Lecture 2

Topics: Arc Length. Functions of 2 or 3 variables. Sketching surfaces. Level curves.

- Find the formula for arc length from first principles using the definition of the definite integral. (More details will be given in the next lecture)
- The physical interpretation of arc length being the integral of speed makes the formula very intuitive
- Note: A more rigorous mathematical treatment of the arclength formual is given in in the *materials* section of MyCourses (somewhat rigorous proof of the arc length formula). This is material is **not required** and is not discussed in lecture, but is available for general interest.
- Computed the arc length of the circle of radius 'a' and obtained the expectd result
- Domains and graphs of functions of two variables.
- How to sketch surfaces by taking slices (traces). Did the example of the paraboloid in detail. Talked about the physical applications of paraboloids such as telescopes, satellite dishes, solar collectors etc. Other examples included the sphere, cone and saddle.
- Introduced the concept of a level curve (and level surface). Sketch a couple of examples. Talked about familiar applications such as the contour lines on topographic maps and isotherms on weather maps and so on.
- Viewed some surface and level curves on maple (code is in the materials sections).

Where to find this material

- Adams and Essex 11.3, 12.1
- Corral, 1.9, 2.1
- Guichard, 13.3, 14.1
- Active Calculus. 9.1, 9.8

Arc Length

Recall from lecture 1:

Vector-valued function

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$= \langle x'(t), y'(t) \rangle$

$=$ a tangent vector

Notation:

Let's use Δt instead of h for the small time step

Let $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

Then $$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

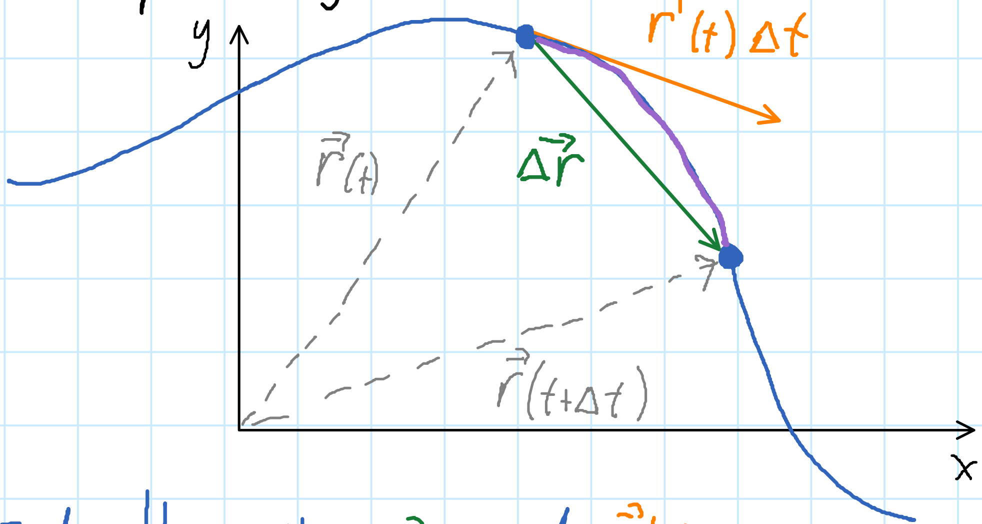
Also, $$\|\vec{r}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$$

<u>Note</u>: For Δt small,

$$r'(t) \approx \frac{\Delta\vec{r}}{\Delta t}$$

or $$\Delta\vec{r} \approx r'(t)\,\Delta t$$

Graphically (This is important!!)



Idea!! Both $\Delta\vec{r}$ and $\vec{r}\,'(t)\Delta$ approximate the curve segment

The approximation improves as $\Delta t \to 0$

This picture will be useful at some other times in the course too.

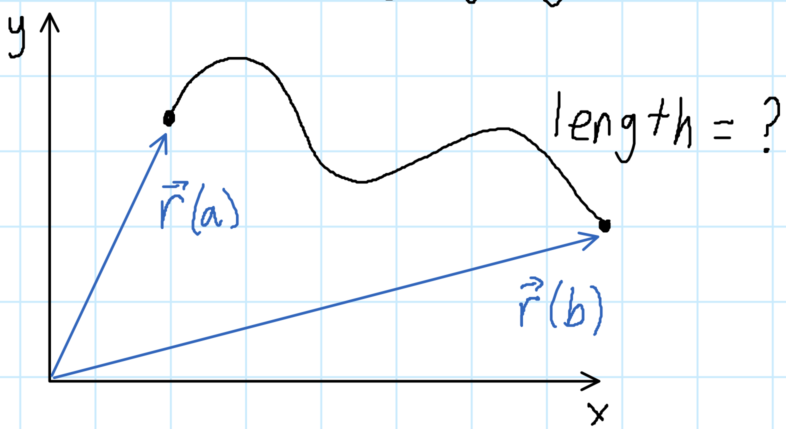
Arc length (2)

Aim:

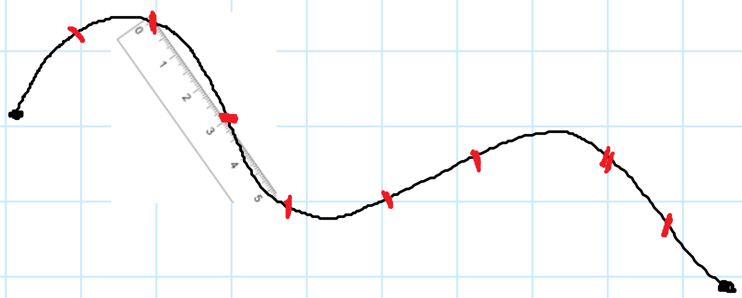
Find a formula for the arc length of a parametric curve.

Setup Curve $\vec{r}(t) = \langle x(t), y(t) \rangle$
$a \le t \le b$

Same in 3D

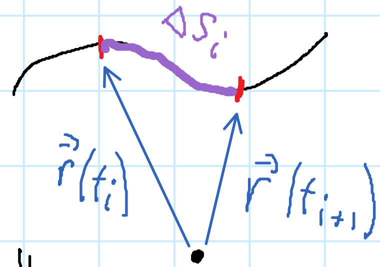


Idea: Say you have a straight ruler. How can find (approximately) the length of the curve



Let's calculate Divide the curve into N pieces of length Δs_i, $i = 1, \dots, N$ corresponding to time steps $\Delta t = \frac{b-a}{N}$

The total arc length $= \sum_{i=1}^{N} \Delta s_i$



$$\approx \sum_{i=1}^{N} \|\vec{r}\,'(t_i)\| \Delta t$$

We expect that in the limit $N \to \infty$ that the approximation becomes exact. This is true, and can me made rigorous, with some (very mild) assumptions on the smoothness/regularity of the curve. See the handoutout linked to on the first page of today's notes.

"$\sum$" $f(x_i) \Delta x = \int_a^b f(x)\,dx$

$$\text{Arc length} = \lim_{N \to \infty} \sum_{i=1}^{N} \|\vec{r}\,'(t_i)\| \Delta t$$

or $\Delta t \to 0$

Oh!! This looks familiar. It is exactly the definition of the definite integral

$$= \int_a^b \|r'(t)\|\,dt$$

Arc length (3)

Conclusion

$$\boxed{\text{arc length} = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Now we (A) give a physical interpretation and (B) do a simple example

(A) If we think of $\vec{r}(t)$ as position then: $\vec{r}\,'(t)$ = velocity (vector)

$\|\vec{r}\,'(t)\|$ = speed (scalar)

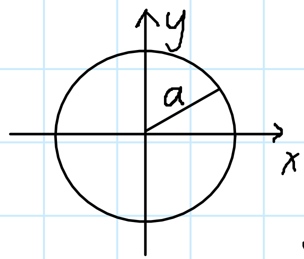
Also, speed $\times$ time = distance

Since our speed is not constant, we summed up all the small distances Δs and obtained

$$\text{Arc length} = \text{Total distace} = \int_a^b \text{speed}\; d\,\text{time}$$

$$= \int_a^b \|r'(t)\|\, dt$$

(A) Let's find the arc length of a circle of radius 'a'. Of course we already know that the anwer is $2\pi a$, so this is a good test case.



As we discussed last class,

$x(t) = a\cos(t)$

$y(t) = a\sin(t)$, $0 \le t \le 2\pi$

So $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$

and $\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t)\rangle$

Now, $\|r'(t)\| = \sqrt{[-a\sin(t)]^2 + [a\cos(t)]^2}$

$= a\sqrt{\sin^2(t) + \cos^2(t)}$

$= a$

It is not a surprise that this is constant. The parametrization of the circle we are using describes motion around the circle at a constant speed!

Therefore we get a very easy integral.

arc length $= \int_0^{2\pi} a\, dt$

$= 2\pi a$ as expected!

Functions of several variables

So far we have studied functions

- $f: \mathbb{R} \to \mathbb{R}$ (in earlier courses)
- $f: \mathbb{R} \to \mathbb{R}^n$ (vector-valued functions we just studied)

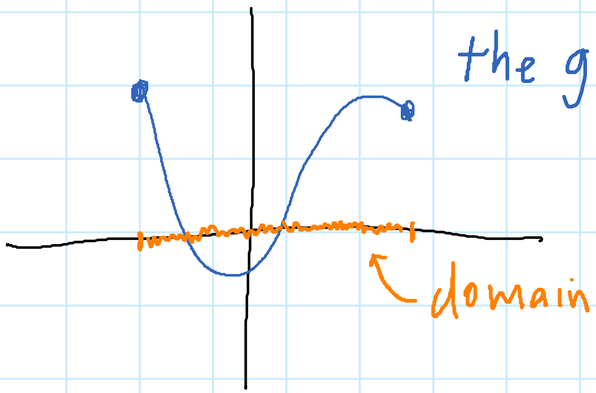
Now we look at functions of several variables $f: \mathbb{R}^n \to \mathbb{R}$

For example : $f(x, y) = 2x + x^2 y$

What are some physical/real-life examples of functions of several variable? **Go to zulip -> lectures/polls**

Domains and graphs

In one variable, $f: \mathbb{R} \to \mathbb{R}$



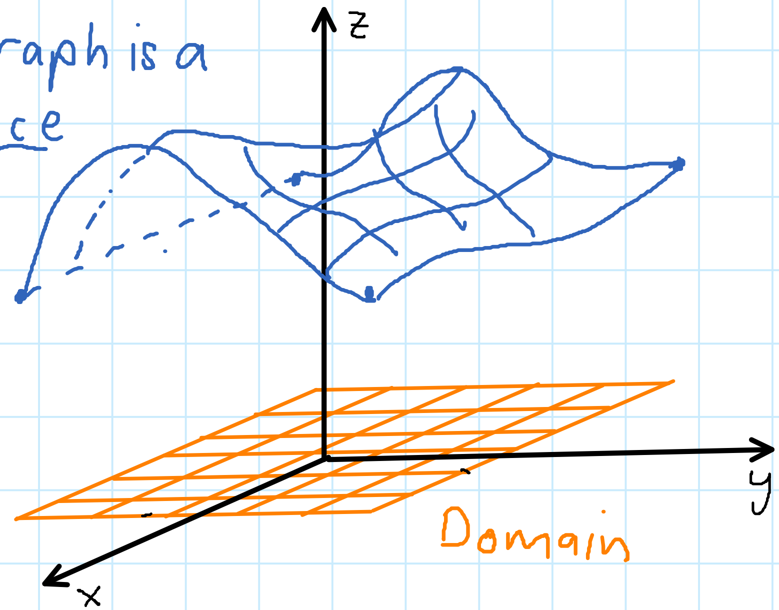
the graph is a curve in $\mathbb{R}^2$

For a function of 2 variables

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

the domain is a region in the xy-plane

the graph is a surface



Functions of several variables (2)

Domain examples

(1) $f(x,y)=\sqrt{1-(x^2+y^2)}$

domain (largest) $=\{(x,y) \mid 1-(x^2+y^2)\geq 0\}$

$=\{(x,y) \mid x^2+y^2\leq 1\}$

$=$ 

(2) $f(x,y)=\sqrt{1/(1-x^2-y^2)}$

domain (largest) $=\{(x,y) \mid x^2+y^2<1\}$

$=$ 

Open and closed regions

Recall: $[-1,1]=\{x \mid -1\leq x\leq 1\}$

 = closed interval (contains endpoints)

$(-1,1)=\{x \mid -1<x<1\}$

 = open interval

closed region in $\mathbb{R}^2$ (contains all its boundary points)

Any disk centered at a boundary point intersects the inside and outside

open region in $\mathbb{R}^2$

For example, this is important in guaranteeing the existence of absolute minima and maxima. A closed interval/region is necessary.

3D sketching

We need a collection of simple surfaces to use as example throughout the course (and Diff Int 3). These surfaces also appear frequently in real life.

Let's introduce the ideas in an example

(1) Our goal is to sketch the graph of the function $f(x,y) = x^2+y^2$

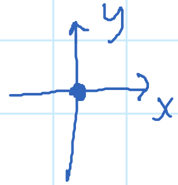
That is, sketch the surface $z = x^2+y^2$

Key idea! Look at $\boxed{\text{traces}}$ (slices with 1 variable held constant)

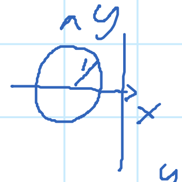
Horizontal traces (fix z)

$z=-1$, $x^2+y^2=-1$ No solution

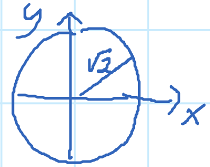
$z=0$, $x^2+y^2=0$



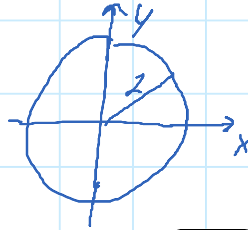
$z=1$, $x^2+y^2=1$



$z=2$, $x^2+y^2=2$

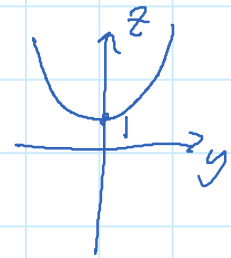


$z=4$, $x^2+y^2=4$

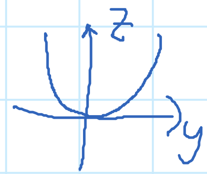


Vertical traces

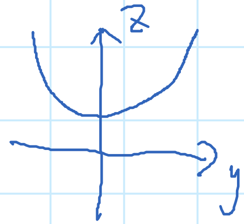
$x=-1$, $z=1+y^2$



$x=0$, $z=y^2$



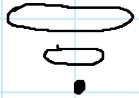
$x=1$, $z=1+y^2$



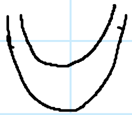
3D sketching (2)

Summary

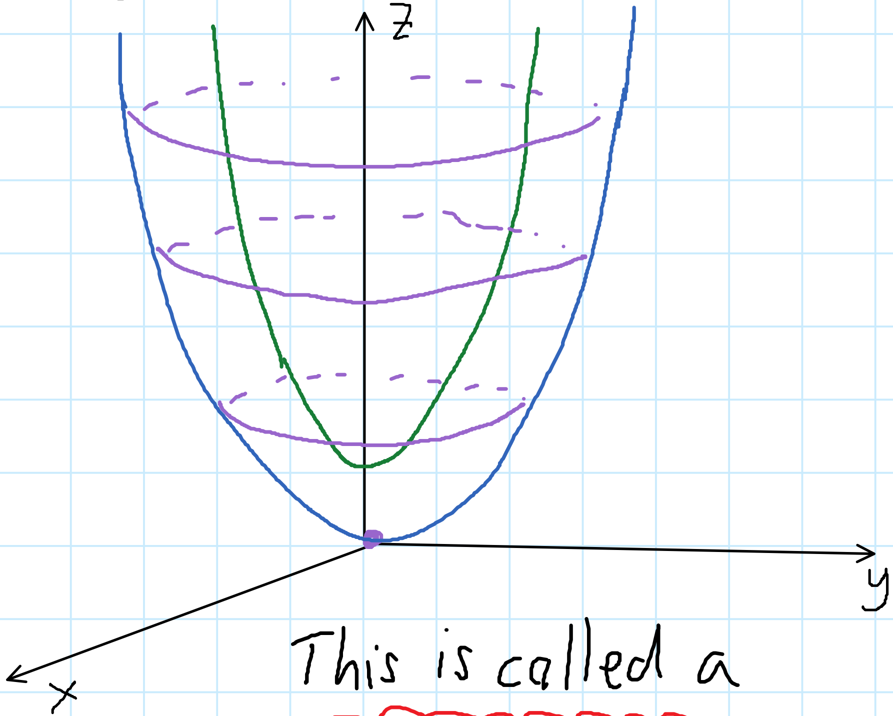
Horizontal :



Vertical



Finally we can sketch the surface



This is called a

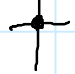
PARABOLOID

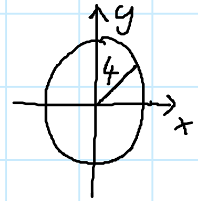
More examples

(2) $z = \sqrt{x^2+y^2}$ (on $z^2 = x^2+y^2$)

$\hookrightarrow z = \pm\sqrt{x^2+y^2}$

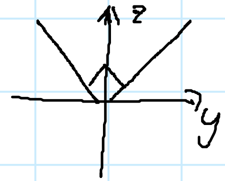
Horizontal traces ?

$z=0$  $z=1$ 

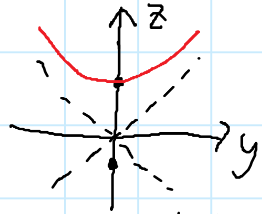
$z=4$ 

Vertical traces ?

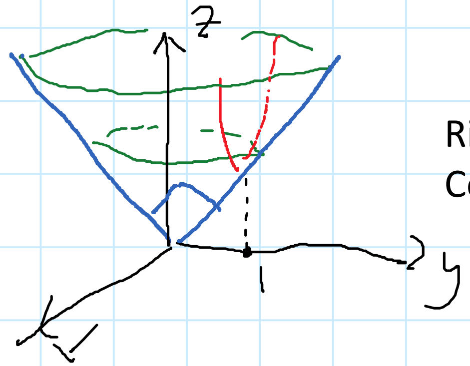
$y=0$, $z = \sqrt{x^2}$
$= |x|$



$y=1$, $z = \sqrt{1+y^2}$
$\Rightarrow z^2 - y^2 = 1$



Hyperbola



Right-circular
Cone (top half)

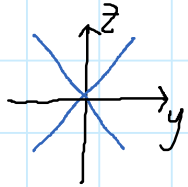
Read about Conic sections

3D sketching (3)

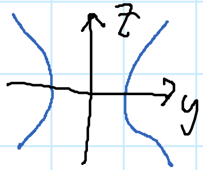
(3) $z = x^2 - y^2$

Horizontal, $z = 0 \Rightarrow x^2 = y^2$

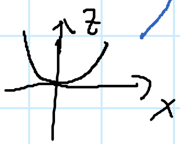
$\Rightarrow |x| = |y|$



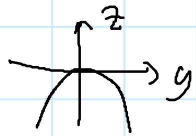
$z = 1$, $x^2 - y^2 = 1$

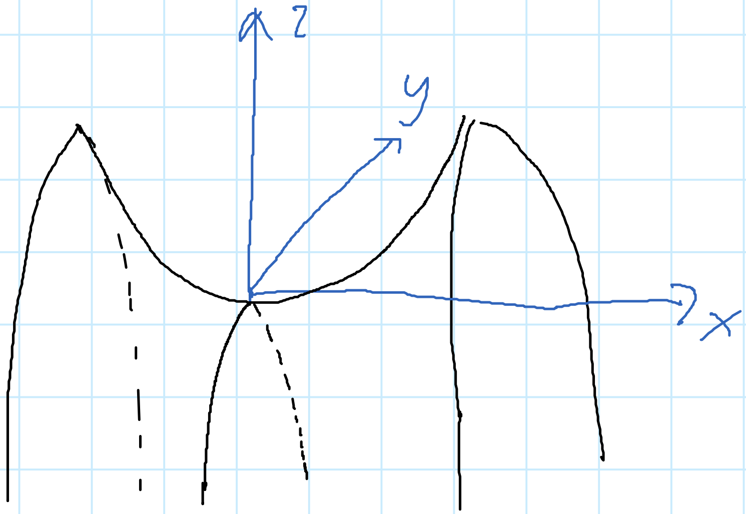


Vertical, $y = 0$, $z = x^2$



$x = 0$, $z = -y^2$



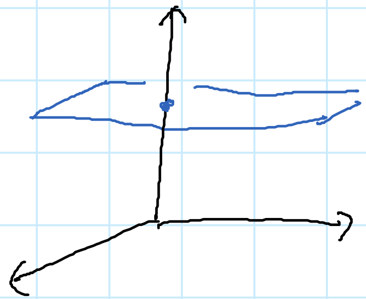


HYPERBOLOID (SADDLE)

(4) $z = f(x,y) = y^2$

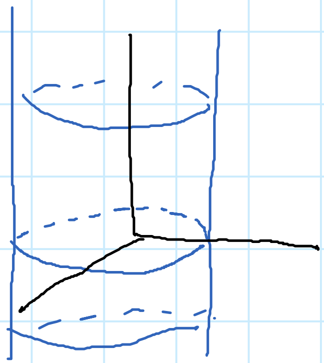
Parabolic cylinder

(5) $z = f(x,y) = 3$



plane

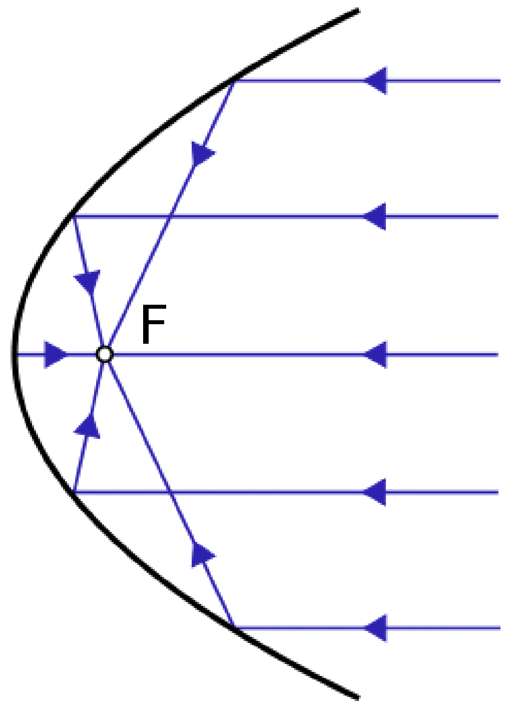
(6) $x^2 + y^2 = 4$ (z = free)



Circular cylinder

Paraboloids

What special geometric/physical property does a paraboloid have that make them extremely useful in real life?

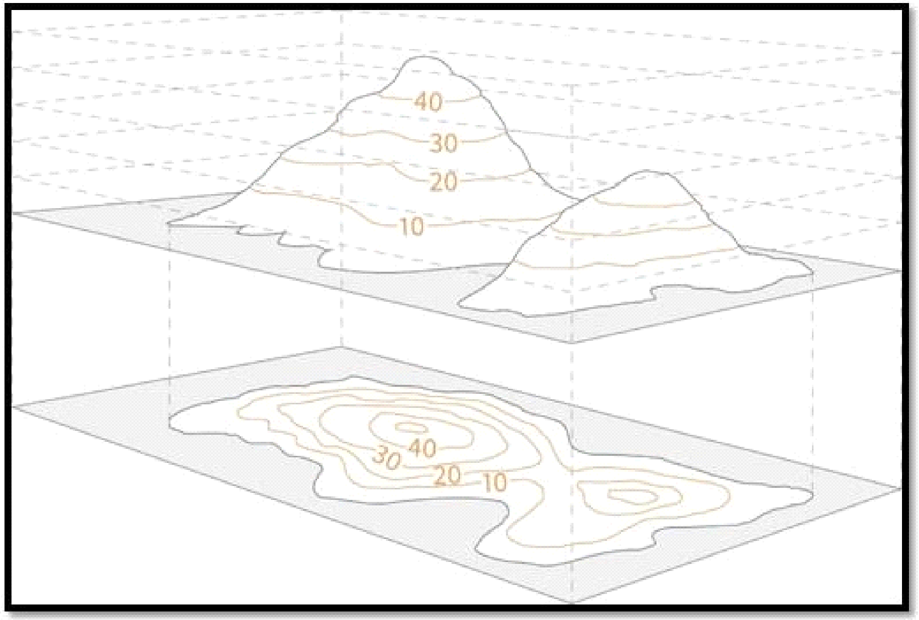


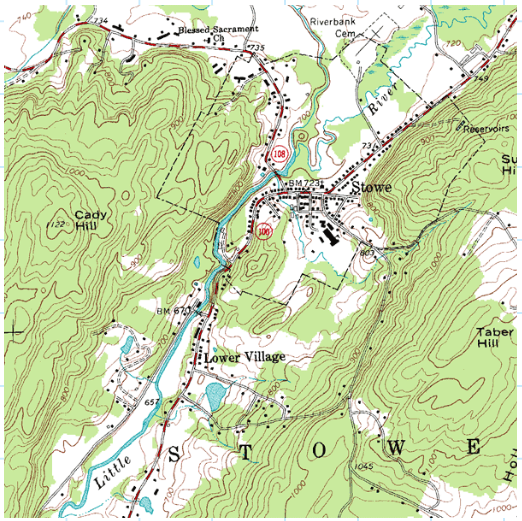
Parabolic things

Level Curves

(We did not have time to cover this in lecture 2. We will go through it in lecture 3)

This is actually a familiar idea. We want to represent a surface in terms of a plot in the xy-plane





A (set) level curve of $f(x,y)$ at level c is the set of points

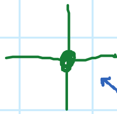
$$\{(x,y) \mid f(x,y) = c\}$$

That is, the solutions to $f(x,y) = c$

Example Sketch some level curves of $f(x,y) = x^2 + y^2$

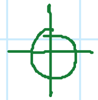
$f(x,y) =$ negative value No solutions

$f(x,y) = 0$

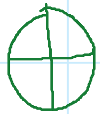


level curves

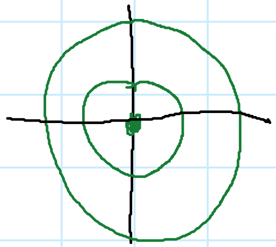
$f(x,y) = 1$



$f(x,y) = 4$



Combining these



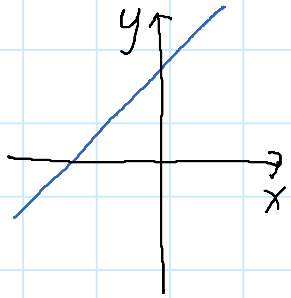
contour plot

Level curves (2)

A conceptual point (useful later in the course)

Let $g(x) = x+1$ which is a function of one variable

The graph of $g(x)$ is

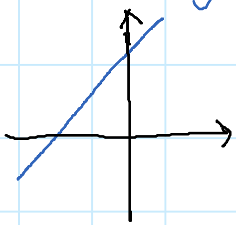


We can rewrite this line as follows

$$y = x+1 \iff y - x = 1$$

Let $f(x,y) = y - x$ which is a function of two variable

The level curve $f(x,y) = 1$ is the same line



Level surfaces

If we have a function of 3 variable, f(x, y, z), then we can not sketch its graph as we would need a 4th dimension to record the values of f.

However, we can sketch the level set (surface) of f(x,y,x)

Example Let $f(x,y,z) = e^{x^2+y^2+z^2}$

sketch the level surface $f = 6$