverge to $\pm\infty$. The trajectories are forced to increase or decrease monotonically: In a first-order system there can be no oscillations!

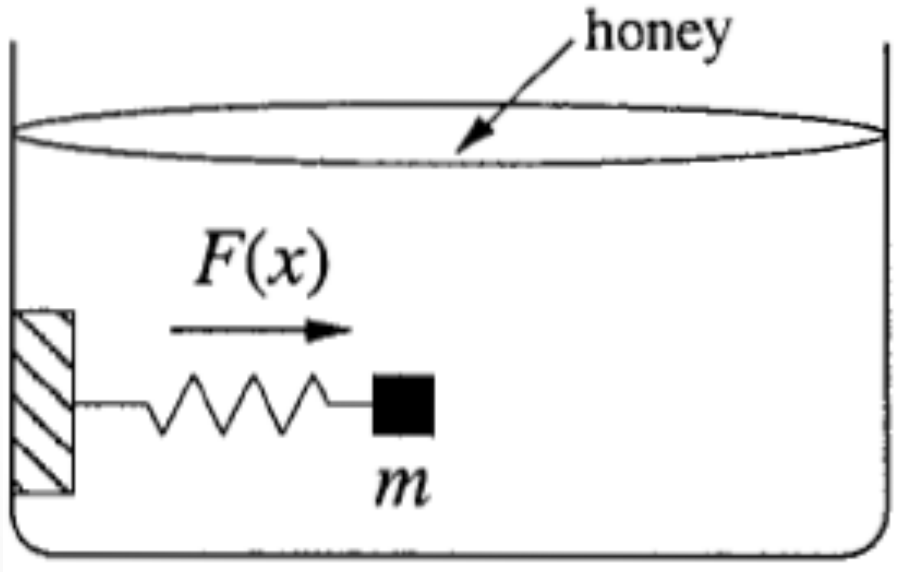


In other words, no periodic solutions for $\dot{x} = f(x)$.
(Flow on the line. Vector field on a circle is different.)

Impossibility of oscillations

Mechanical analog: overdamped systems. If in Newton's equation damping dominates over inertia,

$$m\ddot{x} + b\dot{x} = F(x) \Rightarrow b\dot{x} \approx F(x).$$



Strong viscous damping.

Potentials

$$\dot{x} = f(x)$$

If $f(x)$ is well behaved (e.g. continuous), it is integrable, so one can introduce the potential $V(x)$ of "the force" $f(x)$

$$f(x) = -\frac{dV}{dx}$$

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

Conclusion: $V(t)$ decreases along trajectories $\rightarrow$ the particle always moves towards lower potential.

Potentials

$$f(x) = -\frac{dV}{dx} \qquad \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

$$\frac{dV}{dt} = 0 \;\rightarrow\; \frac{dV}{dx} = 0 \;\rightarrow\; f(x) = 0$$

The potential stays constant in time only at equilibrium (fixed) points, which correspond to extrema of V

Minimum of V $\rightarrow \dfrac{d^2V}{dx^2} > 0 \rightarrow f'(x) < 0 \rightarrow$ stable fixed point

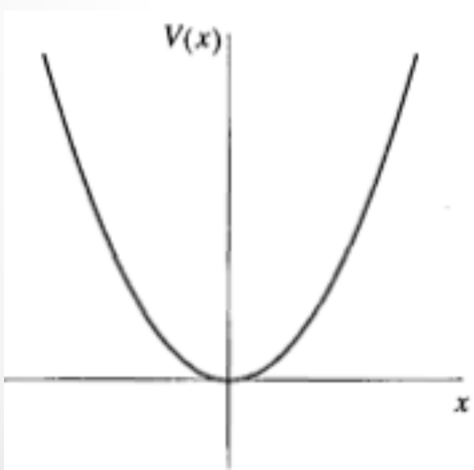
Maximum of V $\rightarrow \dfrac{d^2V}{dx^2} < 0 \rightarrow f'(x) > 0 \rightarrow$ unstable fixed point

Potentials: Example I

$$\dot{x} = -x$$

$$-\frac{dV}{dx} = -x \quad \rightarrow \quad V(x) = \frac{1}{2}x^2 + C \qquad (C \text{ is arbitrary})$$

$$C = 0$$



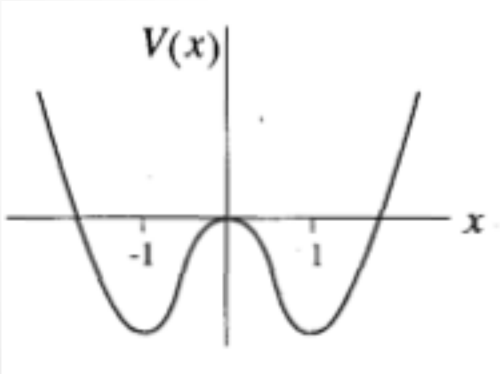
Fixed point at $x = 0$, stable (minimum of V)

Potentials: Example II

$$\dot{x} = x - x^3$$

$$-\frac{dV}{dx} = x - x^3 \quad \rightarrow \quad V(x) = -\frac{x^2}{2} + \frac{x^4}{4} + C \qquad (C \text{ is arbitrary})$$

$$C = 0$$



Stable fixed points at $x = \pm 1$, (minima of V)

Unstable fixed point at $x = 0$, (maximum of V)

This system is **bistable**.

Solving equations on the computer

Numerical integration

$$\dot{x} = f(x), \qquad x(t_0) = x_0$$

Euler's method

Idea: in a small time interval Δt after time t_0 the velocity of the particle/system is approximately the same as at t_0

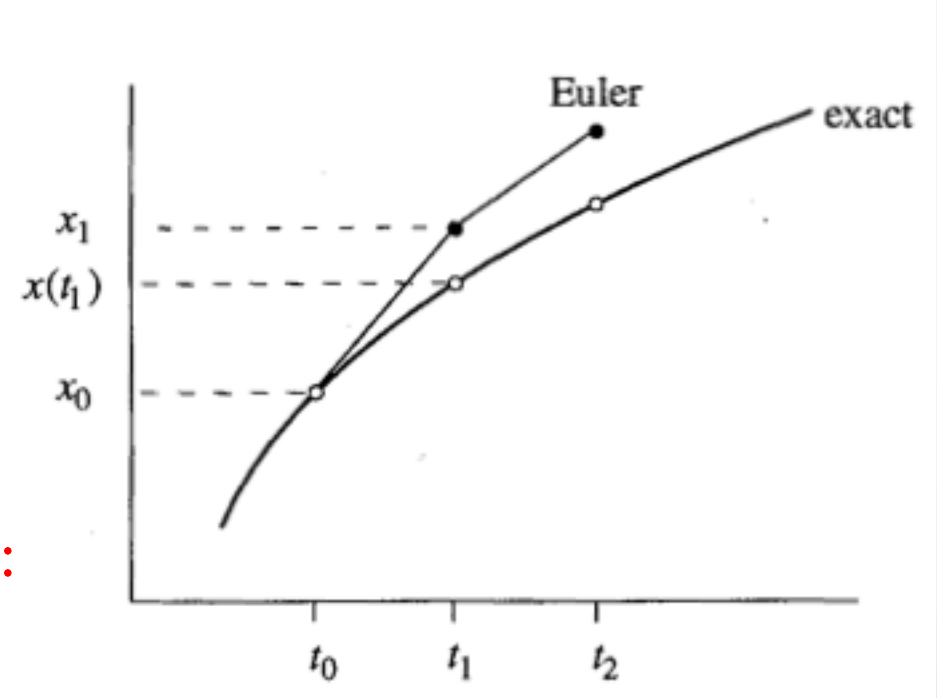
$$x(t_0 + \Delta t) \sim x_1 = x_0 + f(x_0)\Delta t$$

$$x_{n+1} = x_n + f(x_n)\Delta t$$

Solving equations on the computer

Euler's method

Problem: the method gets bad quickly, unless Δt is really small (but then it takes a long time to create the trajectory).



Error (for a given stepsize):

$$E = |x(t_n) - x_n|$$

For Euler's method: $E \propto \Delta t$

Solving equations on the computer

Improved Euler's method

Weakness of Euler's method: velocity is taken at the beginning of the interval $[t_n, t_n + \Delta t]$

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t \qquad \text{(trial step)}$$

$$x_{n+1} = x_n + \frac{1}{2}[f(x_n) + f(\tilde{x}_{n+1})]\Delta t \qquad \text{(real step)}$$

For Improved Euler's method: $E \propto (\Delta t)^2$

Solving equations on the computer

Runge-Kutta method

Tradeoff: high-order methods are more precise but require additional computations

$$
\begin{aligned}
k_1 &= f(x_n)\Delta t \\
k_2 &= f(x_n + \tfrac{1}{2}k_1)\Delta t \\
k_3 &= f(x_n + \tfrac{1}{2}k_2)\Delta t \\
k_4 &= f(x_n + k_3)\Delta t
\end{aligned}
$$

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For Runge-Kutta method: $E \propto (\Delta t)^4$

Solving equations on the computer

There are several ways for numerically solving the relevant differential equations. It's all about numerical integration in time.

1. You can write your own algorithm. See numerical integration methods e.g. in Press, et al: Numerical Recipes, lecture notes in the Computational Science course (MyCourses).
2. Use numerical methods packages like Matlab, Mathematica, or Maple.
3. Use softwares specifically designed for solving and visualising systems of nonlinear dynamics: Pplane and XXP.
4. **Write your own algorithm using high-level library functions. This you will be doing some in Python.**

Visualisation: Mathematica, Matlab, Maple, **Python**, …

Solving equations on the computer

Exercise 2.8.1 in Strogatz:

Solve system $\dot{x} = x(1-x)$ numerically; slope field an numerically integrated $x(t)$ starting from different $x(0)$.

Now, take a look at the instructions and the example Jupyter notebook under Materials.

The numerical exercises are to be returned as Jupyter notebooks.

Next time: Bifurcations.

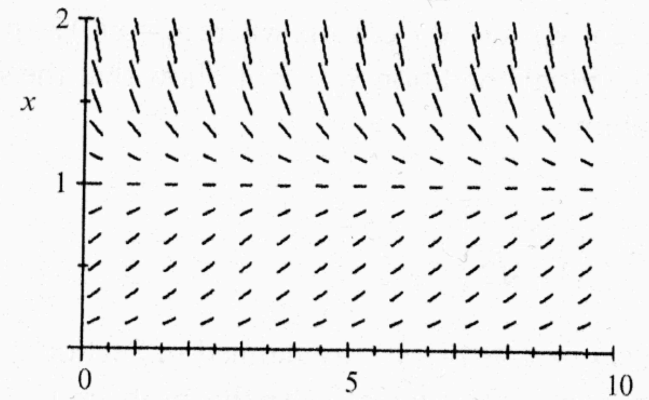


Figure 2.8.2

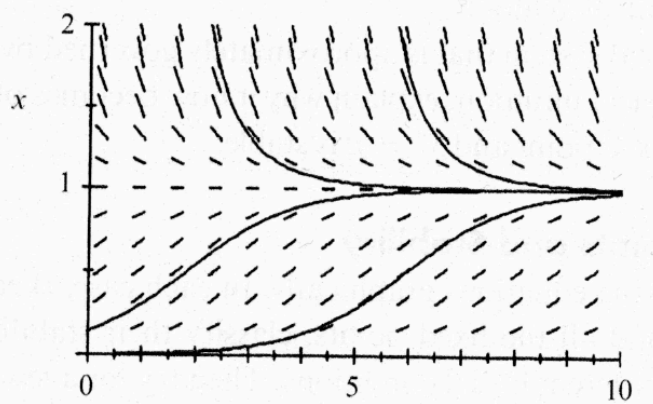


Figure 2.8.3