Lecture 4

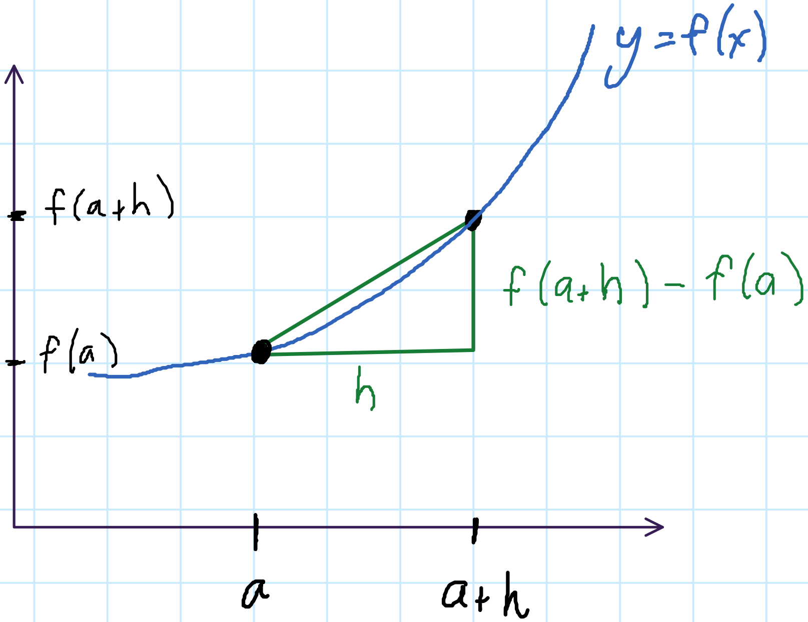
- Defined the partial derivative using the limit definition in analogy with the one-variable case
- Computer some first and second partial derivatives. Stated the theorem that that $f_{xy} = f_{yx}$ is true whenever the 2nd partial derivatives are continuous (slightly stronger theorems are possible)
- Reviewed the chain rule in one variable, stated it in two variables. Gave the intuition for the proof using the limnit definition of the derivative and some algebra.
- Reviewed the general equation of the plane through the point (x_0, y_0, z_0) and with normal $\langle a,b,c \rangle$. It is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$. The example of finding the equation of the plane through 3 given points was given in the first homework assigenemnt.
- Found the equation for the tangent plane to the surface $z = f(x,y)$ at (a,b) in the following way. We looked at the curves on the surface parallel to the x-axis and y-axis. We parametrized these curves. For example the curve parallel to the x-axis is $\langle t, b, f(t,b) \rangle$. Differentiating gives tangent vectors parallel to the surface. The cross product of these tangent vectors produces a normal vector $\langle -f_x(a,b), -f_y(a,b), 1 \rangle$. We remarked that we should remember this method as it will be useful also later (for deducing the surface area formula).
- Computed the tangent plane in a simple example.

Where to find this material

- Adams and Essex 10.4, 12.3 - 12.5
- Corral, 1.5, 2.2 (chain rule not covered) , 2.3
- Guichard, 12.5, 14.3, 14.4, 14.6
- Active Calculus. 9.5, 10.2 - 10.5

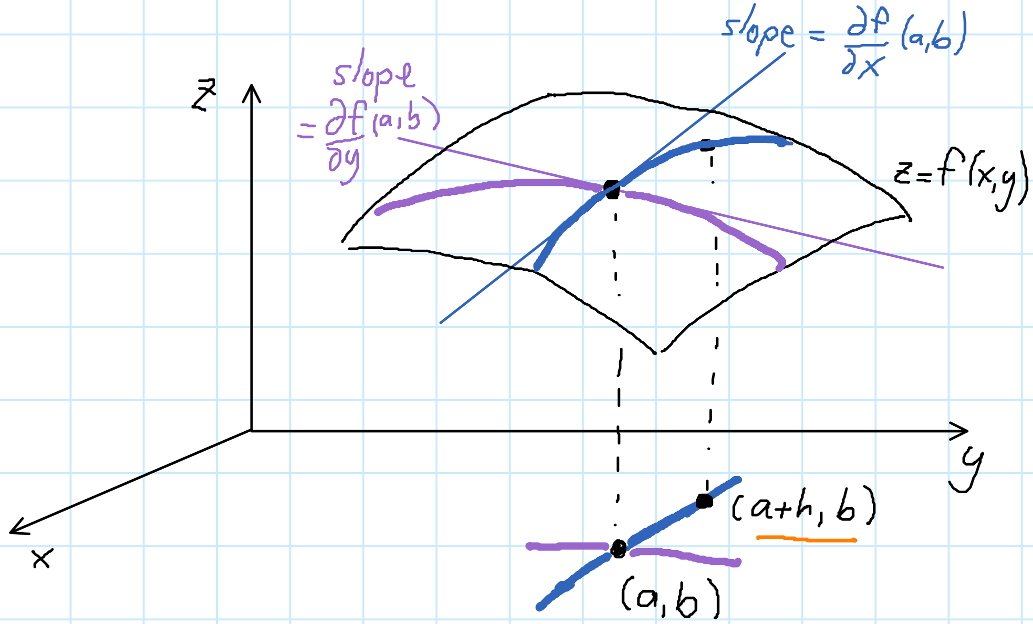
Partial derivatives

Recall the 1 variable derivative defintion.



$$\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition of the partial derviates of a function $f(x, y)$



Slope in the x-direction

$$= \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

Slope in the y-direction

$$= \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Partial derivatives (2)

How to compute the partial derivates?

The good news is that this is easy.

- To compute $\frac{\partial f}{\partial x}$ we think of y as fixed and differentiate as usual in with respect to x.
- To compute $\frac{\partial f}{\partial y}$ we think of x as fixed and differentiate as usual in with respect to y.

Example: Let $f(x,y) = x^2 + y^3 + 7xy$

$$\frac{\partial f}{\partial x} = 2x + 0 + 7y$$

$$\frac{\partial f}{\partial y} = 0 + 3y^2 + 7x$$

Notation

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

Cannot use f'

Higher derivatives

$f(x,y)$

- diff in x: $f_x = \frac{\partial f}{\partial x}$
  - x: $f_{xx} = \frac{\partial^2 f}{\partial x^2}$
  - y: $f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x}$
- diff in y: $f_y = \frac{\partial f}{\partial y}$
  - x: $f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y}$
  - y: $f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$(f_x)_y$$

Example: Let $f(x,y) = x\cos(y)$

$x\cos(y)$

- $f_x = \cos(y)$
  - $f_{xx} = 0$
  - $f_{xy} = -\sin(y)$
- $f_y = -x \cdot \sin(y)$
  - $f_{yx} = -\sin(y)$
  - $f_{yy} = -x\cos(y)$

$$f_{xy} = f_{yx}$$

Partial derivatives (3)

Theorem(Symmetry of second derivatives)

If all the 2nd partial derivativ`es of a function $f(x, y)$ exist and are continuous then

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$$

Note: Analogous results are true for 3rd derivates and higher.

Chain Rule (for differentiating a compositon of functions)

Recall the 1 variable case.

Notation: $(f \circ g)(x) = f(g(x))$

ex. $g(x) = x^2$, $f(x) = \sin(x)$

$(f \circ g)(x) = f(g(x)) = \sin(x^2)$

Note. $(g \circ f)(x) = g(f(x)) = [\sin(x)]^2$

$= \sin^2 x$

chain rule $\quad \dfrac{d\, f(g(x))}{dx} = \dfrac{d\, f(g)}{dg} \cdot \dfrac{dg}{dx}$

$= f'(g) \cdot g'(x)$

ex: $f(g) = \sin(g)$, $\quad \dfrac{df}{dx} = \cos(g)$

$g(x) = x^2$, $\quad \dfrac{dg}{dx} = 2x$

Then

$\dfrac{d}{dx} \cos(x^2) = \dfrac{d\, f(g(x))}{dx} = \cos(g) \cdot 2x$

$= \cos(x^2) \cdot 2x$

Chain rule in 2 variables

<u>Compositions</u>: $u(x,y)$, $v(x,y)$

$$f(u,v) = f(u(x,y), v(x,y))$$

<u>Let's look at a simpler case</u>

functions $u(t)$, $v(t)$, $f(x,y)$

<u>Aim</u>: what is the derivative of the 1-variable function

$$z(t) = f(u(t), v(t))$$

<u>ex</u> $u(t) = \cos(t)$, $v(t) = t^3$, $f(x,y) = xy^2$

The $f(u(t),v(t)) = u(t)[v(t)]^2$

$= \cos(t)\, t^6$ $= (t^3)$

We can diff. directly to get

$$\frac{df}{dt} = -\sin(t)\cdot t^6 + \cos(t)\cdot 6t^5$$

=

Now, let's look at this using the <u>chain rule</u>

$$\begin{tikzcd} & u(t) \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & f(u(t), v(t)) \\ & v(t) \arrow[ur] & \end{tikzcd}$$

- A change in t causes a change in u, and this change in u causes a change in f. Chain of events
- Also, a change in t causes a change in v, and this change in v causes a change in f.

$$\frac{df}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

<u>ex</u> $\frac{\partial f}{\partial u} = v^2$, $\frac{du}{dt} = -\sin(t)$

$\frac{\partial f}{\partial v} = 2uv$, $\frac{dv}{dt} = 3t^2$

$$\frac{df}{dt} = v(t)^2(-\sin(t)) + 2u(t)v(t)\, 3t^2$$

$$= t^6(-\sin(t)) + 2\cos(t)\, t^3\, 3t^2$$

Chain rule(2)

Intuitive "proof"

Set-up: Functions, $u(t), v(t)$, and $f(t) = f(u(t), v(t))$

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(u(t+\Delta t), v(t+\Delta t)) - f(u(t), v(t))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(u(t+\Delta t), v(t+\Delta t)) \overbrace{- f(u(t), v(t+\Delta t)) + f(u(t), v(t+\Delta t))}^{=0} - f(u(t), v(t))}{\Delta t}$$

Let $\Delta u = u(t+\Delta t) - u(t)$

$$= \lim_{\Delta t \to 0} \frac{f(u+\Delta u, v(t+\Delta t)) - f(u, v(t+\Delta t))}{\Delta u} \cdot \frac{\Delta u}{\Delta t} + \text{" same trick with } \Delta v \text{ "}$$

$$= \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

NOTE: This is not a complete proof. There are two issues

1. Δu could be zero
2. We need some assumptions on the smoothness of u, v and f. In particular to get from the 2nd last line to the last line.

The proof of the 1-variable chain rule is explained well here. (Not required in this course)

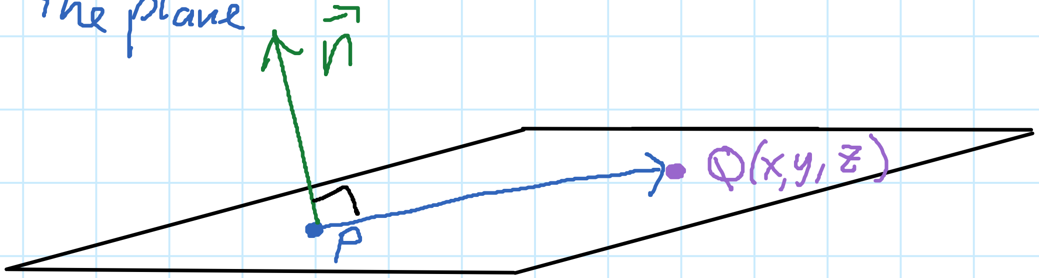
https://web.williams.edu/Mathematics/lg5/A37W12/Chain.pdf

Planes (quick review)

Given (1) $P(x_0, y_0, z_0)$ on the plane

(2) $\vec{n} = \langle a, b, c \rangle$ = a normal vector

Find the equation of the plane



Q lies on the plane $\iff$ $\vec{PQ}$ orthogonal to $\vec{n}$

$\iff \vec{PQ} \cdot \vec{n} = 0$

$\iff \langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a,b,c \rangle = 0$

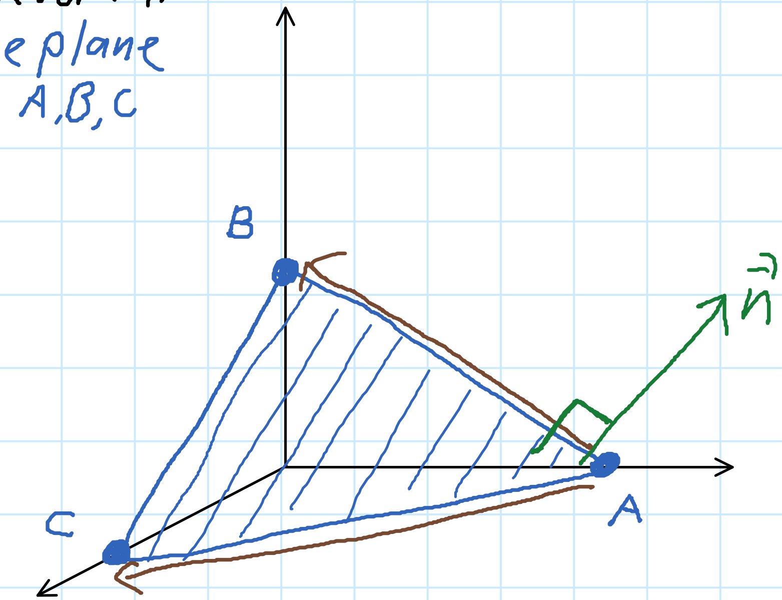
$\iff a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\iff ax + by + cz = d$

where $d = ax_0 + by_0 + cz_0$

In homework #1

Find the plane through A, B, C

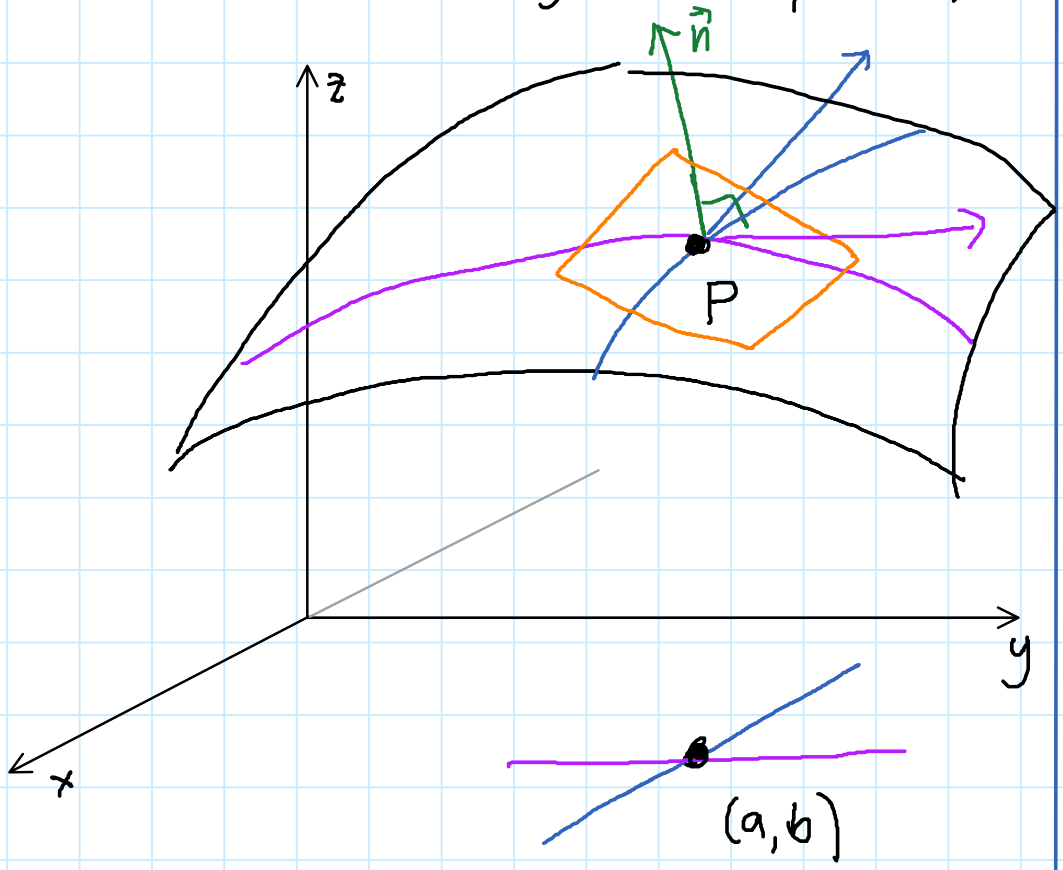


$\vec{AB}$ and $\vec{AC}$ are parallel to the plane

So $\vec{n} = \vec{AB} \times \vec{AC}$ is orthogonal to the plane

Tangent plane to a surface

<u>Aim</u> Find the tangent plane to the surface $z = f(x,y)$ at the point (a,b)



To get the equation of the plane we need

- Point on the plane ✓ $P(a,b,f(a,b))$
- Normal vector = ?

Steps to find a normal vector

1. Parametrize the curves
2. Find the tangents
3. normal = cross product

① $\vec{r}_2(t) = \langle a, t, f(a,t) \rangle$

$\vec{r}_1(s) = \langle s, b, f(s,b) \rangle$

② $\vec{r}_2\,'(t) = \langle 0, 1, \frac{d}{dt} f(a,t) \rangle$ $\quad \frac{\partial f}{\partial y}(a,t)$

$\vec{r}_2\,'(b) = \langle 0, 1, \frac{\partial f}{\partial y}(a,b) \rangle$

$\vec{r}_1\,'(a) = \langle 1, 0, \frac{\partial f}{\partial x}(a,b) \rangle$

Tangent plane (2)

$$\vec{n} = \vec{r}_1{}'(a) \times \vec{r}_2{}'(b)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix}$$

$$= \left\langle -\frac{\partial f}{\partial x}(a,b),\ -\frac{\partial f}{\partial y}(a,b),\ 1 \right\rangle$$

Note: This is a special normal vector. Its length turns out to be important. We will see this later in the course when we study surface area. Also in Diff Int 3.

We had $P = (a, b, f(a,b))$

The equation of the tangent plane is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z - f(a,b)) = 0$$

$$\Downarrow$$

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

What does this remind you of?

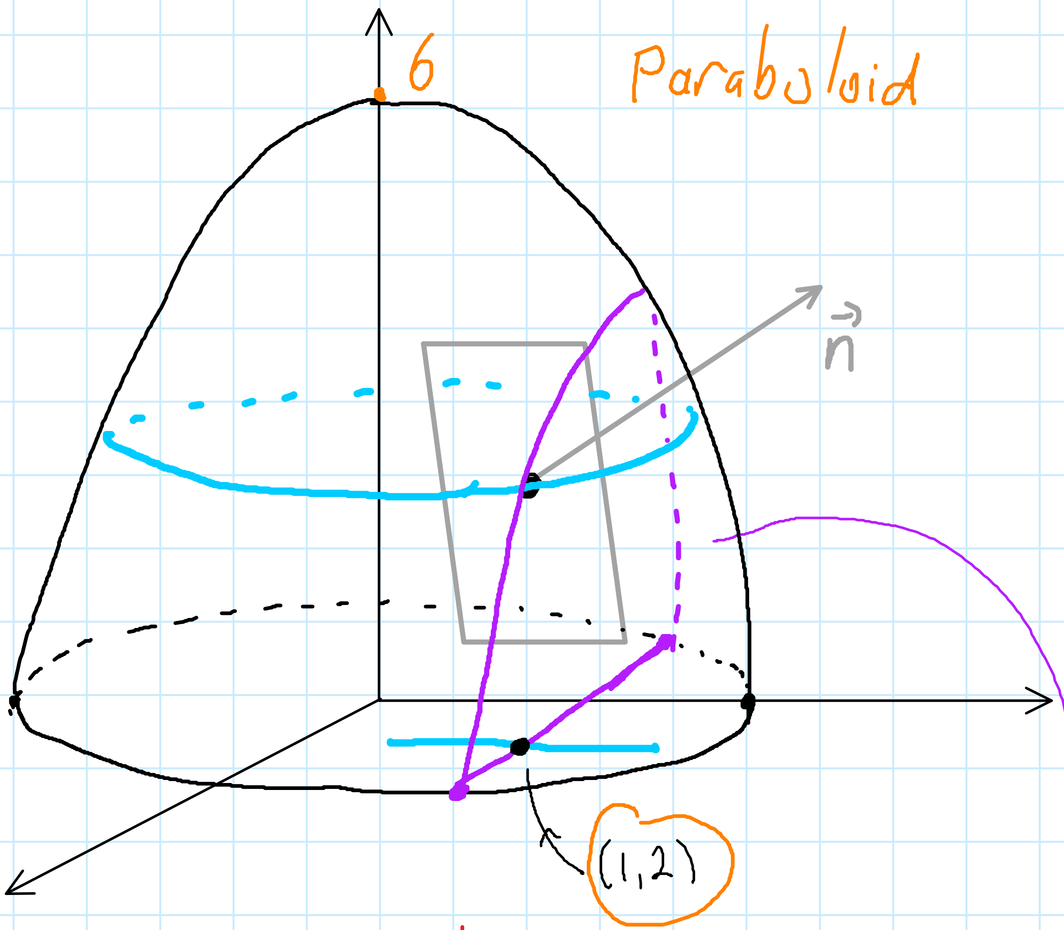
Eq of a line

$$y - y_0 = m(x - x_0)$$

$$y = \underset{\underset{f(x_0)}{\parallel}}{y_0} + \frac{df}{dx}(x_0)(x - x_0)$$

Tangent plane example

Find the tangent plane to the surface $z = 6 - x^2 - y^2$ at the point $(1,2,1)$



→ Zulip poll

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Compute $\dfrac{\partial f}{\partial x} = -2x$, $\dfrac{\partial f}{\partial x}(1,2) = -2$

$\dfrac{\partial f}{\partial y} = -2y$ $\quad \dfrac{\partial f}{\partial y}(1,2) = -4$

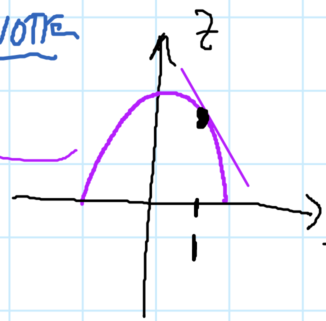
So $\vec{n} = \langle 2, 4, 1 \rangle$ (Looks roughly correct in the sketch)

The tangent plane equation is:

$$z = \underbrace{1}_{= f(1,2) = 6-1-4=1} + (-2)(x-1) + (-4)(y-2)$$

$$\Leftrightarrow \quad 2x + 4y + z = 11$$

NOTE



when $y = 2$

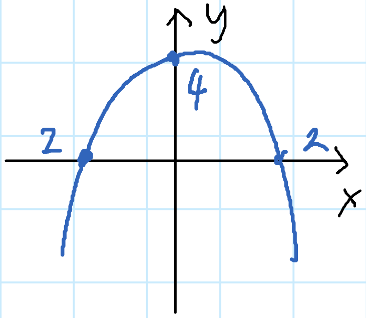
$z = 2 - x^2$

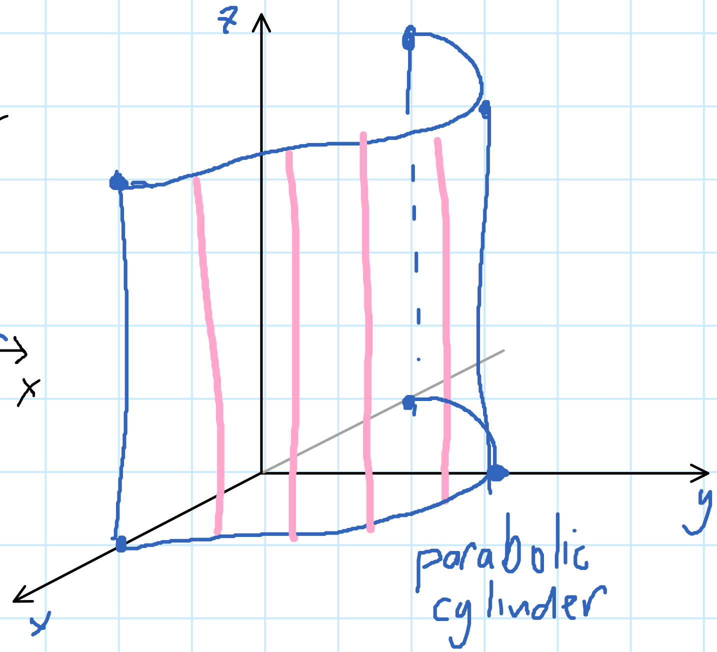
slope $= z'(1) = -2$

Homework 2, #1

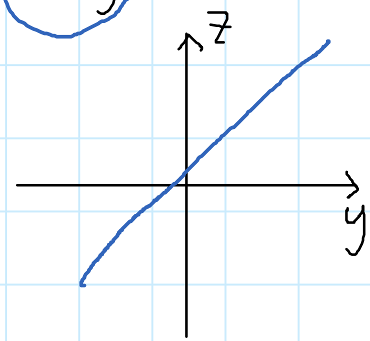
$z=0, \ z=y, \ y=4-x^2$ Homework 2, #1

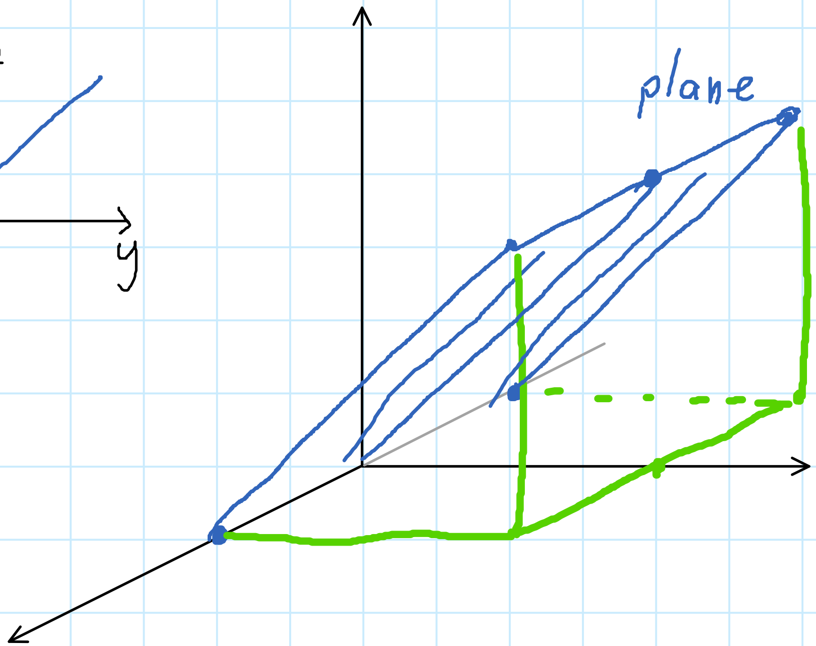
$y = 4 - x^2$

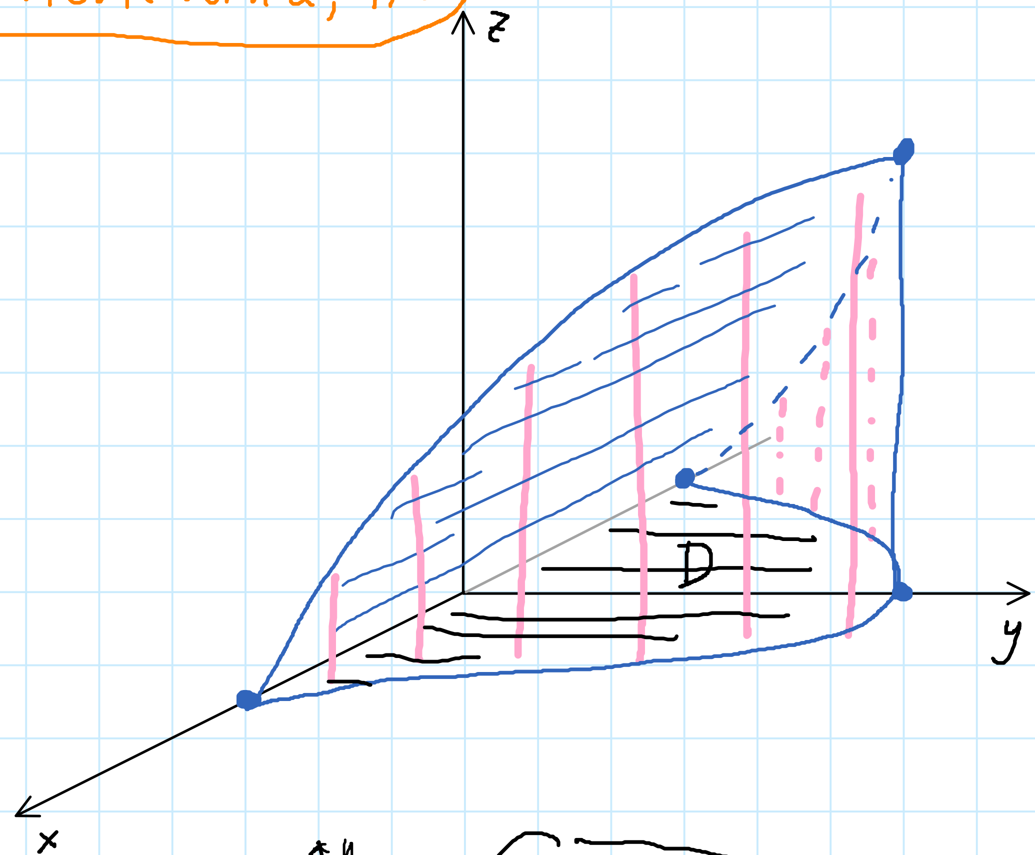


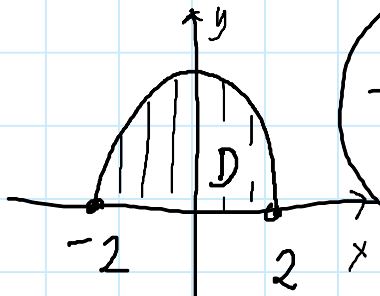


$z=y$









$-2 \le x \le 2$

$0 \le y \le 4 - x^2$