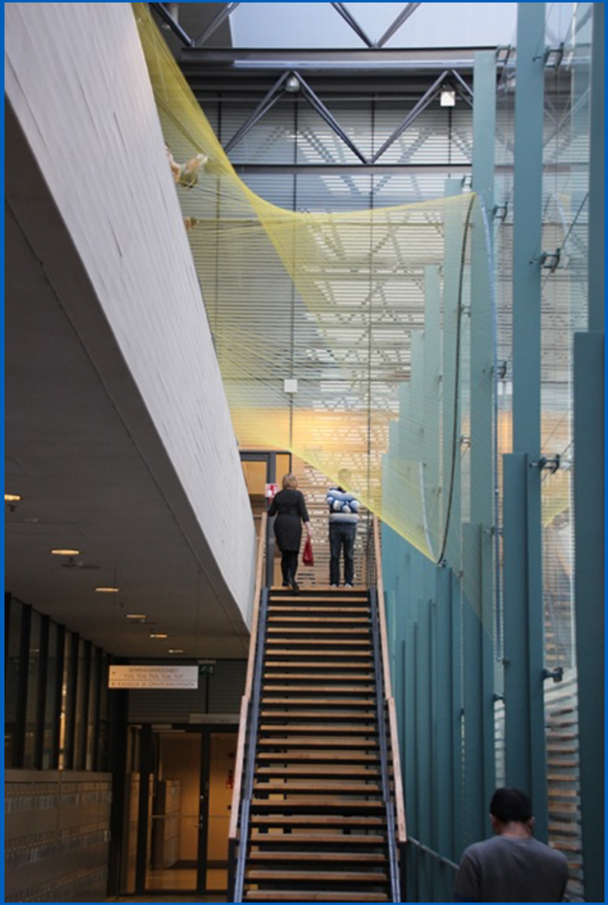
Aalto University

Curvature through ruled surfaces

Kirsi Peltonen,
Crystal Flowers, 21.1.2021



Program schedule for Jan 21st

15:15 Basic examples of ruled surface

- Basic properties
- Use in architecture
- Developable surfaces

16:00 Break

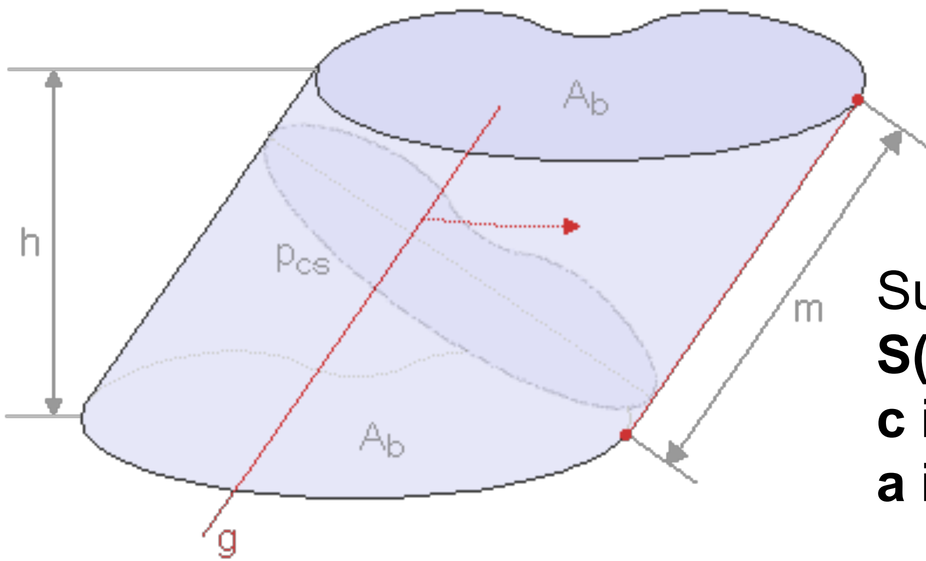
16:15 What is curvature ?

- principal curvatures
- Gaussian curvature
- Mean curvature

17:00 Break

17:15 Marco: What is 'Concept' in the context of an art exhibition?

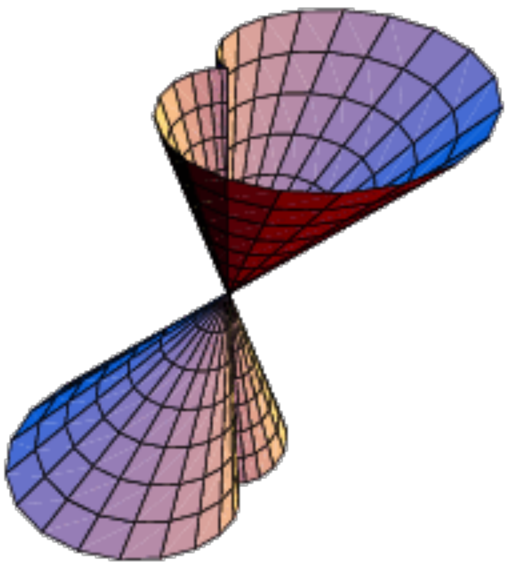
Generalized cylinders



Surface parametrization
S(u,v)=c(u)+va
c is a space curve (need not be closed)
a is a fixed vector (ruler)

Note: Can be rolled from a flat piece of paper

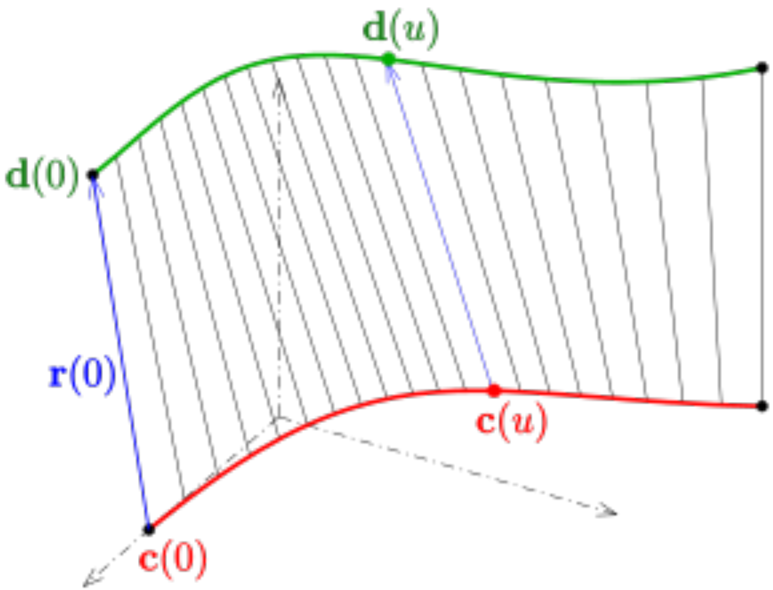
Generalized cones



Surface parametrization
$S(u,v)=p + vd(u)$
p fixed point (tip of the cone)
d a space curve
Rulers: $S(.,v)=p + vd(.)$

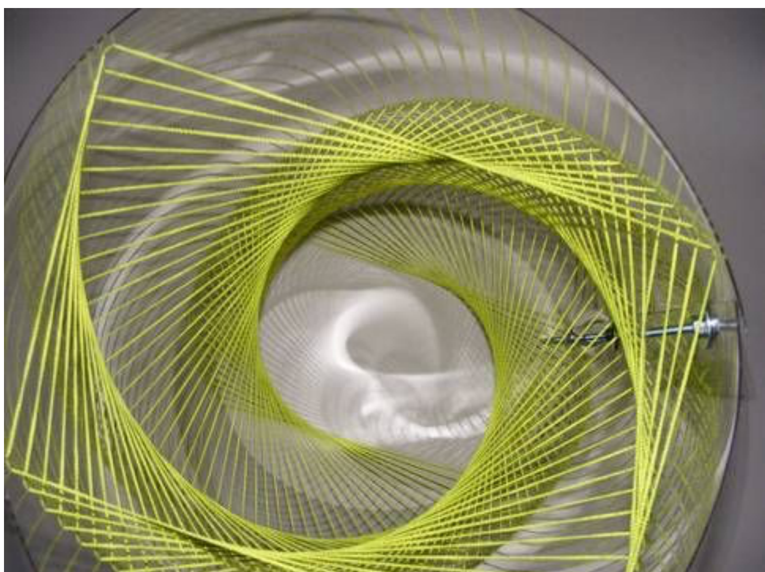
Note: Can be rolled from a flat piece of paper

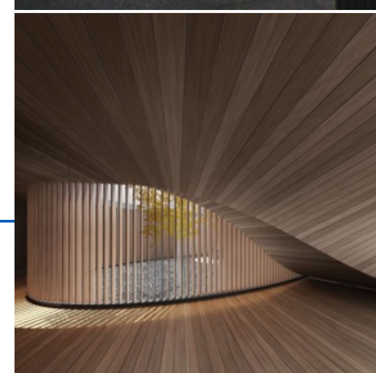
General ruled surface



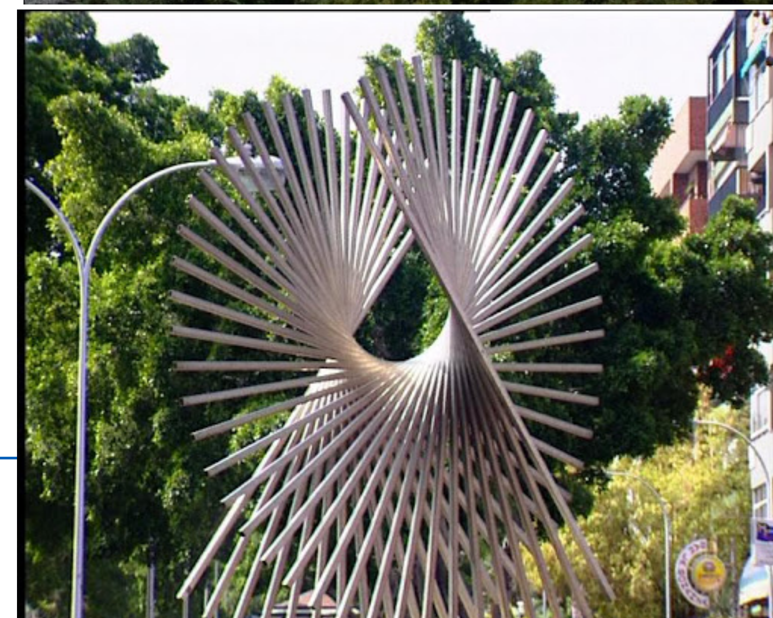
S(u,v)=c(u)+vd(u)

- c,d space curves
- Ruler S(.,v)=c(.)+vd(.)
- For generalized cone d(u)=a constant
- For generalized cylinder c(u)=p constant





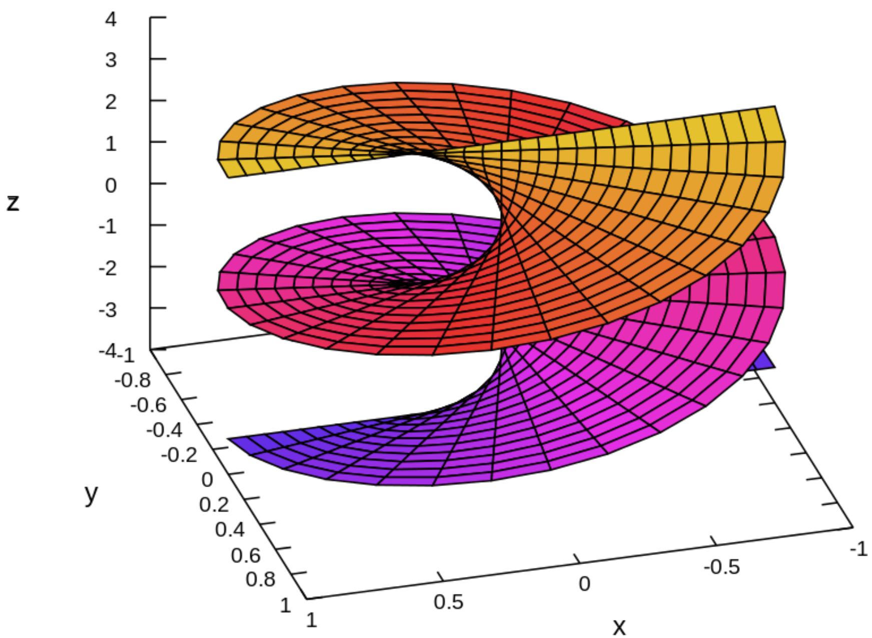

COURTESY HENNING LARSEN ARCHITECTS



Charles O. Perry, Andréu Alfaro,…

Helicoid

Euler 1774, Jean Babtiste Meusnier 1776



S(u,v)=(vcos u, vsin u, ku)
=(0,0,ku) + v(cos u, sin u, 0)

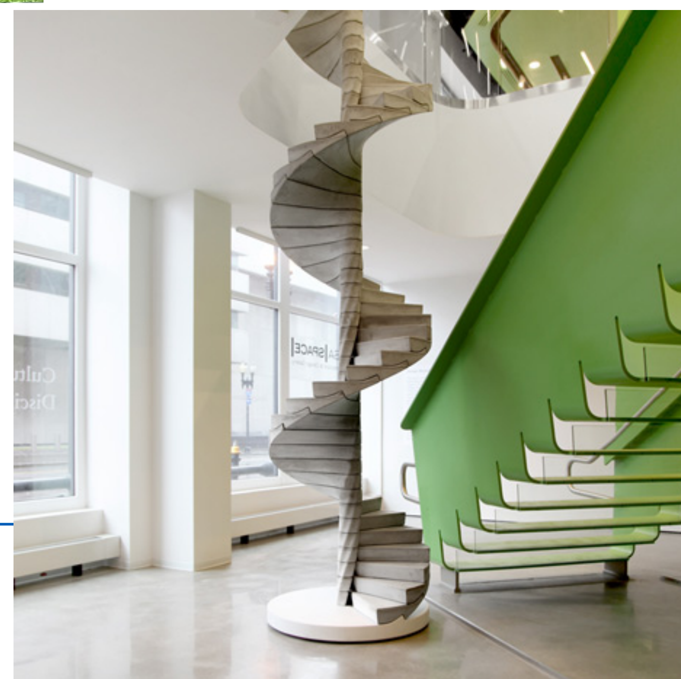
- c(u)= (0,0,ku)
- d(u)=(cos u, sin u, 0)

Fixed v parametrizes a helix

Also a minimal surface !

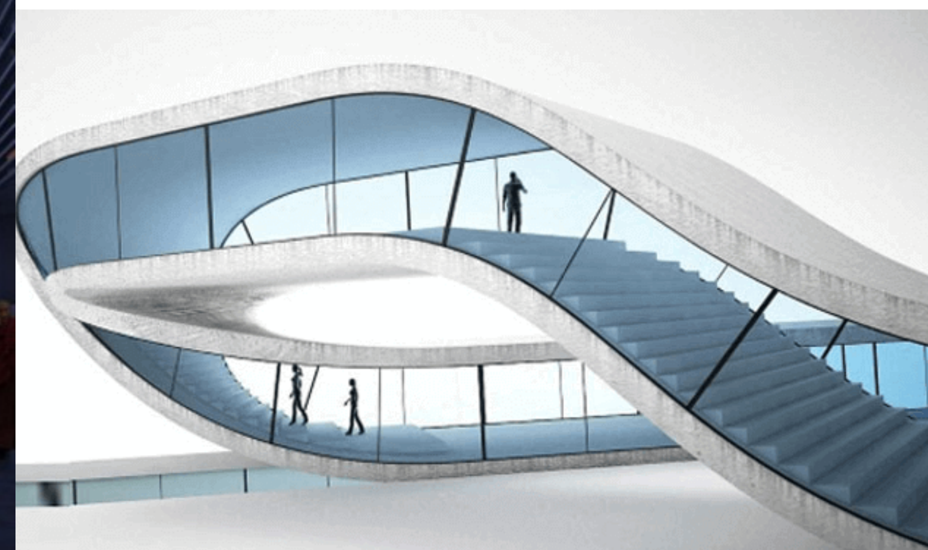


KANDINSKY

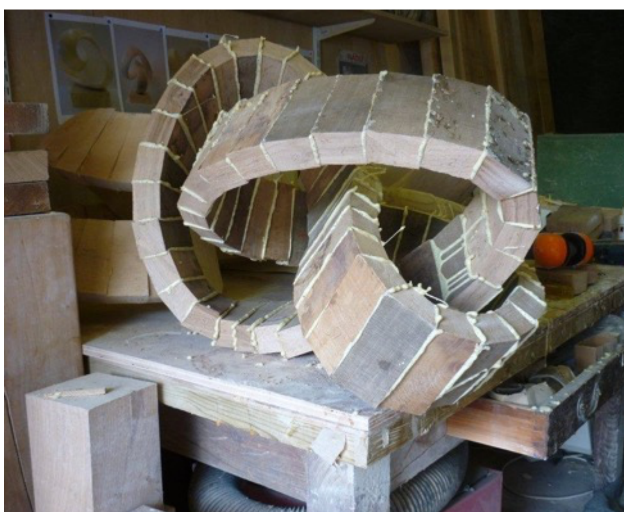



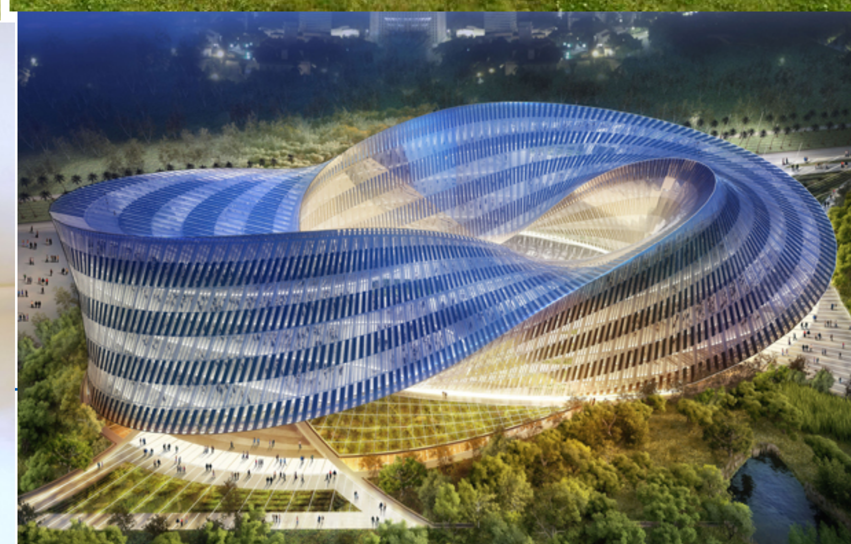

Möbius strip

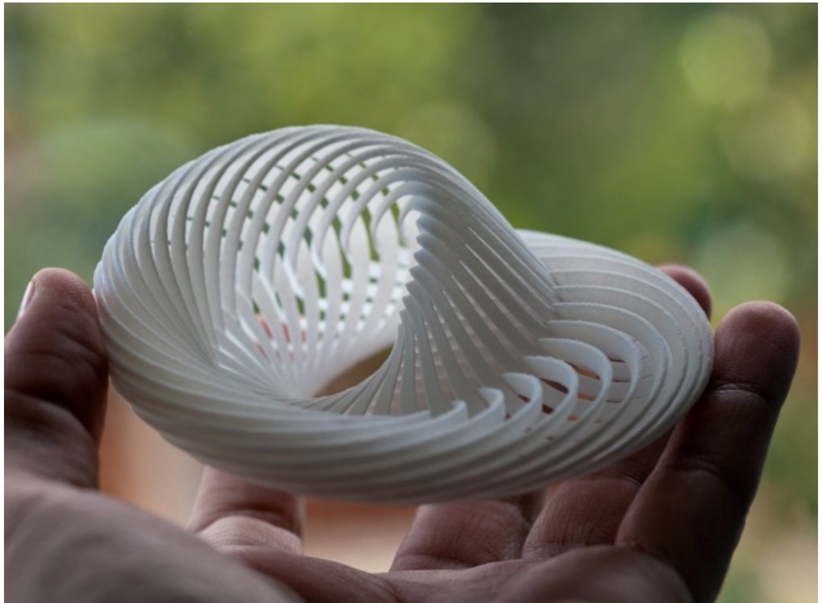
- Möbius, Listing 1858
- Roman mosaics 200-220 CE
- Nonorientable
- **S(u,v)=((1+vcos u/2)cos u,**
  (1+vcos u/2)sin u, v sin u/2)
  =(cos u, sin u, 0) +
  v(cos u/2 cos u, cos u/2 sin u, sin u/2)
- **c(u)=(cos u, sin u, 0)**
- **d(u)=(cos u/2 cos u, cos u/2 sin u, sin u/2)**











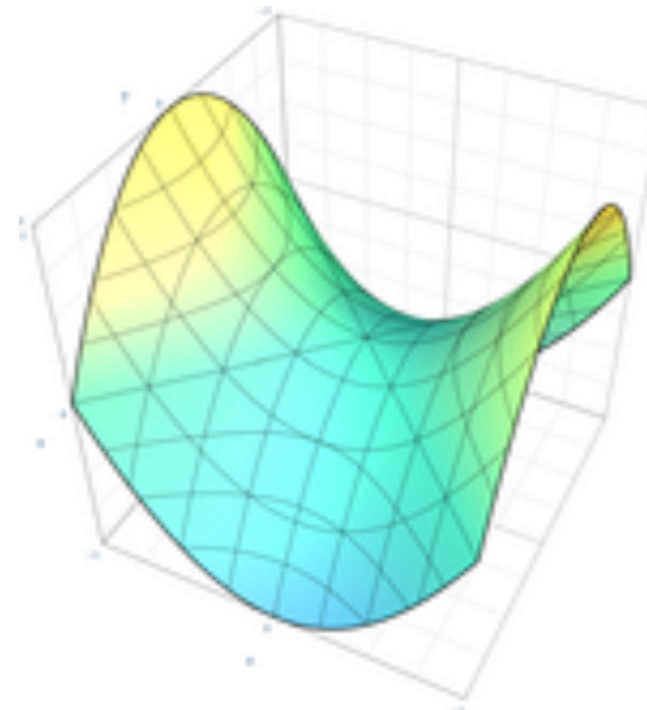

Keizo Ushio


SHALL BECOME

Hyperbolic paraboloid

$z = (x/a)^2 - (y/b)^2$ (a quadric)

Where are the rulings ?

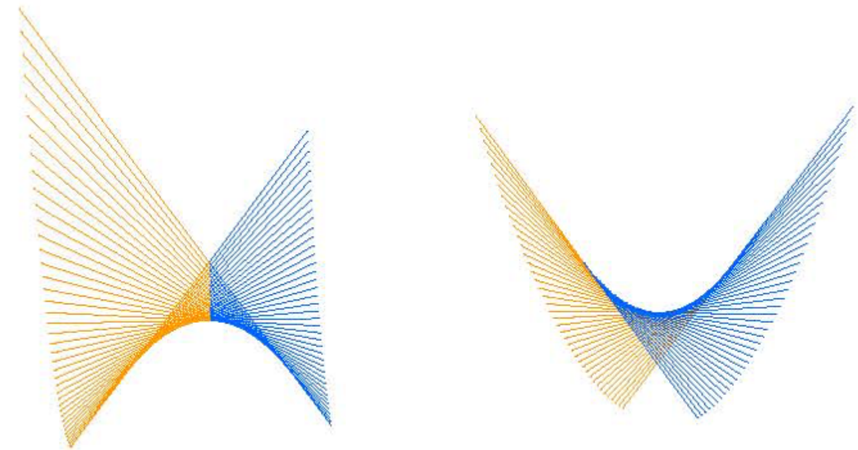


In fact it is doubly ruled (two one-parameter families of lines)

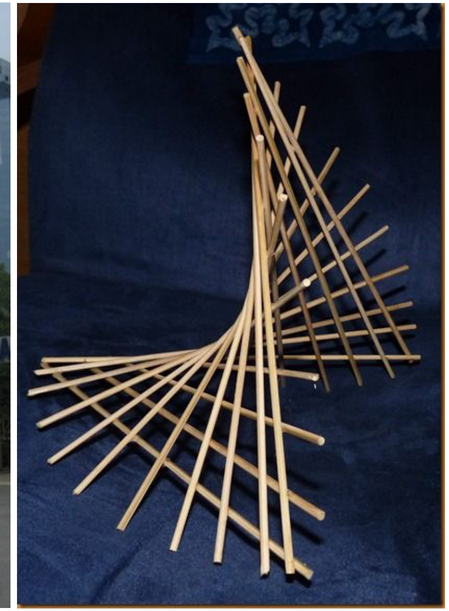
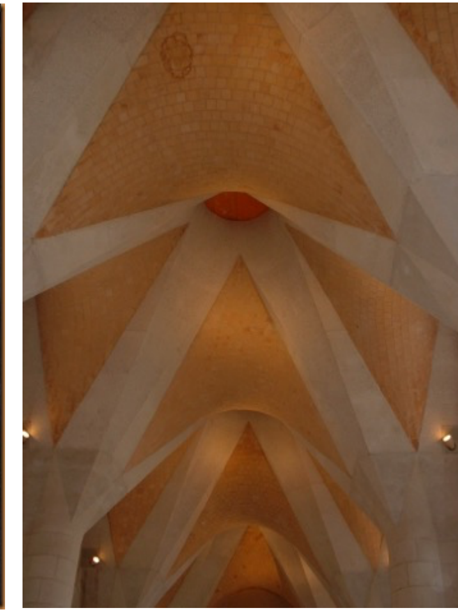
Hyperbolic paraboloid as a ruled surface

$$S^{\pm}(u,v) = (a(u+v), \pm bv, u^2+2uv)$$
$$= (au,0,u^2)+v(a, \pm b,2u)$$

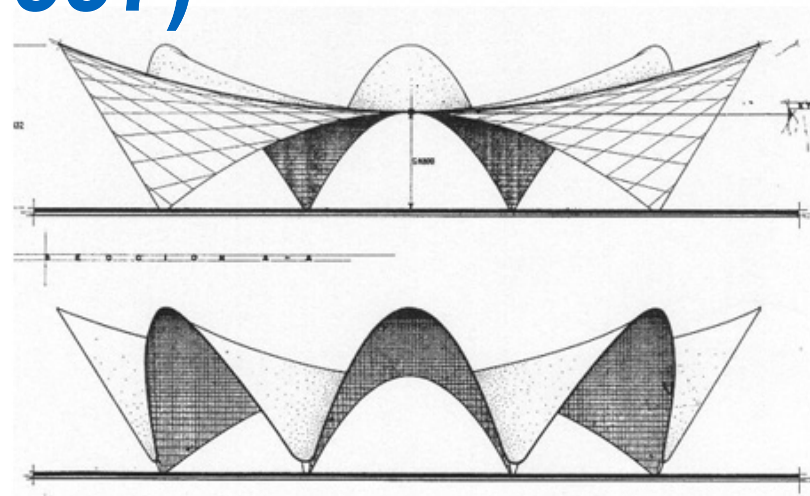
- $c(u) = (au,0,u^2)$
- $d(u) = (a, \pm b,2u)$

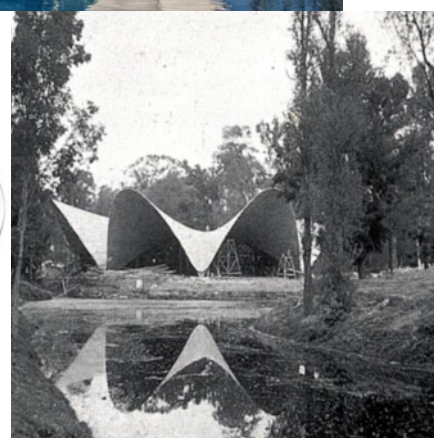


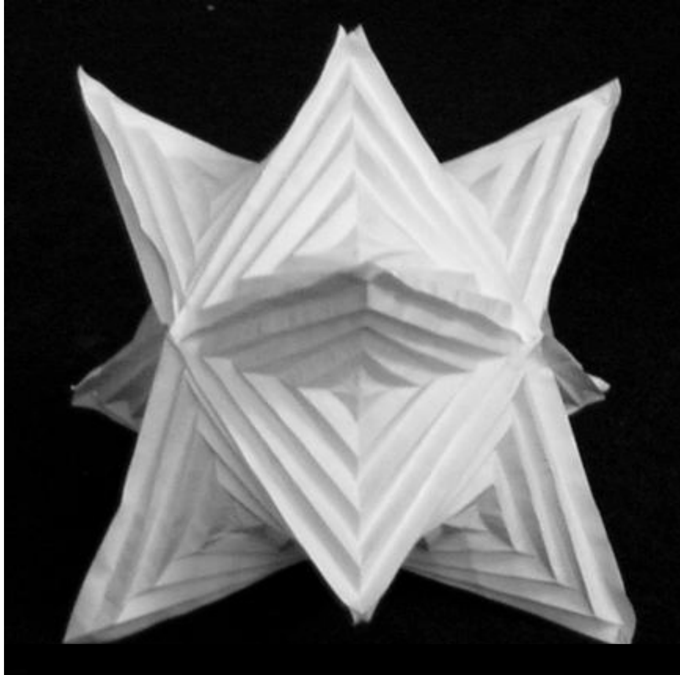
George Francis, Illinois Feb 2020

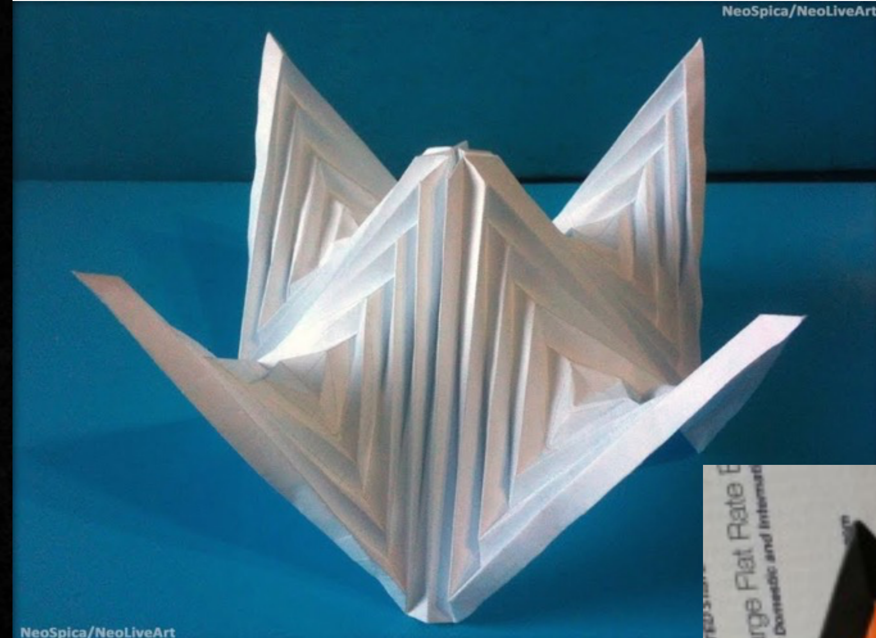

WKD




Félix Candela (1910-1997)








NeoSpica/NeoLiveArt
NeoSpica/NeoLiveArt


Large Flat Rate
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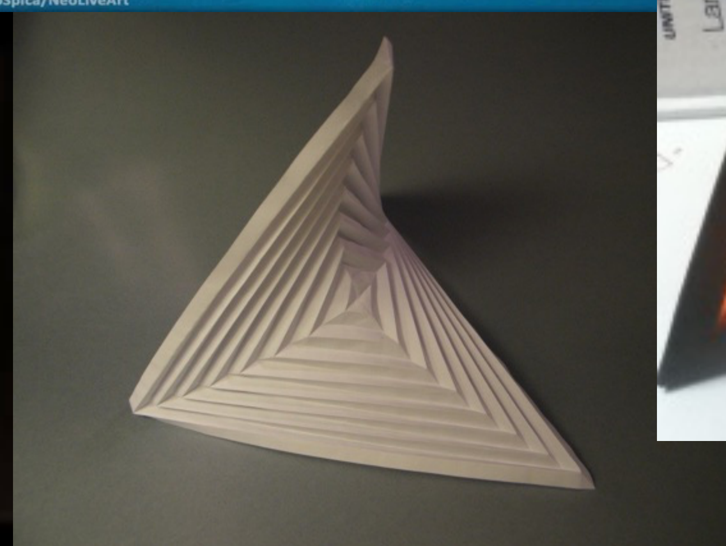
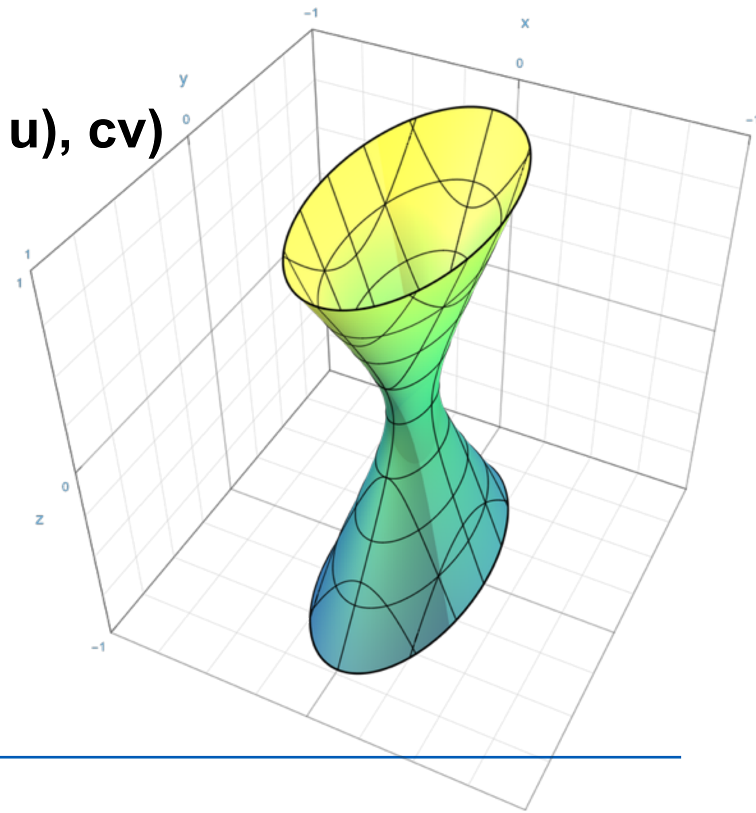


Hyperboloid of one sheet

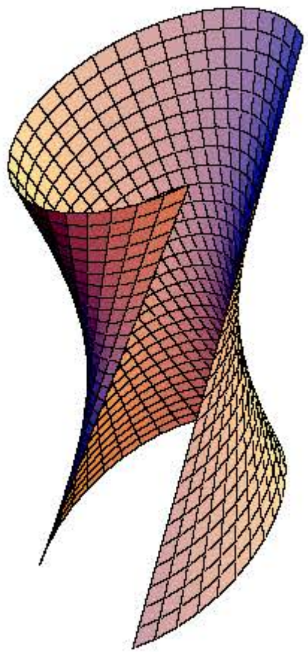
$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$ (a quadric)

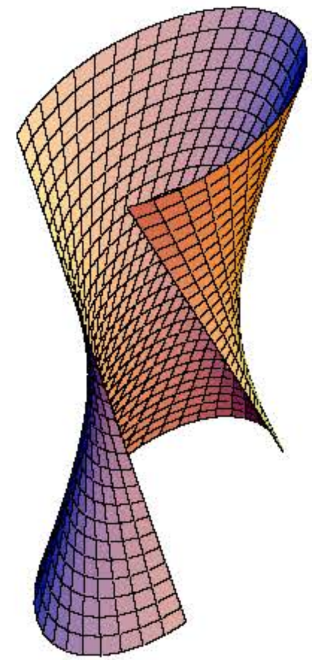
$S^{\pm}(u,v) = (a(\cos u \mp v \sin u), b(\sin u \pm v \cos u), cv)$
$= (a \cos u, b \sin u, 0) +$
$v(\mp a \sin u, \pm b \cos u, c)$

- $c(u) = (a \cos u, b \sin u, 0)$
- $d(u) = (\mp a \sin u, \pm b \cos u, c)$



Double rulings of a hyperboloid





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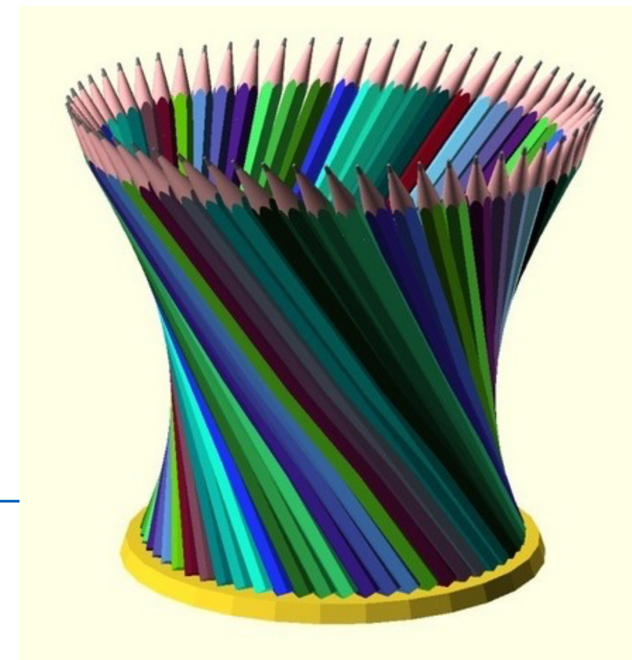
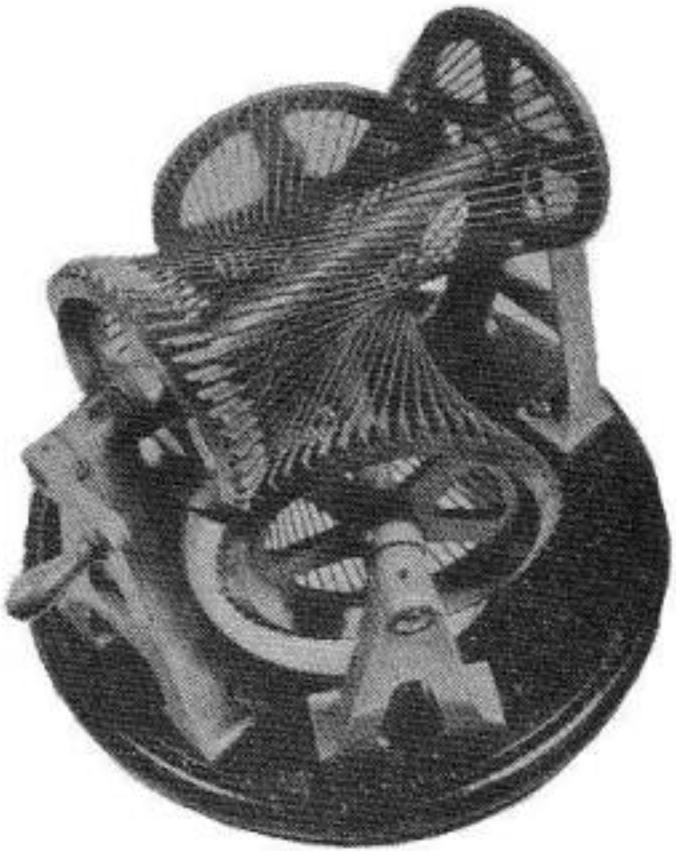
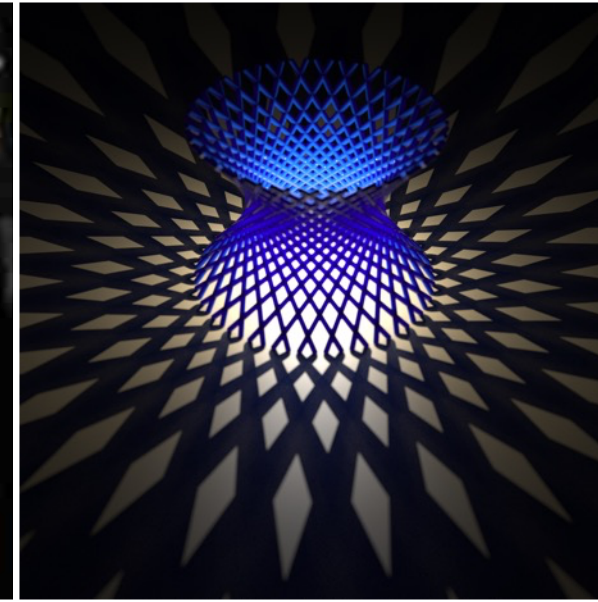
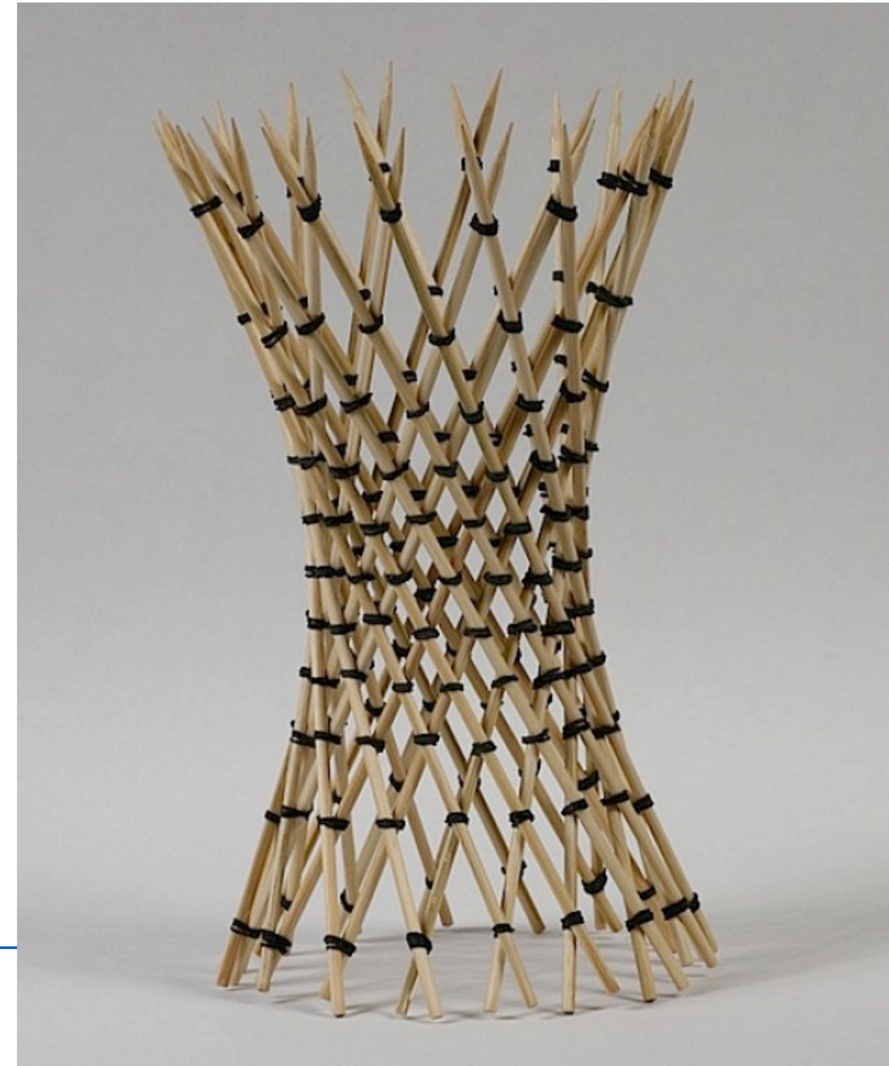




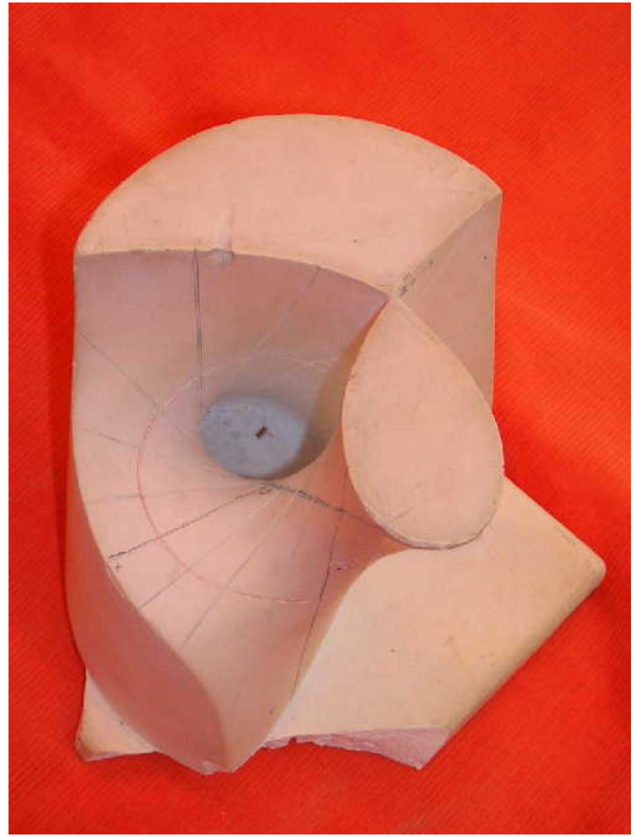


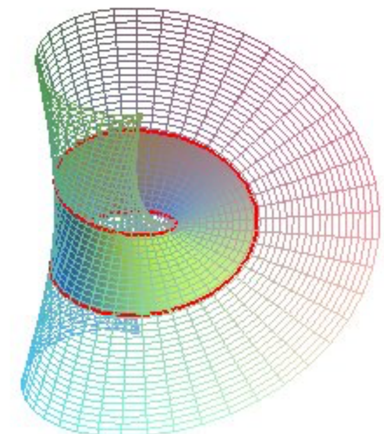

NeoSpica
NeoSpica



A ‘quartic scroll’ by R. Baker

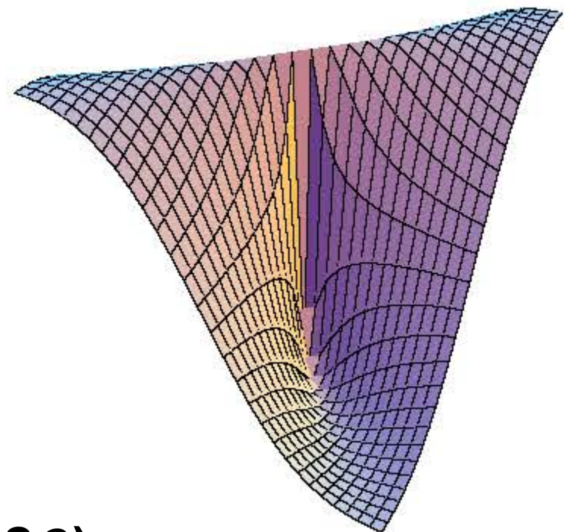
A ruled cubic (Schilling VII, nr 20)

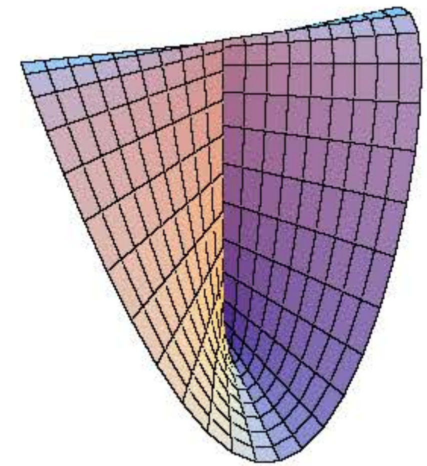




Möbius surface?

Plücker conoid with 2 folds





$z = 2xy/(x^2+y^2)$

In polar coordinates (r, θ)

$x = r\cos\theta$, $y = r\sin\theta$

$$\mathbf{S}(r, \theta) = (r\cos\theta,\ r\sin\theta,\ \sin 2\theta)$$
$$= (0, 0, \sin 2\theta) + r(\cos\theta, \sin\theta, 0)$$

$$\mathbf{c}(\theta) = (0, 0, \sin 2\theta)$$

$$\mathbf{d}(\theta) = (\cos\theta, \sin\theta, 0)$$

https://en.wikipedia.org/wiki/Pl%C3%BCcker%27s_conoid
http://www.josleys.com/htmlgalleries/curvanims/anim118.mov

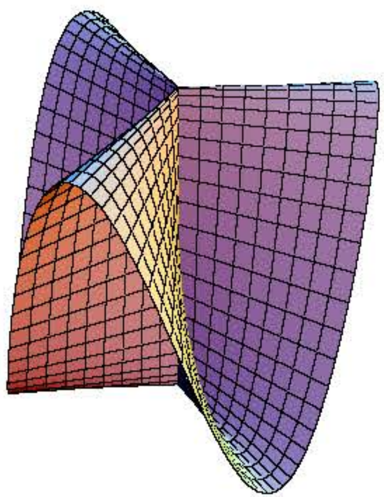
Plücker conoid with n folds

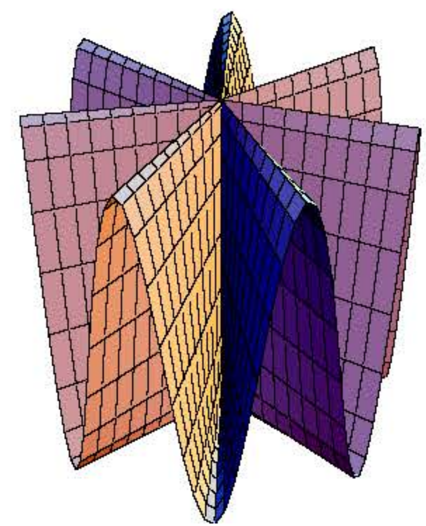
$S(r, \theta)=(r \cos\theta, r \sin\theta, \sin n\theta)$

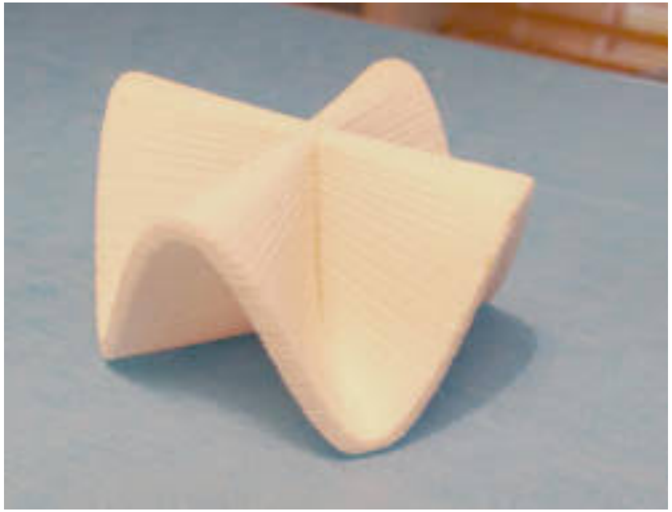
$\quad =(0, 0, \sin n\theta)+r(\cos\theta, \sin\theta, 0)$

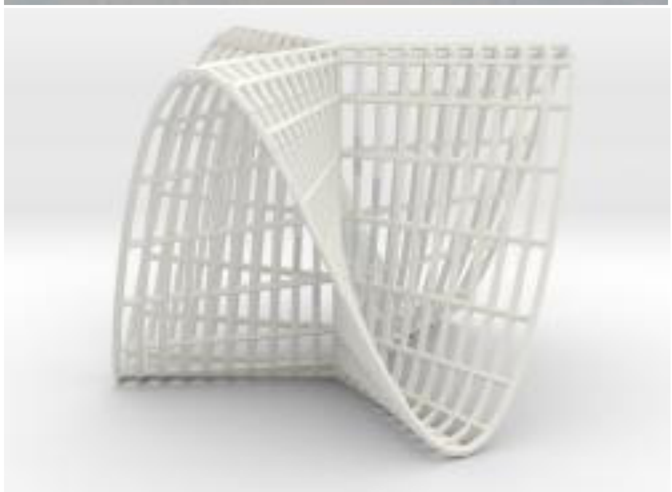
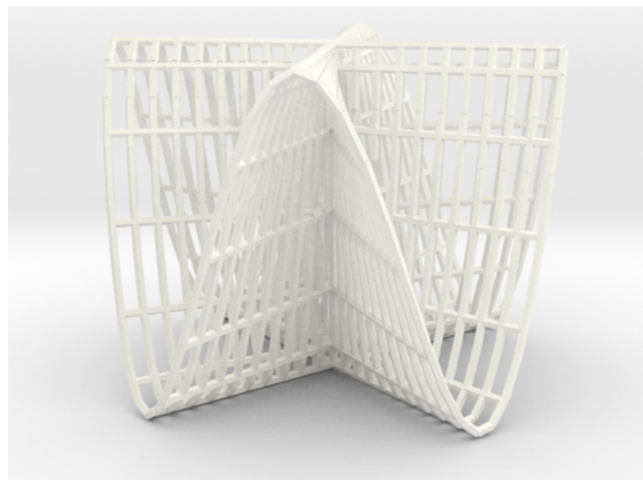
$c(\theta)=(0, 0, \sin n\theta)$

$d(\theta)=(\cos\theta, \sin\theta, 0)$







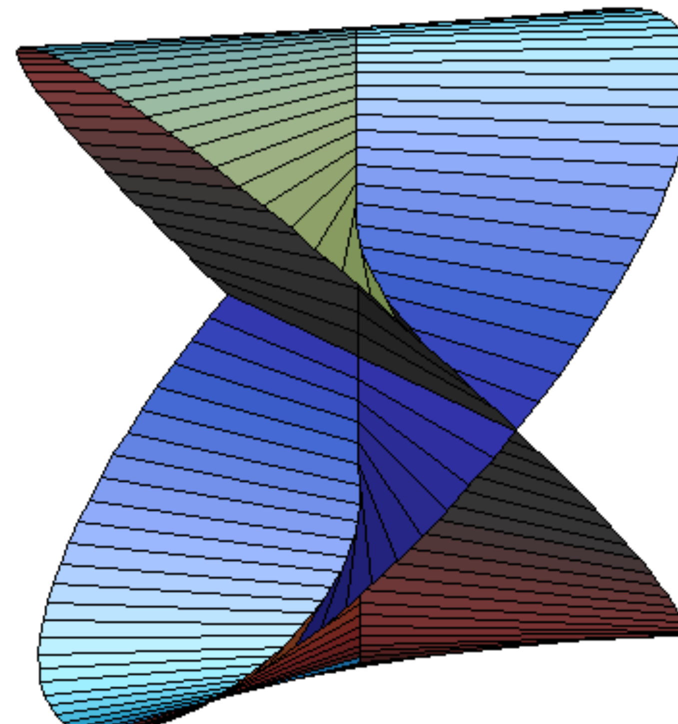



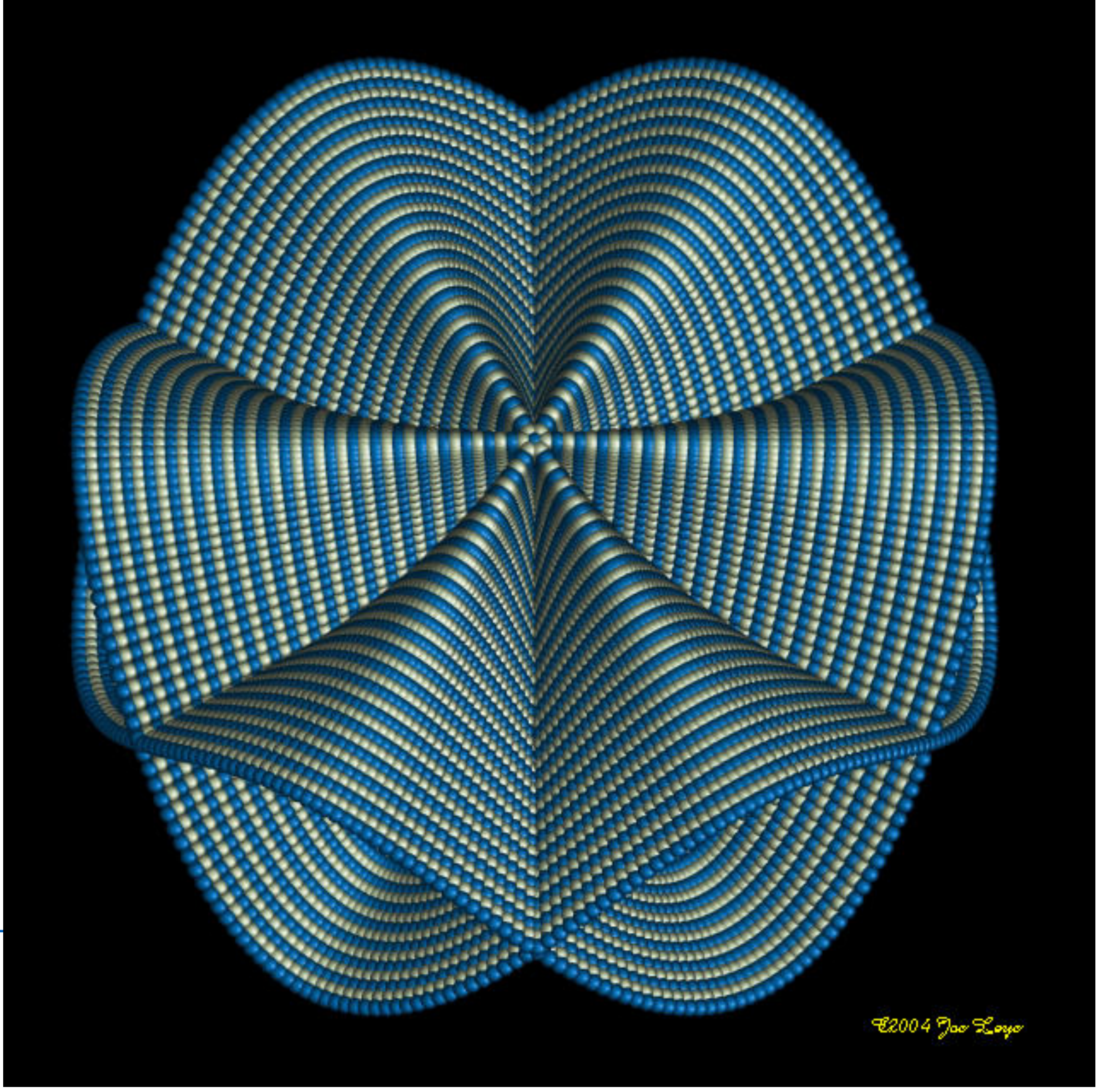
Right conoids

All ruled surfaces with rulings parallel to a plane passing through a line that is perpendicular to the plane.

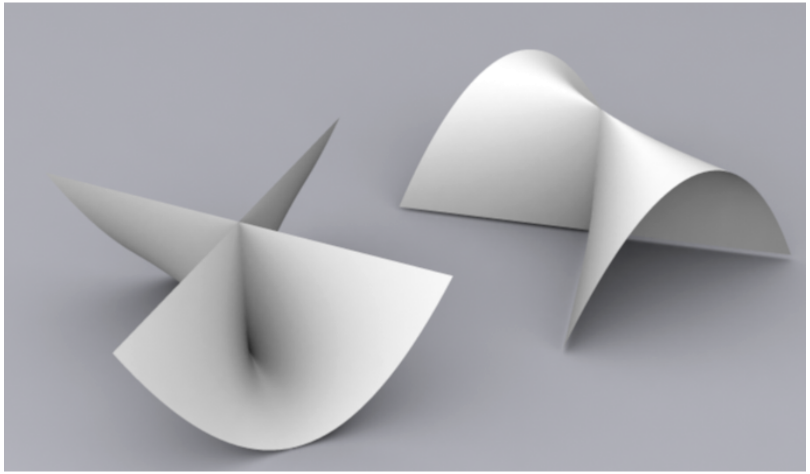
Ex. Take xy-plane and z-axis, then

$$S(u,v) = (v\cos\theta(u), v\sin\theta(u), h(u))$$
$$= (0,0,h(u)) + v(\cos\theta(u), \sin\theta(u), 0)$$





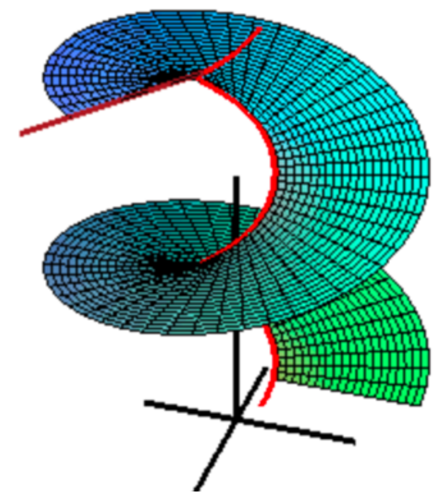
http://www.josleys.com



Catalan ruled surface

S(u,v)=c(u) + v d(u)

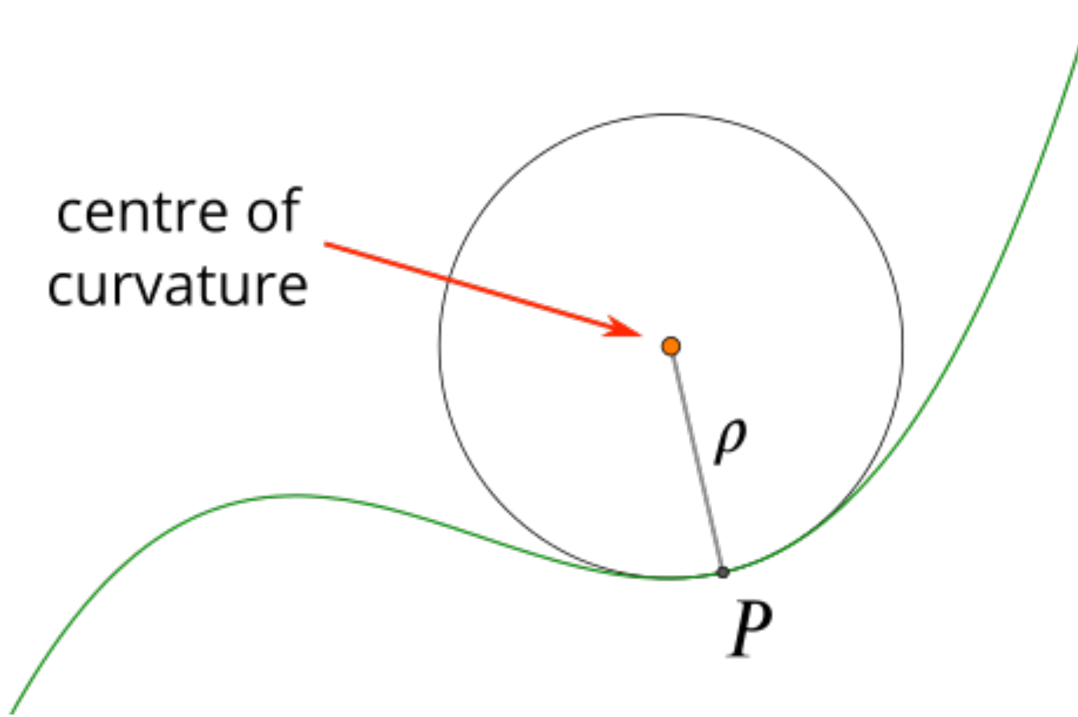
- c(u)= space curve
- d(u)= unit vector of the ruling parallel to a **fixed** plane



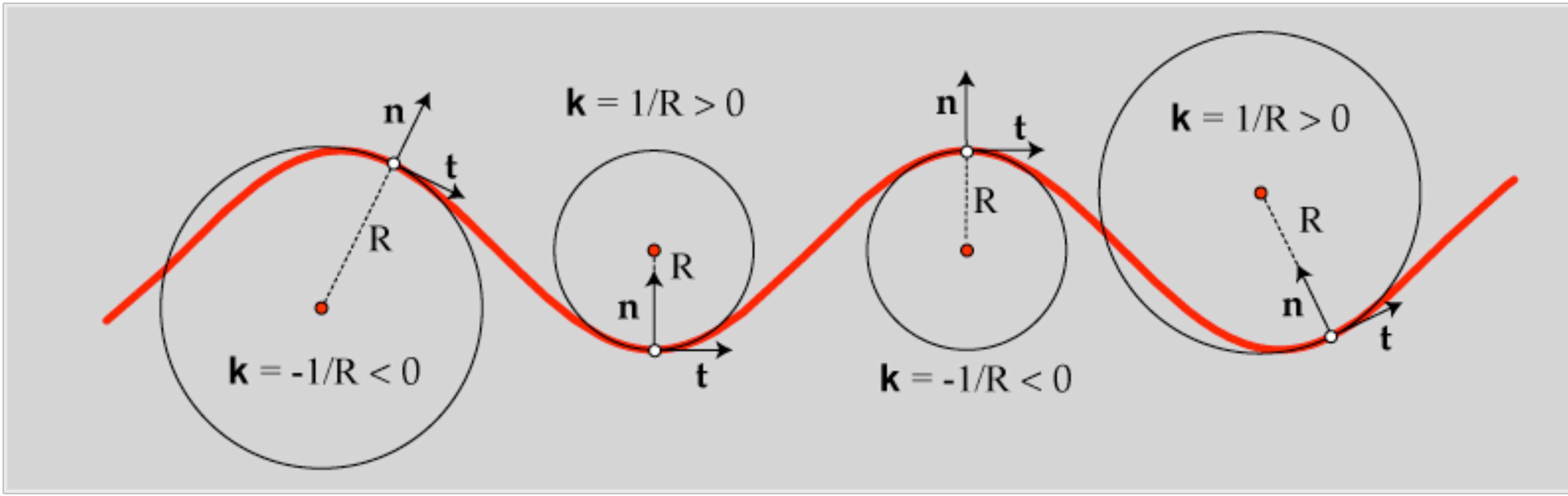
What is curvature ?

Curvature of a smooth planar ***curve*** at point **P** is $\varkappa(P)=1/\boldsymbol{\rho}$

- works also for curves in space or higher dimensions
- points should be approachable with circles
- ***extrinsic*** quantity



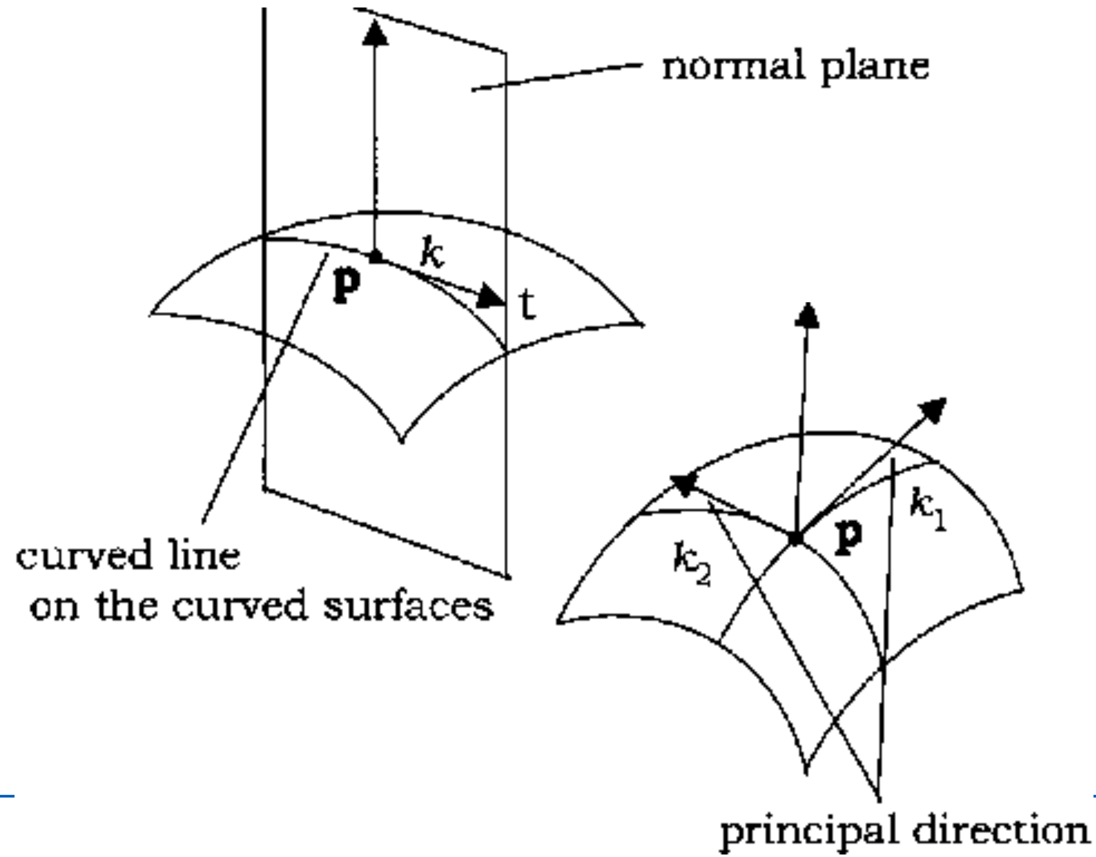
Curvature of a (parametrized planar) curve has a sign



What is curvature of a surface ?

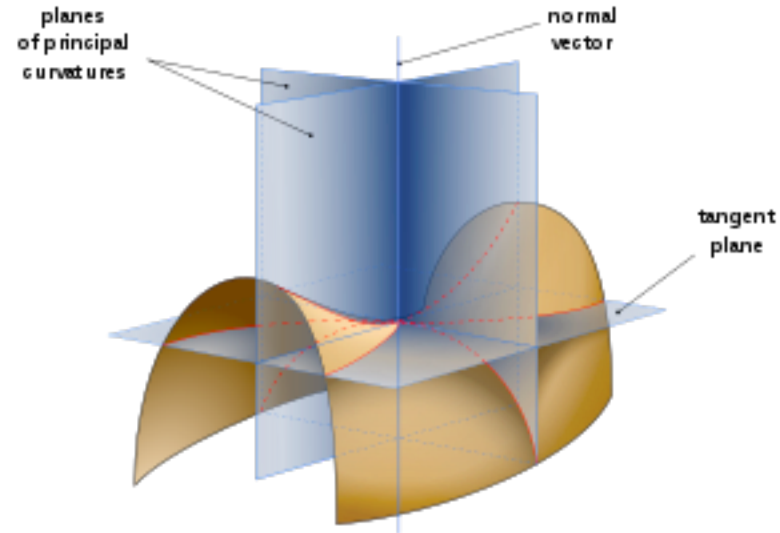
Gauss curvature

$K(p) = \varkappa_1(p)\, \varkappa_2(p)$

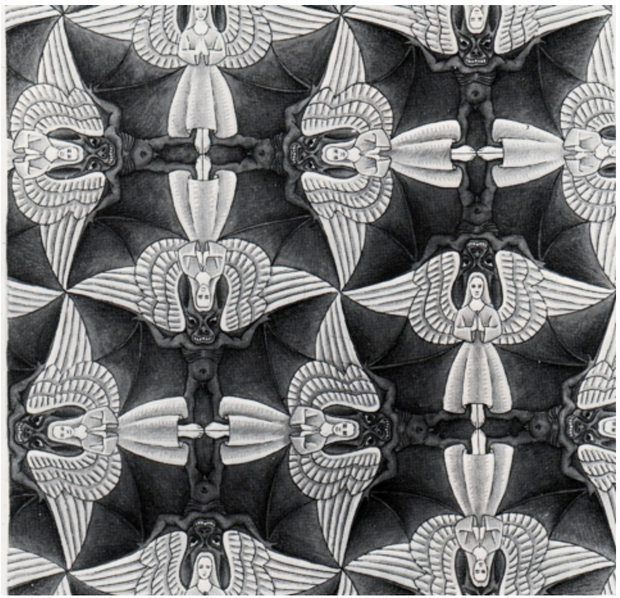


Theorema Egregium (Gauss, 1827)

Curvature K is an *intrinsic* quantity !



Euclidean (=flat), spherical and hyperbolic models of 2D geometry



K = 0

K>0

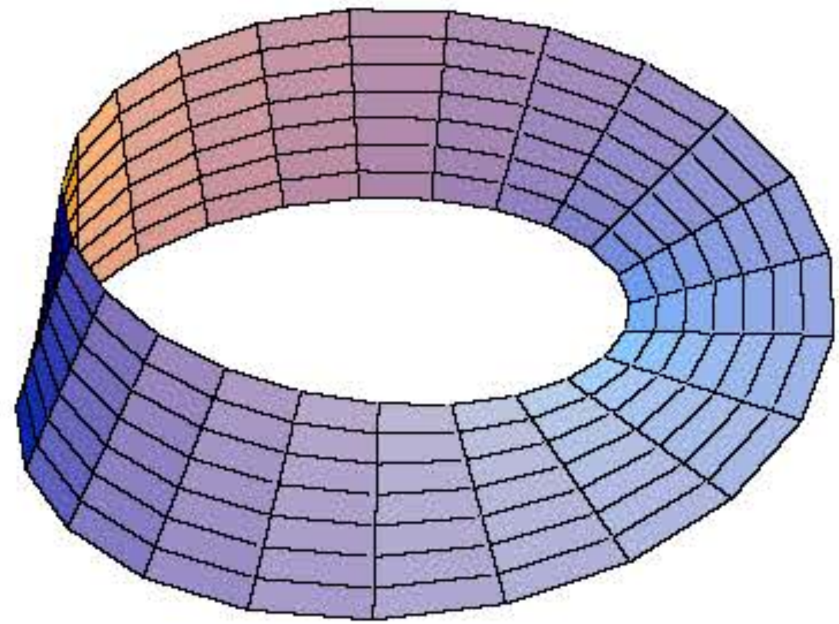
K<0

Gaussian curvature of a ruled surface

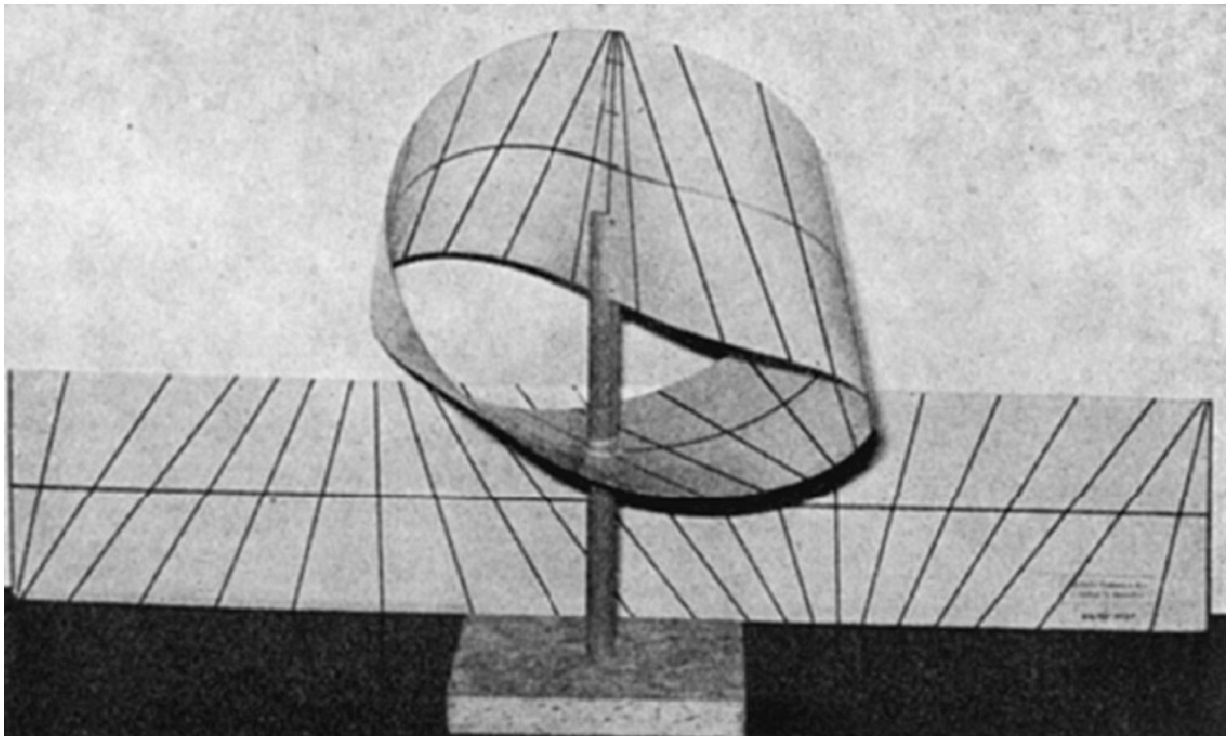
$S(u,v)=c(u)+vd(u)$

$K(u,v)= -(d'(u).\ N)/(EG-F^2) \leq 0$ (!)

Ex: Möbius band, with the parametrization earlier is nowhere flat !
Especially: This parametrization cannot be formed from a flat strip of paper



Ruled Möbius band with K=0



W. WUNDERLICH, 1962

Mean curvature $H(p) = \frac{1}{2}(\varkappa_1(p) + \varkappa_2(p))$

- **Not an intrinsic quantity !**
- **H=0: minimal surfaces**

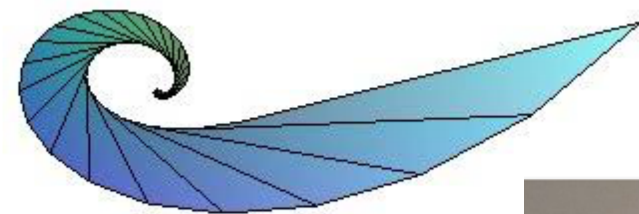
The only minimal ruled surfaces are plane and helicoid

- **Helicoid**
- **Hyperbolic paraboloid**
- **Hyperboloid**
- **Plücker Conoid**
- **Right conoids**

All have (varying) negative curvature !

Are there other flat (K=0) ruled surfaces than plane, generalized cylinders and cones ?

Three classes of flat (K=0) ruled surfaces



= Developable surfaces

- Generalized cones
- Generalized cylinders
- Tangent developables: $S(u,v)=c(u)+vc'(u)$

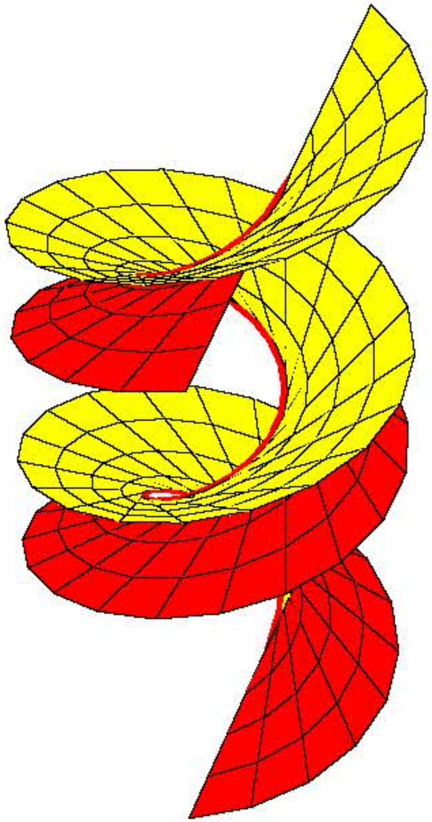
Aristotle (384-322 B.C.):

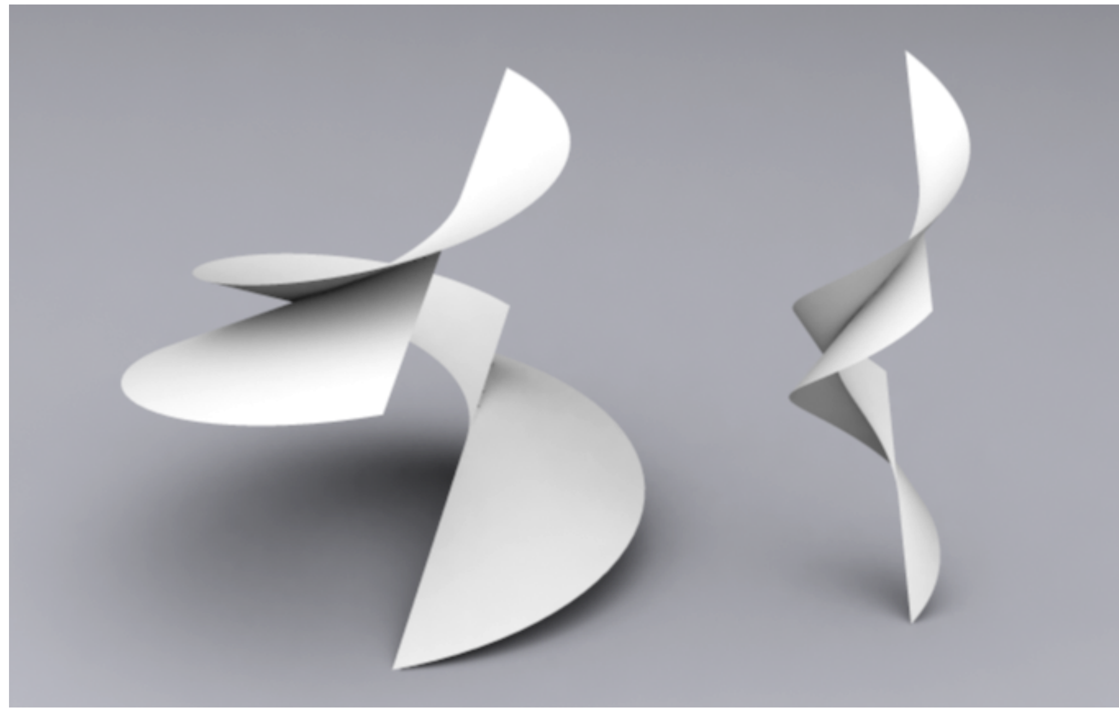
'a line by its motion produces a surface'

Monge (1746-1818): a principle to generate surfaces => seeds to **'descriptive geometry'**

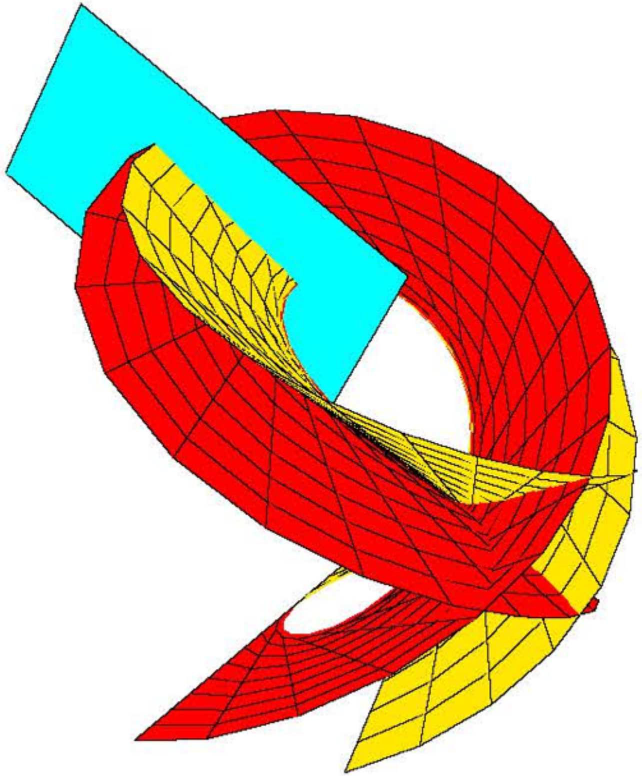
Antoine Pevsner, 1938

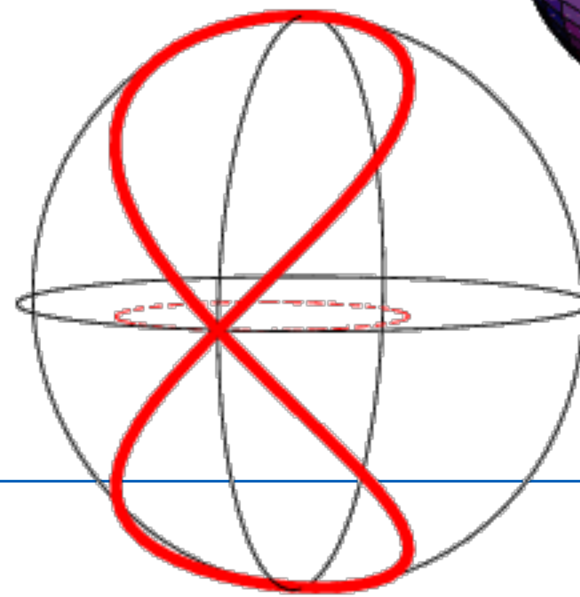
Tangent developable to a circular helix

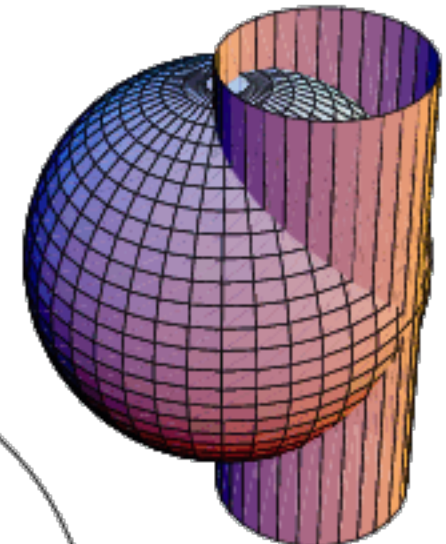




Tangent developable to a Viviani's curve

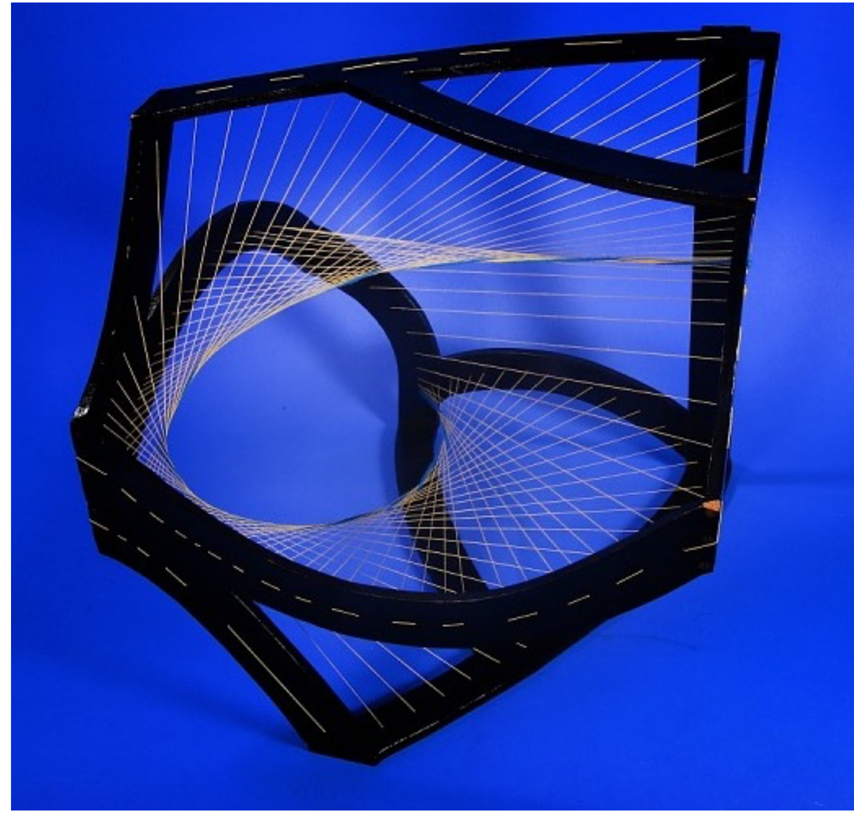






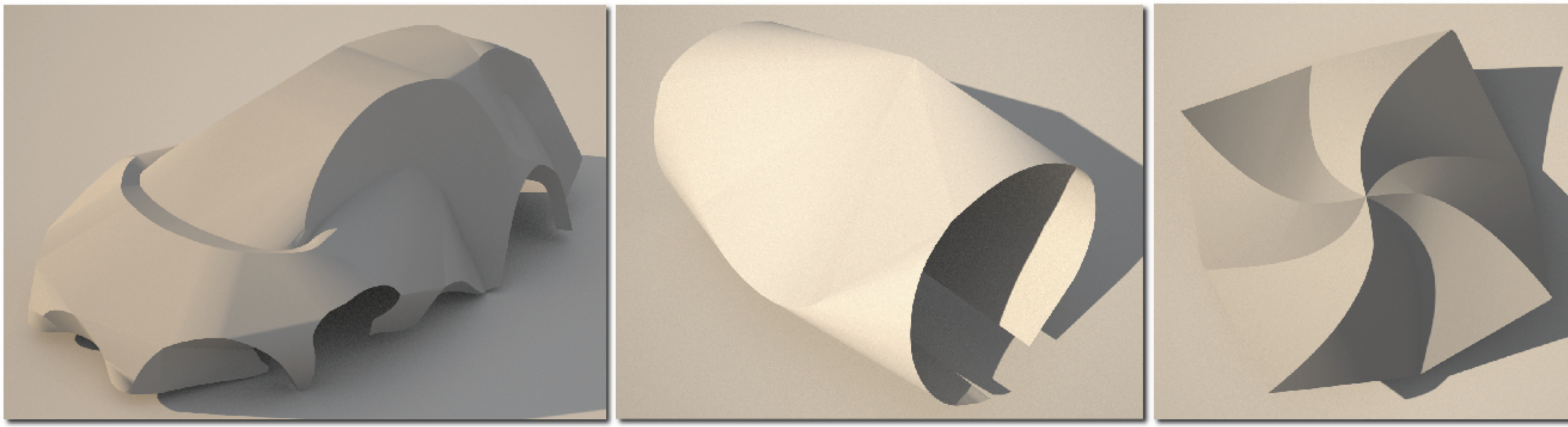
Quartic Surface Developable of a Twisted Cubic

Richard P. Baker



Antoine Pevsner

Folds in tangent developable surfaces provide a bridge to folded objects that have K=0 between the folds……



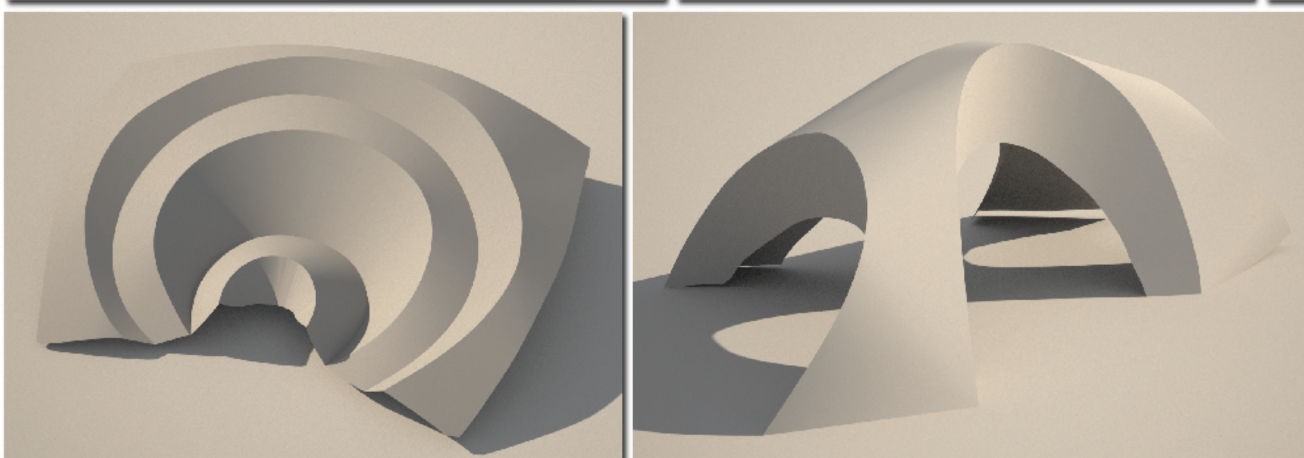


Figure 7: *A gallery of digital paper models. Models were computed with scans of real paper models as reference surfaces. Reconstructed models exhibit curved and straight folds and can be isometrically unfolded into the plane. Several special cases like cone singularities (top row – middle) and converging curved folds (top row – right) are shown.*

Some further history about surfaces that can be developed into plane

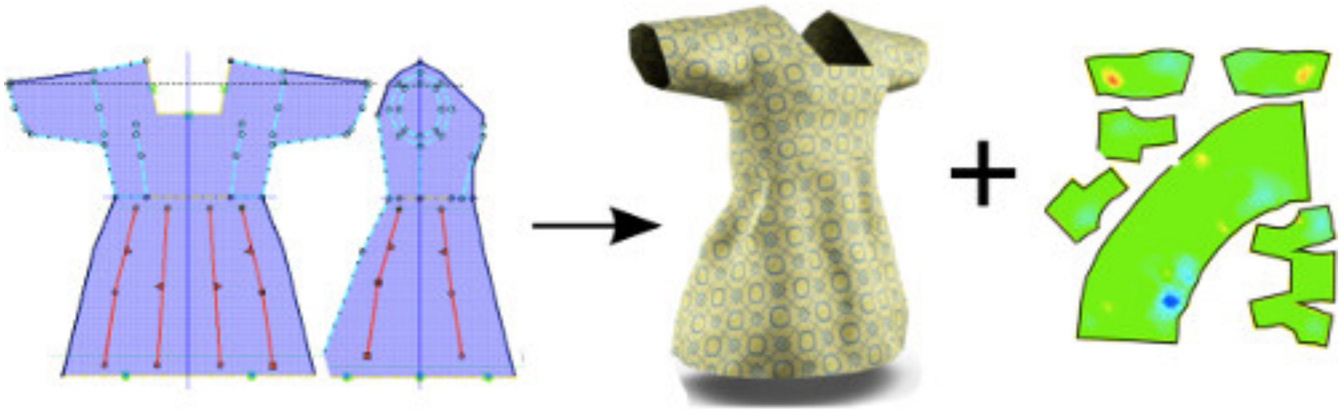
- **William Hawney** (author on surveying): 1717 described the cylinder as a surface ‘rolled over a plane so that all its points are brought into coincidence with the plane’**.**
- **1737 Amédée François Frézier** (1682-1773) also considered the rolling of the plane to form a circular cylinder and cone
- **Euler (1707-1783) & Monge** more systematic treatment of

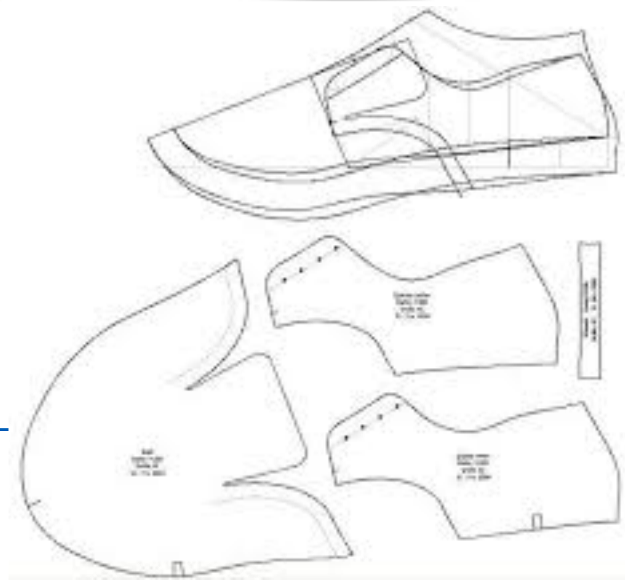
developable surfaces via differential calculus (= ’study of change’) =>

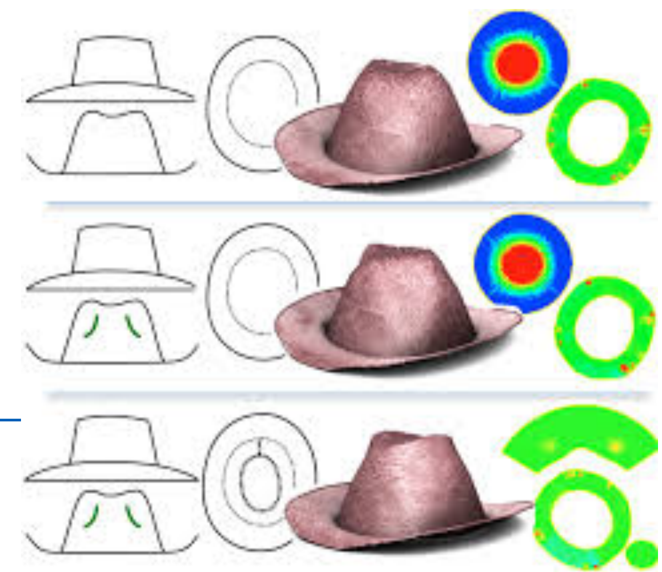
- **1886** term **“differential geometry”** was coined by **Luigi Bianchi**

Developable surfaces in ship building

….Cloth fabrication…..







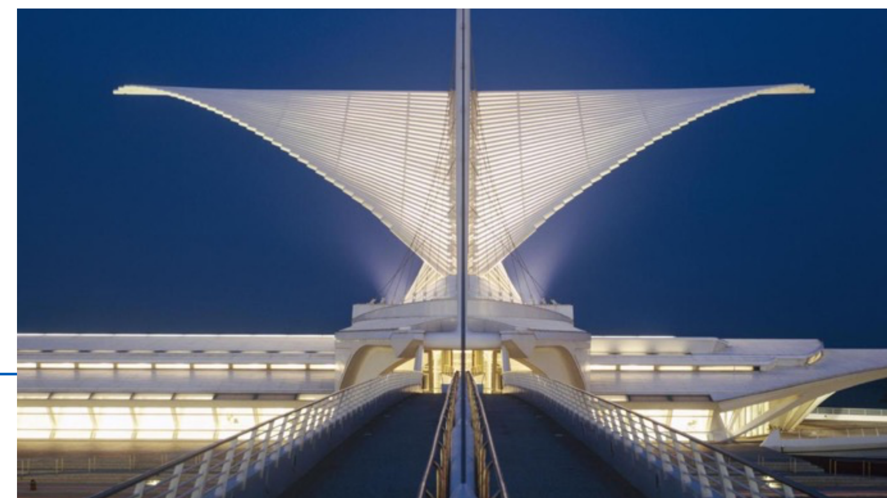
….Gehry architecture

Aalto University

Peter E. Lewis Building, Case Western, Cleveland

….Hans Hollein architecture

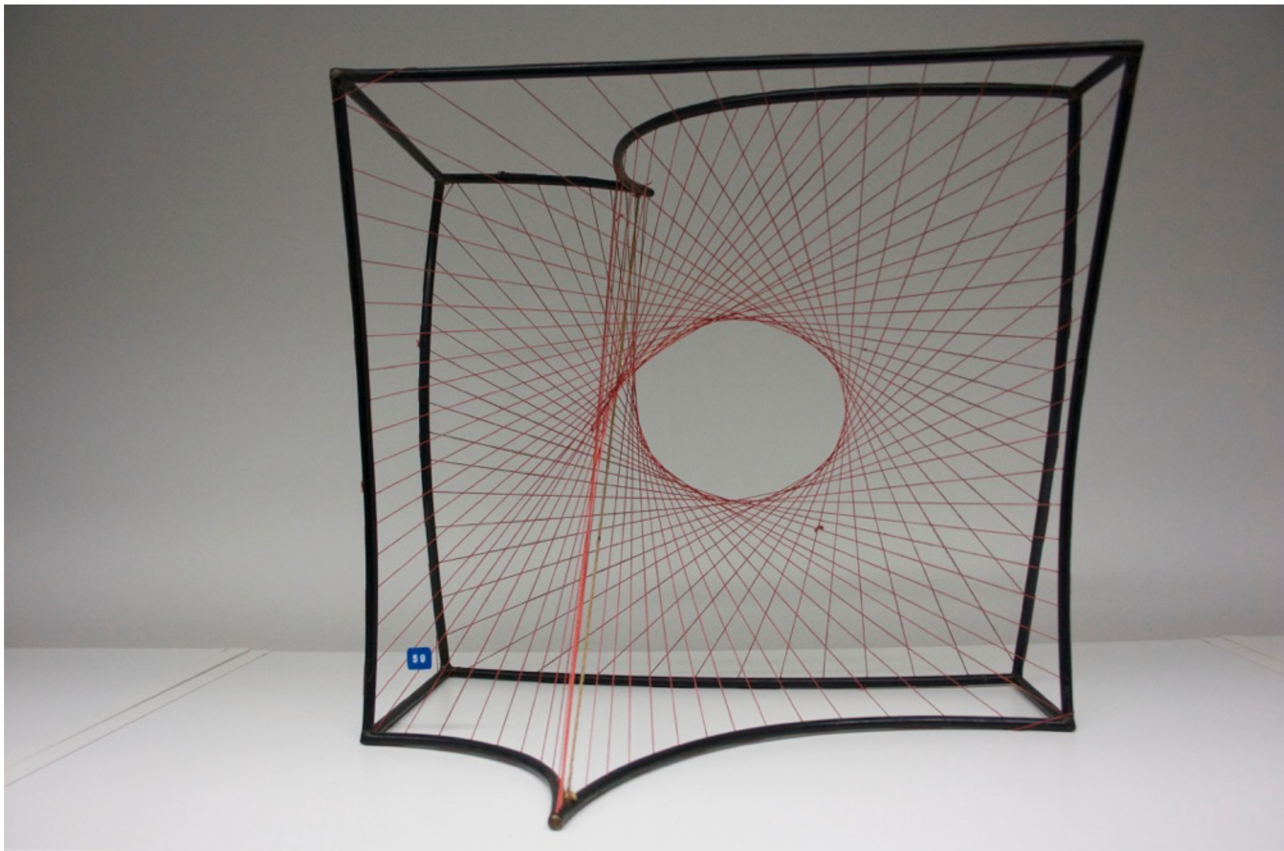
Santiago Calatrava



Some models on display at Math Dept vitrine (Otakaari 1, M wing, 2nd floor)

http://math.aalto.fi/models/

A tangent developable

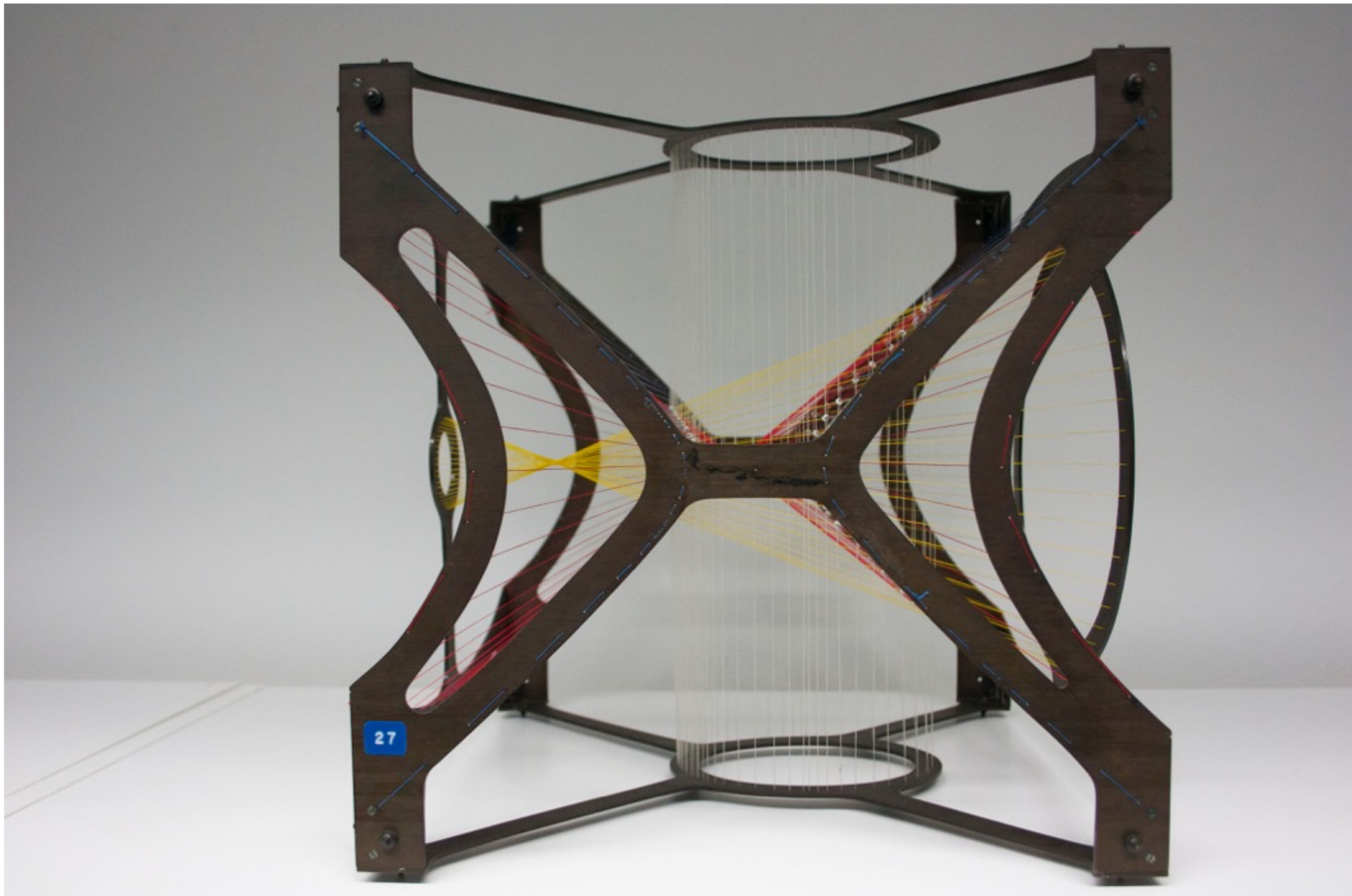


XXVIII,5; 156

A quartic ruled surface

Schilling XIII,9; 114

Quartic curve in the intersection of cylinder and two cones



XII,1; 159

A Ruled generalized helicoid

- Rulers meet the axis of the screw line
- Rulers are not orthogonal to the axis
- Surface intersects itself

Schilling XX,5; 132

Some References

Glaeser & Gruber: *Developable surfaces in contemporary architecture,* Journal of Mathematics and Arts, 2007

Lawrence: *Developable surfaces: Their history and applications,* Journal of Mathematics and Arts, 2010

W. Kühnel: *Differential geometry: Curves-Surfaces-Manifolds,* AMS, 2000