Q_1 - show that a hermition 2×2 matrix $\underline{A}$ is positive definite if and only if $tr(A) > 0$ and $\det(A) > 0$?

- Solution:

since A is a hermition matrix, the eigenvalues of A are real, i.e, λ_1 and λ_2 are real.

$$\underline{A} \text{ is positive definite} \overset{?}{\longleftrightarrow} \begin{matrix} tr(A) > 0 \\ \det(A) > 0 \end{matrix}$$

part one: given the left statement, we prove the right claim is correct. Thus we assume that A is P.D, then we have $\begin{cases} \lambda_1 > 0 \\ \lambda_2 > 0 \end{cases}$

Moreover, we know that $\begin{cases} tr(A) = \sum_{i=1}^{2} \lambda_i \\ \det(A) = \prod_{i=1}^{n} \lambda_i \end{cases}$

$$\Rightarrow \begin{cases} tr(A) = \lambda_1 + \lambda_2 > 0 \\ \det(A) = \lambda_1 \cdot \lambda_2 > 0 \end{cases} \checkmark$$

part two: given $\begin{cases} tr(A) > 0 \\ \det(A) > 0 \end{cases}$, we prove that A is P.D.

Assume that $\begin{cases} tr(A) > 0 \\ \det(A) > 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 + \lambda_2 > 0 & (*) \\ \lambda_1 \cdot \lambda_2 > 0 & (**) \end{cases}$

from (**), we conclude that $\begin{cases} \lambda_1 < 0, \lambda_2 < 0 \\ \text{or} \\ \lambda_1 > 0, \lambda_2 > 0 \end{cases}$

If we assume that $\lambda_1 < 0, \lambda_2 < 0 \Rightarrow \lambda_1 + \lambda_2 < 0$ ✗

contradict with (*)

Thus, the other assumption is true, i.e, $\lambda_1 > 0, \lambda_2 > 0$

which results in $\underline{A}$ being a P.D matrix. ✓

*Q2 – Find all eigenvalues and their corresponding eigenvectors of the matrix $\underline{A} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

– solution:

For λ and u which are eigenvalue and eigenvector of A, we have:

$$\underline{A}\,\underline{u} = \lambda\,\underline{u} \implies (\underline{A} - \lambda \underline{I})\,\underline{u} = \underline{0}$$

So, the matrix $\underline{B} = \underline{A} - \lambda \underline{I}$ is a singular matrix

and $\boxed{\det(B) = 0}$

$$\det(B) = \det\left(\begin{bmatrix} 1-\lambda & -3 & 0 \\ -3 & 1-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 6-\lambda \end{bmatrix}\right) - (-3)\cdot\det\left(\begin{bmatrix} -3 & 0 \\ 0 & 6-\lambda \end{bmatrix}\right)$$

$$\cancelto{0}{+\,0\cdot\det\left(\begin{bmatrix} -3 & 1-\lambda \\ 0 & 0 \end{bmatrix}\right)}$$

$$= (1-\lambda)\left[(1-\lambda)(6-\lambda)\right] + 3\cdot\left[(-3)(6-\lambda)\right]$$

$$= (1-\lambda)^2(6-\lambda) - 9(6-\lambda) = \left[(1-\lambda)^2 - 9\right]\cdot(6-\lambda)$$

$$\Rightarrow \det(B) = \left[(1-\lambda)^2 - 9\right](6-\lambda) = 0$$

$$\Rightarrow \begin{cases} 6-\lambda = 0 \Rightarrow \boxed{\lambda_1 = 6} \\ (1-\lambda)^2 - 9 = 0 \Rightarrow (1-\lambda)^2 = 9 \Rightarrow 1-\lambda = \pm 3 \Rightarrow \begin{cases} \boxed{\lambda_2 = 4} \\ \boxed{\lambda_3 = -2} \end{cases} \end{cases}$$

After calculating eigenvalues of A, we find the eigen vectors using λ_1, λ_2 and λ_3. Thus, we have

$$\underline{A}\,\underline{u}_1 = \lambda_1 \underline{u}_1 \Rightarrow \underline{A}\,\underline{u}_1 = 6\,\underline{u}_1 \qquad \text{if } \underline{u}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 6\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} a-3b \\ -3a+b \\ 6c \end{bmatrix} = \begin{bmatrix} 6a \\ 6b \\ 6c \end{bmatrix}$$

From the two first equations, we have:

$$\begin{cases} a - 3b = 6a \longrightarrow 5a = -3b \longrightarrow a = -\frac{3}{5}b \quad (*) \\ -3a + b = 6b \xrightarrow{*} \frac{9}{5}b = 5b \Rightarrow \boxed{b=0} \quad (**) \end{cases}$$

using $(*)$ and $(**)$, $\boxed{a=0}$ $\Rightarrow \underline{u}_1 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$

If we constrain $\|u_1\| = 1 \Rightarrow \sqrt{0+0+c^2} = 1 \Rightarrow \boxed{c = \pm 1}$

$$\Rightarrow u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad u_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

similarly, we can write:

$$\underline{A}\,\underline{u}_2 = \lambda_2 \underline{u}_2 \Rightarrow \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 4 \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a - 3b \\ -3a + b \\ 6c \end{bmatrix} = \begin{bmatrix} 4a \\ 4b \\ 4c \end{bmatrix} \xrightarrow[\text{equation}]{\text{third}} 6c = 4c \Rightarrow \boxed{c = 0}$$

From the first equation, $a - 3b = 4a \Rightarrow -3b = 3a \Rightarrow \boxed{a = -b}$ (*)

If we constrain $\|u_2\| = 1 \Rightarrow \sqrt{a^2 + b^2 + 0} = 1$

$$\xrightarrow{(*)} a^2 + a^2 = 1 \Rightarrow 2a^2 = 1 \Rightarrow a^2 = \frac{1}{2} \Rightarrow \boxed{a = \pm \frac{1}{\sqrt{2}}} \ (**)$$

using (*) and (**), we obtain $\boxed{b = \mp \frac{1}{\sqrt{2}}}$

$$\Rightarrow u_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{or} \quad u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

For the third eigenvalue $\lambda_3 = -2$, we have

$$\underline{A}\,\underline{u}_3 = \lambda_3 \underline{u}_3 \Rightarrow \begin{bmatrix} a - 3b \\ -3a + b \\ 6c \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \\ -2c \end{bmatrix} \xrightarrow[\text{equation}]{\text{third}} \boxed{c = 0}$$

From the first equation, we have $a - 3b = -2a \Rightarrow \boxed{a = b}$ (*)

$$\|u_3\| = 1 \Rightarrow \sqrt{a^2 + b^2 + 0} = 1 \xrightarrow{(*)} 2a^2 = 1 \Rightarrow \boxed{a = \pm \frac{1}{\sqrt{2}}} \ (**)$$

From (*) and (**), we obtain that $\boxed{b = \pm \frac{1}{\sqrt{2}}}$

$$\Rightarrow u_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{or} \quad u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Two additional property :

1) As it can be seen, $tr(A) = \sum_{i=1}^{3} \lambda_i = 8$

2) $u_i^H \cdot u_j = 0 \quad , \quad i \neq j \longrightarrow$ since A is a hermition (symetric in real domain) matrix.

*Q3 - Find the Least-squares solution of $\underline{A}\,\underline{x} = b$

where $\underline{A} = \begin{bmatrix} 1+j & 1-2j \\ 2+3j & -2+j \\ 4+j & 3-j \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 1+j \\ 1-j \\ 0 \end{bmatrix}$.

- Solution:

As It is included in the lecture note, $x_{LS} = (\underline{A}^H \underline{A})^{-1} \underline{A}^H \underline{b}$

$$A^H A = \left(\begin{bmatrix} 1+j & 1-2j \\ 2+3j & -2+j \\ 4+j & 3-j \end{bmatrix} \right)^H \cdot \begin{bmatrix} 1+j & 1-2j \\ 2+3j & -2+j \\ 4+j & 3-j \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1-j & 1+2j \\ 2-3j & -2-j \\ 4-j & 3+j \end{bmatrix} \right)^T \cdot \begin{bmatrix} 1+j & 1-2j \\ 2+3j & -2+j \\ 4+j & 3-j \end{bmatrix}$$

$$= \begin{bmatrix} 1-j & 2-3j & 4-j \\ 1+2j & -2-j & 3+j \end{bmatrix} \begin{bmatrix} 1+j & 1-2j \\ 2+3j & -2+j \\ 4+j & 3-j \end{bmatrix}$$

$$= \begin{bmatrix} 2+13+17 & (1-2j-j-2)+(-4+2j+6j+3)+(12-4j-3j-1) \\ (1+j+2j-2)+(-4-6j-2j+3)+(12+3j+4j-1) & 5+5+10 \end{bmatrix}$$

$$\Rightarrow A^H A = \begin{bmatrix} 32 & 9-2j \\ 9+2j & 20 \end{bmatrix}$$

$$\Rightarrow (A^H A)^{-1} = \frac{1}{\det(A^H A)} \begin{bmatrix} 20 & -9+2j \\ -9-2j & 32 \end{bmatrix}$$

$$= \frac{1}{(32\times 20)-(81+4)} \begin{bmatrix} 20 & -9+2j \\ -9-2j & 32 \end{bmatrix} = \frac{1}{555} \begin{bmatrix} 20 & -9+2j \\ -9-2j & 32 \end{bmatrix}$$

$$\underline{A}^H \underline{b} = \begin{bmatrix} 1-j & 2-3j & 4-j \\ 1+2j & -2-j & 3+j \end{bmatrix} \begin{bmatrix} 1+j \\ 1-j \\ 0 \end{bmatrix} = \dots = \begin{bmatrix} 1-5j \\ -4+4j \end{bmatrix}$$

$$\Rightarrow x_{LS} = (A^H A)^{-1} A^H b = \frac{1}{555} \begin{bmatrix} 20 & -9+2j \\ -9-2j & 32 \end{bmatrix} \begin{bmatrix} 1-5j \\ -4+4j \end{bmatrix}$$

$$= \dots = \begin{bmatrix} 0.0865 - 0.2595j \\ -0.2649 + 0.3081j \end{bmatrix}$$

$$\Rightarrow \hat{b} = A x_{LS} = \begin{bmatrix} 0.6973 + 0.6649j \\ 1.1730 - 1.1405j \\ 0.1189 + 0.2378j \end{bmatrix}$$

while $b = \begin{bmatrix} 1+j \\ 1-j \\ 0 \end{bmatrix}$