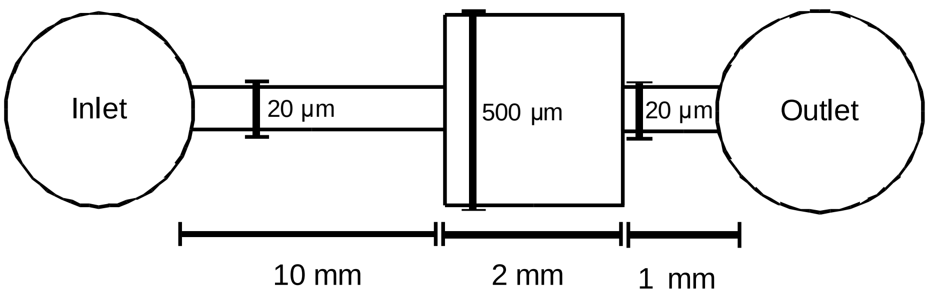
1. Hagen-Poiseuille's Law

A microfluidic chip consists of an inlet, an outlet, two 20 microns wide channel segments and a 500 microns wide reaction chamber as shown in the picture (top view). The channel depth throughout the chip is 80 µm and the lengths and widths of the three segments have been marked to the picture. The liquid used on the chip is water at 20°C (µ= 1 mPa*s).

a) Calculate the fluidic resistance of the chip using the hydraulic radius approximation.



Step 1: calculate the hydraulic radius

$$r_H = \frac{2A}{P} = \frac{2wh}{2w+2h} = \frac{wh}{w+h}$$

Hydraulic radii of the elements:

$$r_1 = r_3 = \frac{20\,\mu m \cdot 80\,\mu m}{20\,\mu m + 80\,\mu m} = 16\,\mu m$$

$$r_2 \approx 69\,\mu m$$

Step 2: Calculate the flow resistance of each of the 3 parts. We are estimating the the rectangular channels are equivalent to a circle cross sectional channel with a radius matching the hydraulic radius. We can thus utilize the flow resistance formula for a cylindrical channel:

$$R_H = \frac{8\mu L}{\pi r^4}$$

hint: Express the hydraulic resistance in units of Pa*s/mm^3

Flow resistances of the elements:

$$R_{H1} = \frac{8\mu L}{\pi r_H^4} = \frac{8 \cdot 1mPas \cdot 10mm}{\pi(0.016mm)^4} \approx 390000 Pas / mm^3$$

$$R_{H3} \approx 39000 Pas / mm^3$$

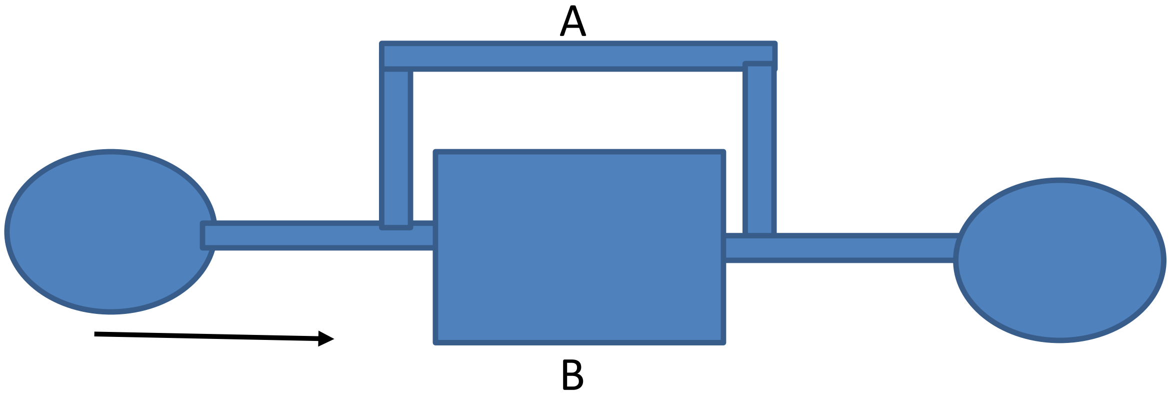
$$R_{H2} \approx 200 Pas / mm^3$$

Step 3: Calculate the flow resistance of the series.

Total flow resistance of a **series**: $R_H = R_{H1} + R_{H2} + R_{H3} \approx 430\,000$ Pas / mm^3

Notice how the smallest channels are dominant in a series.

In a series, the smallest component was the most significant, what about a parallel? Which one matters more for the flow rate, the hydraulic resistance of A or B?   Answer: B



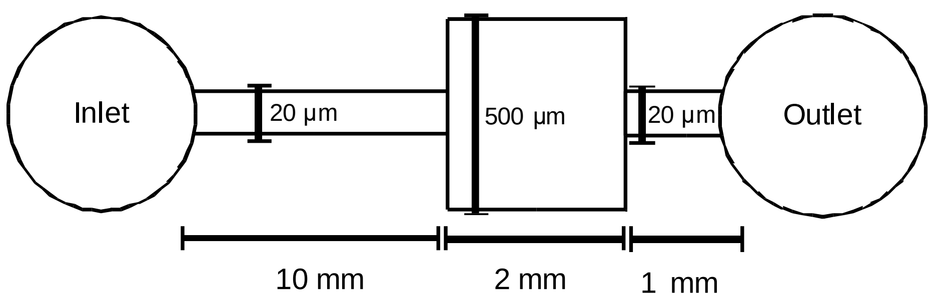
Also, note that in a series, all components have the same flow through them, but this is NOT the case with a parallel configuration.

b) Calculate the volumetric flow rate (in **μl/min**) when a 1000 Pa pressure difference is applied between the inlet and the outlet.

Hagen-Poiseuille: $$\Delta P = \frac{8\mu L}{\pi R^4} Q = R_H Q$$

$$Q = \frac{\Delta P}{R_H} = \frac{1000 Pa}{430000 Pas/mm^3} \approx 2.3 \cdot 10^{-3} mm^3/s \approx 0{,}14 \mu l/\min$$

The average linear velocities:
$v_{ave} = Q/A \approx 1.4$ mm/s on the narrow part and 0.06 mm/s for the broad.



c) The chip inlet is connected to the pressure pump with tubing that has circular cross section and inner radius of 1 mm. The length of the tubing however varies from experiment to experiment between 10 cm and 30 cm. Does this variation have a big effect on the flow rate?

Again, lets calculate the hydraulic resistances. This time the channel actually is cylindrical so no approximation needed. Radius r= 1 mm, L=10cm – 30cm

$$R_H = \frac{8\mu L}{\pi r^4}$$

For the 10cm radius the hydraulic resistance is:

$$R_H = \frac{8\mu L}{\pi r^4} = \frac{8 \cdot 1mPas \cdot 100mm}{\pi (1mm)^4} \approx 0.25 Pas / mm^3$$

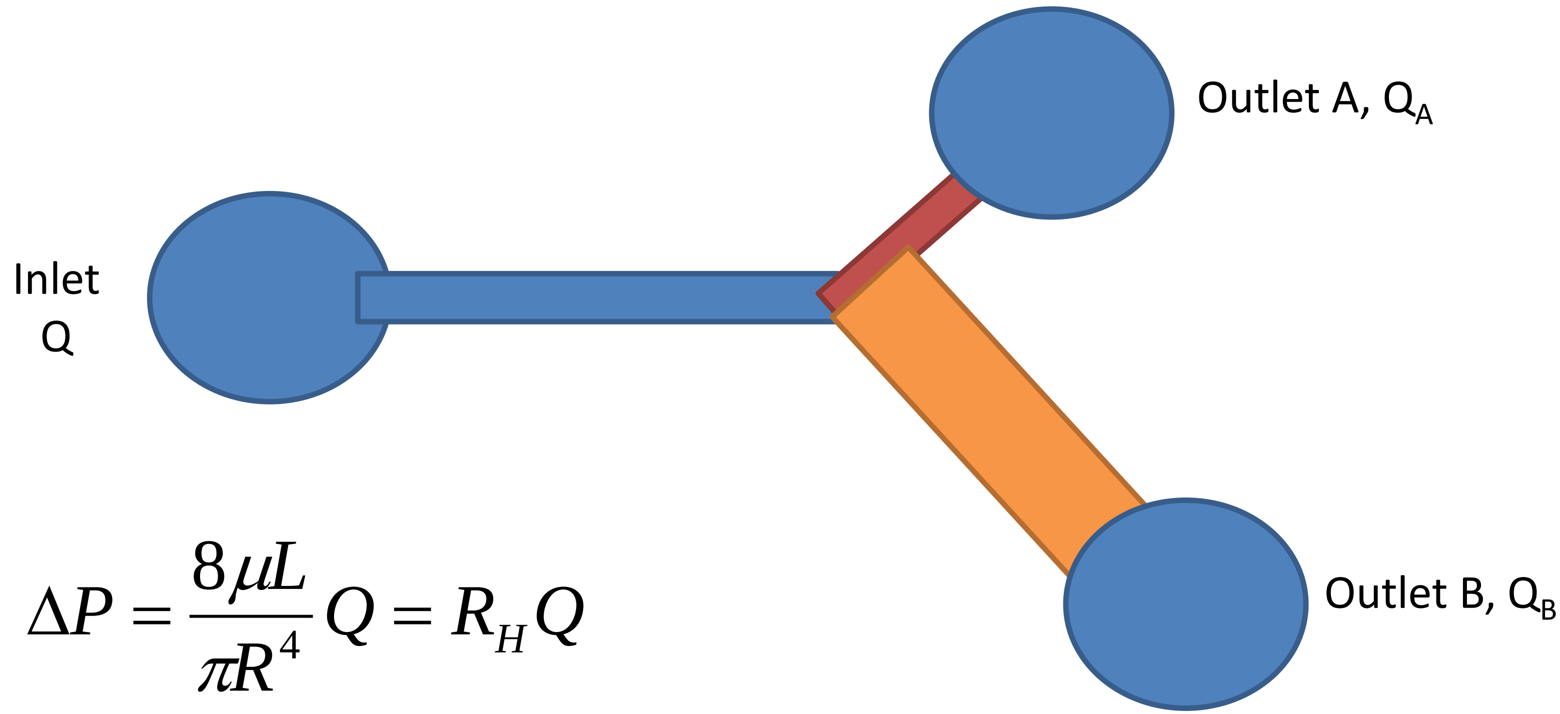
And for the 30cm radius it would be 3 times that, 0.75 Pas $/mm^3$

However, remember that the overall chip hydraulic resistance was 430 000 Pas $/mm^3$, so the effect of the tubing to the total resistance is negligible.

Inlets do cause a lot of disturbance in other ways since their volumes are much higher than that of the chip. So gravity can affect the tubing even if it does not affect the chip. Or any bubbles or disturbances can have dramatic effects inside the chip.

2. A microfluidic circuit and scaling

Liquid with volumetric flow rate Q is pumped with a syringe pump to Y shaped channel. After the shared channel, the channel splits into two. Channel A is has dimensions width w, height h and length L. Channel B has dimensions width 2w, height 2h and length 2L. Which fraction of the fluid flow goes to outlet A and outlet B respectively?



$$\Delta P = \frac{8\mu L}{\pi R^4} Q = R_H Q$$

Orange channel: all dimensions are doubled compared to the red channel.

Relative flow resistances:

For channel A and B, lets write:

$$R_{HA} = \frac{8\mu L}{\pi r^4} \qquad R_{HB} = \frac{8\mu(2L)}{\pi(2r)^4} = \frac{2}{2^4} R_{HA} = \frac{1}{8} R_{HA}$$

Thus: hydraulic resistance of channel A is 8 times higher than the hydraulic resistance of channel B.

Since the pressure across both channels is the same, we can write:

$$\Delta P = R_{HA} Q_A = R_{HB} Q_B \quad \text{, from which:} \quad Q_B = \frac{R_{HA}}{R_{HB}} Q_A = \frac{R_{HA}}{1/8 R_{HA}} Q_A = 8 Q_A$$

Since Q_A+Q_B=Q, we get that Q_B =8/9*Q and Q_B =1/9*Q

Key learning point, scaling of hydraulic resistance. Channel B has only 1/8 of the hydraulic resistance

3. Capillary filling

A microfluidic channel has a rectangular cross section of 20 µm x 20 µm and the water contact angle is 20°.

a) What is the bond number (when the channel is in a horizontal orientation)?

$$Bo = \frac{\quad}{\quad}$$

L = characteristic length scale
ρ = density (water 1000kg/m^3)
γ = surface tension (water 72mN/m)
a = acceleration (here, g=9.81m/s^2)

If you approximately remember the formula, you can usually get it exactly right by thinking about what parameters affect the phenomenon, and whether they favor X or Y (here gravity force or surface tension forces).

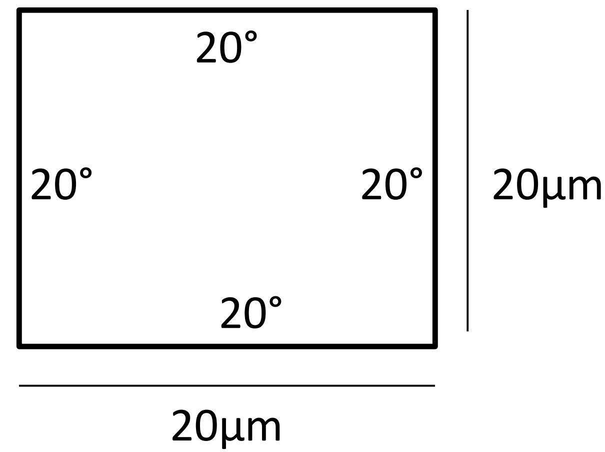
You need to remember some details, like the second power of L (unless you outright derive the Bond number but this was more of a lighter heuristic route to remembering dimensionless numbers)

$$Bo = \frac{\rho a L^2}{\gamma} = \frac{1000 kg/m^3 \cdot 9.81 m/s^2 \cdot (20 \mu m)^2}{72 mN/m} \approx 5 \cdot 10^{-5}$$

<1 so capillary forces are much more significant than gravity.

b) What is the capillary pressure?

γ = surface tension (water 72mN/m)



$$\Delta P = -\gamma_{lv}\left(\frac{\cos\theta_t}{h} + \frac{\cos\theta_b}{h} + \frac{\cos\theta_l}{w} + \frac{\cos\theta_r}{w}\right)$$

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$$\Delta P = -72mN/m \cdot \frac{4\cos 20°}{20\mu m}$$

$$\Delta P = -13.4kPa$$

c) How high would the liquid rise if the channel was turned into vertical orientation. (1 mm of water is about 10 Pa)

$$\Delta P_{capillary} = -\Delta P_{hydrostatic} = -\rho g h$$

$$h = \frac{13.4kPa}{1000kg/m^3 \cdot 9.81m/s^2}$$

$$h \approx 1.37m$$

$$Pa = \frac{N}{m^2} = \frac{kgm/s^2}{m^2}$$

Rule of thumb: 1mm of water is ≈ 10 Pa

The dimensions in the exercise are not unlike the dimensions of xylem in trees that pull water from the roots to the tops of trees.
Clearly though the xylem at the top of the tree have to be smaller than 20 µm though.
(and there are also other components for the water transport in trees)

Bond number and channel orientation:

$$Bo = \frac{Volume\ forces}{Surface\ forces} = \frac{Hydrostatic\ pressure}{Capillary\ pressure} = \frac{\rho a x}{\gamma/R}$$

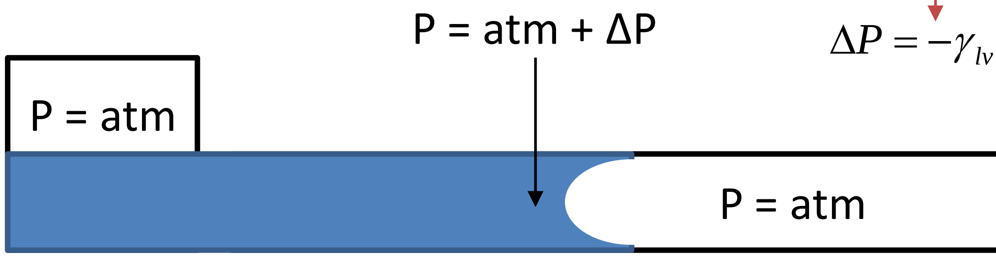
Bond number in its usual form is sensible when the characteristic length scale for hydrostatic pressure x (height of liquid column) is the same as the characteristic length scale for capillary pressure R (radius of curvature of the liquid).

On a horizontal orientation, both are close to the height of the microchannel. So $x \approx R \approx L$

On a vertical orientation, capillary length scale remains the same but the length scale for the hydrostatic pressure is now the length of the chip instead.

Capillary pressure sign conventions:

"Physicist point of view":



$$\Delta P = -\gamma_{lv}\left(\frac{\cos\theta_t}{h} + \frac{\cos\theta_b}{h} + \frac{\cos\theta_l}{w} + \frac{\cos\theta_r}{w}\right)$$

"Microfluidic engineer point of view":

$$\Delta P = \gamma_{lv}\left(\frac{\cos\theta_t}{h} + \frac{\cos\theta_b}{h} + \frac{\cos\theta_l}{w} + \frac{\cos\theta_r}{w}\right)$$

The physicist view is technically correct, but the engineer view is perhaps more easy to understand since it equates a capillary pressure to a pressure applied at the inlet by a pump.

Either way, the pressure gradient in the channel is the same.