Aalto University

MS-A0211 / Period III 2020

Final Exam, 17.02.2020

No calculators or notes of any kind are allowed.

This exam consists of 6 problems, each of equal weight.

Notation for vectors: $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi)$ and "$dV = \rho^2\sin(\phi)$".

Question 1: Here are some unrelated direct questions

(a) Consider the limits $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$, $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$, and $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{e^{x+y}}$.

Find a limit that does not exist and justify why it does not exist (you do NOT have to make any comment about limits that do exist).

(b) Let $\mathbf{r}(t) = \langle t^2, t\rangle$ for $0 \le t \le 2$. Sketch the curve and write an integral (purely in terms of t) for the arc length of this curve. You do NOT have to evaluate the integral.

(c) Compute the double integral of $f(x, y) = x^2y$ over the triangular region with vertices $(0, 0), (-1, 1)$, and $(1, 1)$.

Question 2: Let $f(x, y) = \sin(xe^y) - x + 3$. A given fact is that $f(0.2, 0.1) = 3.01924$ accurate to 5 decimal places.

(a) Compute all the 1st and 2nd order partial derivatives of $f(x, y)$

(b) Taking $(0, 0)$ as the reference point, use *linear approximation* to find an approximation of $f(0.2, 0.1)$.

(c) By refering to the idea of a tangent plane, explain why you obtained the answer you did in part (b)

(d) Use a *2nd order Taylor polynomial* to find an approximation of $f(0.2, 0.1)$. Is this approximation better or worse than the linear approximation?

(e) Find a critical point of $f(x, y)$ and determine if it is a local min, local max or saddle.

Question 3: Let $f(x, y) = 4xy$

(a) Find the absolute extrema (i.e., min and max) of $f(x, y)$ on the elliptical region $4x^2+y^2 \leq 8$. Determine all the locations of the extreme values.

(b) Does $f(x, y)$ have an absolute minimum and absolute maximum on the 1st quadrant (that is, the region $\{(x, y) \mid x \geq 0, y \geq 0\}$). Justify your answer.

Question 4: Let $f(x, y) = y - x^2$.

(a) Write the equations and then make a large sketch of the level curves $f(x, y) = 1$ and $f(x, y) = 2$ on the same axes (that is, make a contour plot with these two level curves).

(b) On the above plot, draw a point P and a vector $\mathbf{u}$ (with initial point at P) such that the directional derivative $D_{\mathbf{u}} f(P)$ is negative.

(c) At the point $(1, 3)$, find the direction (vector) in which $f(x, y)$ is increasing the fastest. Sketch the point and the vector on the above plot. Does it look approximately correct? Why or why not?

Question 5: Two double integral questions.

(a) Reverse the order of integration for $\int_0^1 \int_0^{2x^2+1} f(x, y)\, dydx$. That is write as an integral of the form $\iint \ldots\, dxdy$.

(b) Compute the double integral of the functon $f(x, y) = e^{x^2+y^2}$ over the top half of the disk of radius 3 centered at $(0, 0)$.

Question 6: Two triple integral questions.

(a) Set up an integral in *cylindrical coordinates* to represent the volume of the region in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0$) that lies above the xy-plane, below the plane $z = y - x$ and inside the cylinder $x^2 + y^2 = 4$. Do NOT evaluate the integral.

(b) Let E be the solid region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the plane $z = 1$. Let $d(x, y, z) = z^2$ be the density of E. Sketch E and write a triple integral in *spherical coordinates* that gives the mass of E. You do NOT have to evaluate the integral.