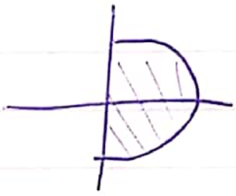
Sample EXAM SOLUTIONS / ANSWERS, 2017

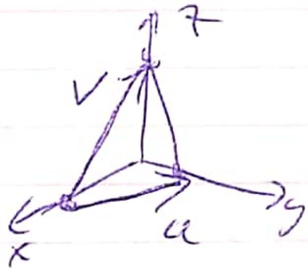
Q1 (a) DOES NOT EXIST. Use the path

(1) $x=0$, (2) $x=y^4$

(b) 

$x \geq 0$

and $x^2+y^2 \leq 4$

(c) 

$\vec{u} = \langle -2, 1, 0 \rangle$

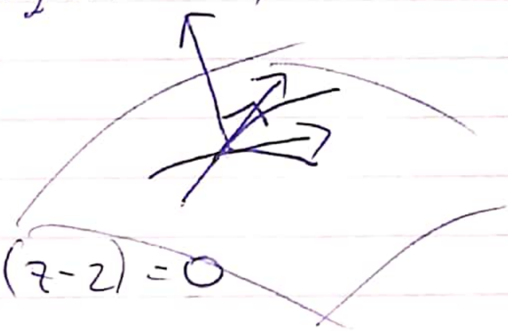
$\vec{v} = \langle -2, 0, 3 \rangle$

$\vec{n} = \vec{u} \times \vec{v} = \langle 3, 6, 2 \rangle$

Plane: $3(x-2) + 6(y-0) + 2(z-0) = 0$

Q2 (a) $\vec{r}_1(0) = \vec{r}_2(1) = \langle 3, 1, 2 \rangle$

$\vec{r}_1{}'(0) = \langle 1, -1, 1 \rangle$, $\vec{r}_2{}'(1) = \langle 1, 1, 2 \rangle$



$\vec{n} = \vec{r}_1{}'(0) \times \vec{r}_2{}'(1)$

$= \langle -3, -1, 2 \rangle$

Tangent plane $-3(x-3) - 1(y-1) + 2(z-2) = 0$

(b) $x = 3.3$, $y = 1.1$ sub into ↑

$-3(0.3) - 1(0.1) + 2z - 4 = 0$

$2z = 4 + 0.9 + 0.1$

$z = 2 + 1/2$

z_0

3

(a) STEP 1: Find critical points

$$\frac{\partial f}{\partial x} = x^2 + y = 0 \Rightarrow y = -x^2 \quad (1)$$

$$\frac{\partial f}{\partial y} = cy + x = 0 \Rightarrow x = -cy \quad (2)$$

sub (2) into (1): $y = -c^2y^2$

$$\Rightarrow y(1 + c^2y) = 0$$

$$\Rightarrow y = 0 \text{ or } y = -\frac{1}{c^2}$$

C.P. $(0,0)$, $\left(\frac{1}{c}, -\frac{1}{c^2}\right)$

2nd Derivative Test

$f_{xx} = 2x$, $f_{yy} = c$

$f_{xy} = f_{yx} = 1$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

- $D(0,0) < 0$ $\therefore$ SADDLE
- $D\left(\frac{1}{c}, -\frac{1}{c^2}\right) = 2\left(\frac{1}{c}\right)(c) - 1 = 1 > 0$

$$f_{xx}\left(\frac{1}{c}, -\frac{1}{c^2}\right) = \frac{2}{c} < 0 \iff c < 0$$

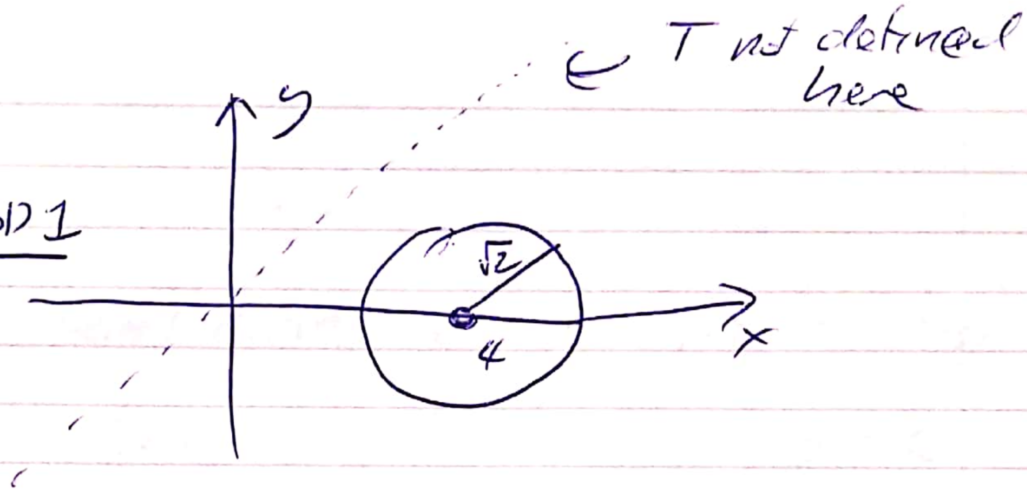
Any value of $c < 0$ gives a local max at $\left(\frac{1}{c}, -\frac{1}{c^2}\right)$

For example, let $c = -1$. C.P is $(1, -1)$

Scanned with CamScanner

3 (b)

METHOD 1



• C.P in the interior: $\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{1}{x+y} \neq 0$ for all (x,y)

$\therefore$ no CRITICAL POINTS

• Boundary: $g(x,y) = (x-4)^2 + y^2 = 2 \quad — \circledast$

method of LAGRANGE MULTIPLIERS

$\nabla f = \lambda \nabla g \Rightarrow \frac{1}{x+y} = \lambda 2(x-4)$

$\frac{1}{x+y} = \lambda 2y$

$\therefore \lambda 2(x-4) = \lambda 2y \quad \left(\lambda \neq 0 \text{ as } \frac{1}{x+y} \neq 0 \right)$

$\therefore x-4 = y$

Sub into $\circledast$ gives $y^2 + y^2 = 2 \Rightarrow y = \pm 1$

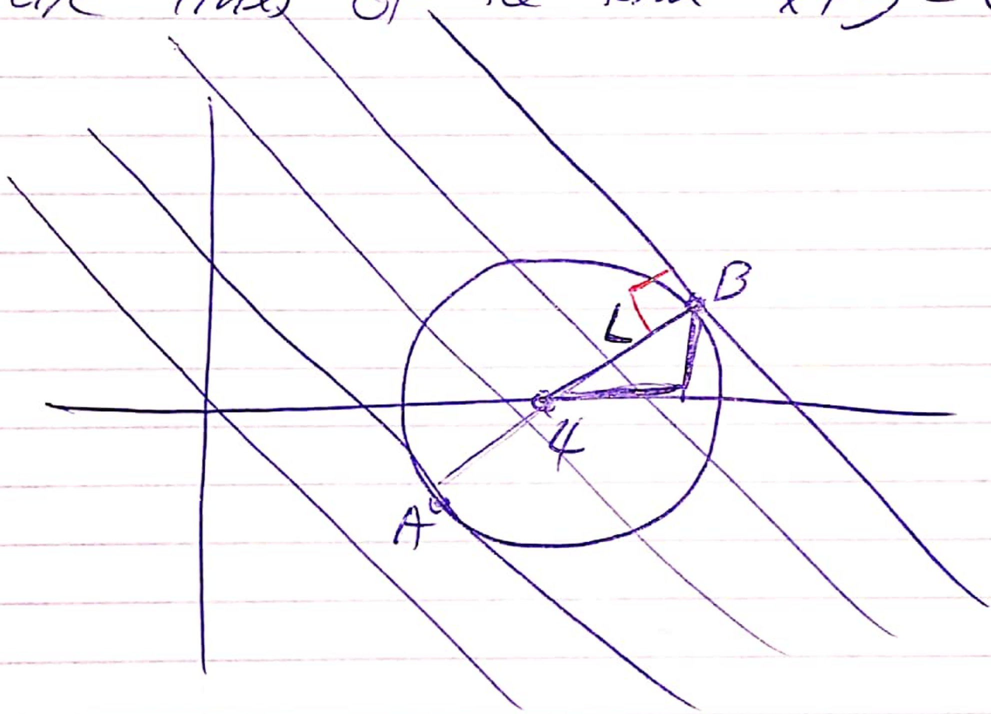
Possible points are $(5,1), (3,-1)$

$T(5,1) = \ln(6) \leftarrow$ MAX

$T(3,-1) = \ln(2) \leftarrow$ MIN

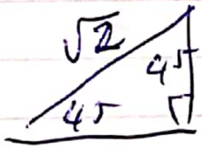
3(b) <u>METHOD 2</u>

The level curves of $T(x,y) = \ln(x+y)$ are lines of the form $x+y=c$



Since the line has slope -1 (-135°) and we know the radius of a circle is perpendicular to the tangent, the radius L in the picture must have slope 1 (45°)

The radius is $\sqrt{2}$. So the triangle

is 

$\therefore$ SIDE LENGTHS ARE BOTH 1

$\therefore$ $A = (3,-1)$ and $B = (5,1)$

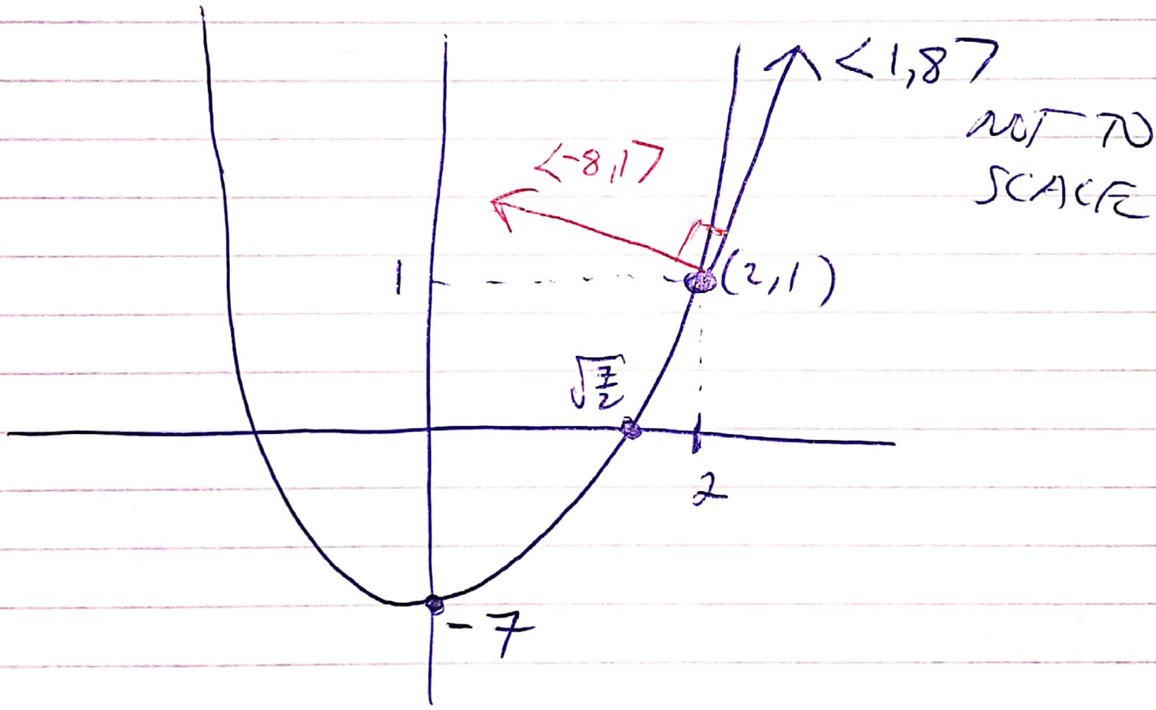
Q4

(a) The level curves are of the form $-2x^2+y=C$

$\Leftrightarrow y=2x^2+C$

$x=2, y=1 \Rightarrow 1=8+C \Rightarrow C=-7$

$\therefore y=2x^2-7$



(b) Let $x=t$ then $y=2t^2-7$

$\vec{r}(t)=\langle t, 2t^2-7\rangle$

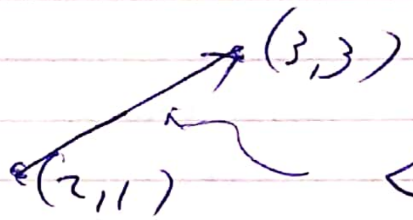
(c) $\vec{r}\,'(t)=\langle 1, 4t\rangle$ when $t=2$

$x=2$

$\vec{r}\,'(2)=\langle 1,8\rangle$ = Tangent VECTOR $y=1$

Q4

(d)



$\langle 3-2,\ 3-1\rangle$

$= \langle 1,2\rangle$

Let $\vec{u} = \dfrac{\langle 1,2\rangle}{\|\langle 1,2\rangle\|} = \dfrac{1}{\sqrt{5}}\langle 1,2\rangle$

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right\rangle$$

$$= 100e^{-2x^2+y}\langle -4x, 1\rangle$$

$$\nabla T(2,1) = \text{CONSTANT}\langle -8,1\rangle$$

$$= 100e^{-7}\langle -8,1\rangle$$

$$D_u T(2,1) = \nabla T(2,1)\cdot\vec{u}$$

$$= \frac{100e^{-7}}{\sqrt{5}}\langle -8,1\rangle\cdot\langle 1,2\rangle$$

$$= \frac{100e^{-7}}{\sqrt{5}}(-8+2)$$

$$= \frac{600e^{-7}}{\sqrt{5}}$$

Q4 (e) $\frac{\nabla T(2,1)}{\|\nabla T(2,1)\|} = \frac{\langle -8,1\rangle}{\|\langle -8,1\rangle\|}$

$= \frac{1}{\sqrt{65}}\langle -8,1\rangle$

(f)

$D_{\left(\frac{\vec{r}'(2)}{\|\vec{r}'(2)\|}\right)} T(2,1) = \text{constant} \langle 1,8\rangle \cdot \langle -8,1\rangle$

$= 0$

this is expected as the rate of change in the tangent direction is always zero since ∇T is orthogonal to the level curve

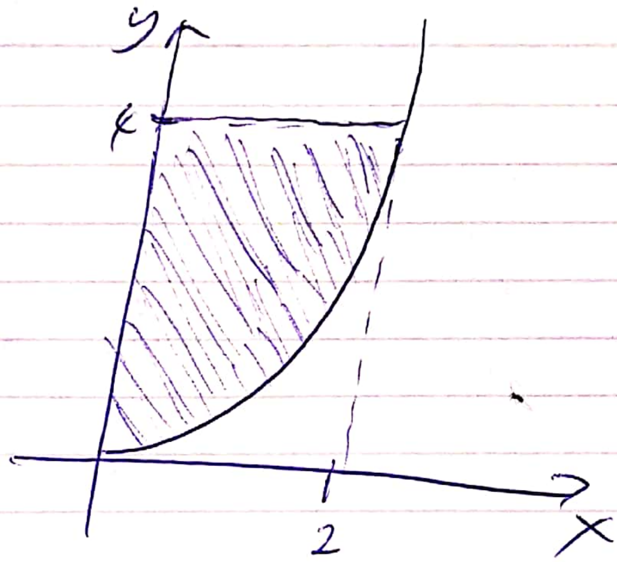
(OR A function does not change as you move along its level curve.

$\therefore$ Directional derivative in the tangent direction should be zero)

Q5

(a) $0 \le x \le 2$

$x^2 \le y \le 4$



Reversing: $0 \le y \le 4$

$0 \le x \le \sqrt{y}$

$$\text{Integral} = \int_0^4 \int_0^{\sqrt{y}} \frac{1}{1+y^{3/2}} \, dx \, dy$$

$$= \int_0^4 \frac{y^{1/2}}{1+y^{3/2}} \, dy$$

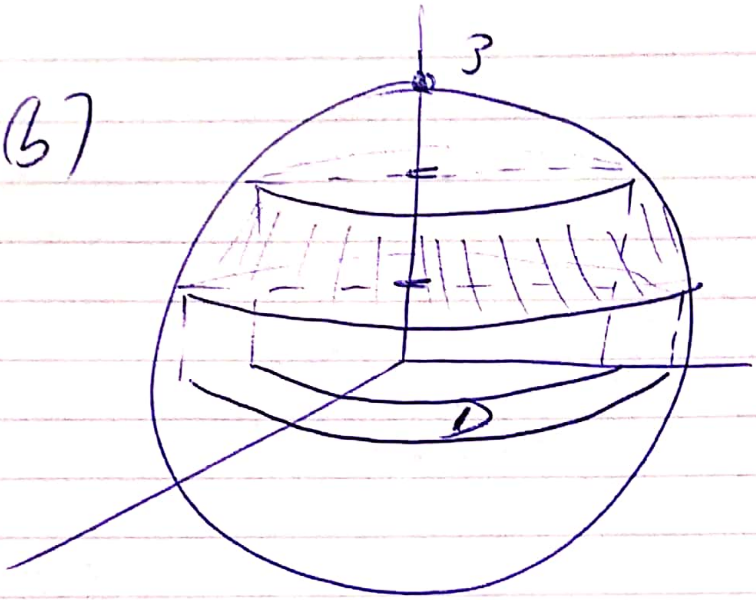
$$= \ln\left(\left|1+y^{3/2}\right|\right) \frac{2}{3} \Big|_0^4$$

$$= \frac{2}{3} \ln(9)$$

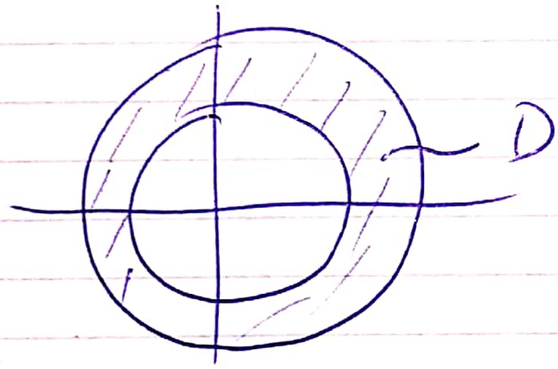
$\boxed{\ln(1)=0}$

$$\left(= \frac{4}{3} \ln(3) \right)$$

Q5 (b)



- when $z=1$, $x^2+y^2=8$
  $z=2$, $x^2+y^2=5$



$0 \le \theta \le 2\pi$

$\sqrt{5} \le r \le \sqrt{8}$

- Surface Area

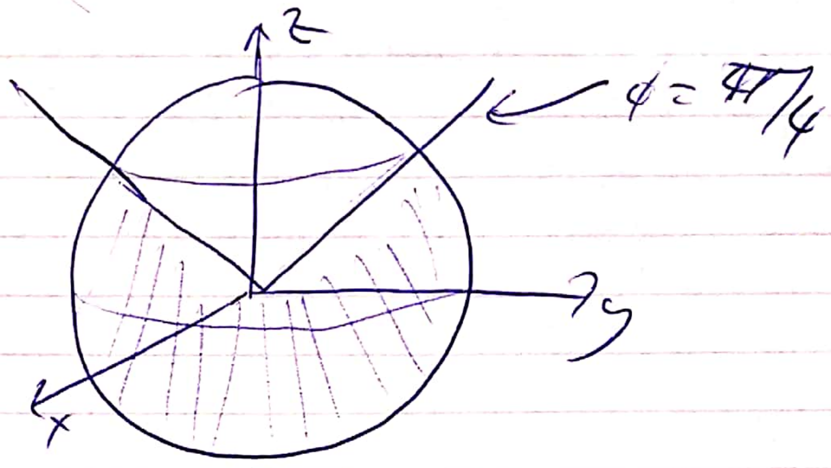
$$= \int_0^{2\pi}\int_{\sqrt{5}}^{\sqrt{8}} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{\sqrt{5}}^{\sqrt{8}} \sqrt{\frac{x^2+y^2}{9-x^2-y^2}+1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{\sqrt{5}}^{\sqrt{8}} \sqrt{\frac{9}{9-r^2}}\; r\,dr\,d\theta$$

$$= \frac{3r}{\sqrt{9-r^2}}$$

6 (a)



$$\text{Vol} = \int_0^{2\pi} \int_{\pi/4}^{\pi} \int_0^{\sqrt{8}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

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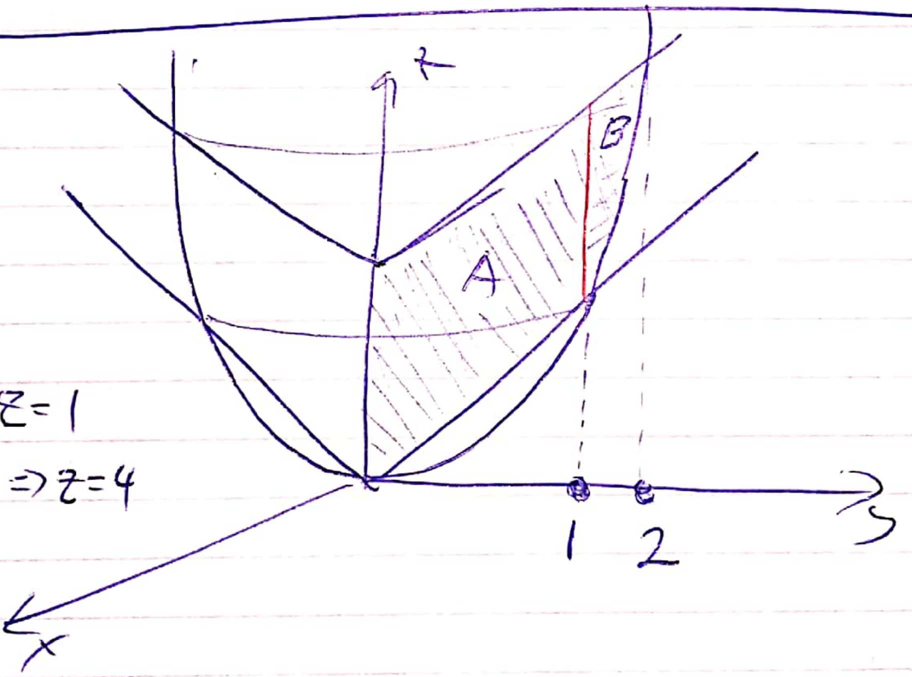
6 (b)



Intersection

$z = \sqrt{z} \Rightarrow z = 1$

$z = 2 + \sqrt{z} \Rightarrow z = 4$

A: $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$, $\sqrt{x^2+y^2} \leq z \leq 2 + \sqrt{x^2+y^2}$

B: $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 2$, $x^2+y^2 \leq z \leq 2 + \sqrt{x^2+y^2}$

$$\text{Vol} = \int_0^{2\pi} \int_0^{1} \int_r^{2+r} r \, dz \, dr \, d\theta + \int_0^{2\pi} \int_1^{2} \int_{r^2}^{2+r} r \, dz \, dr \, d\theta$$