Lecture 5

Linear approximation and differentials, differentiability, the directional derivative

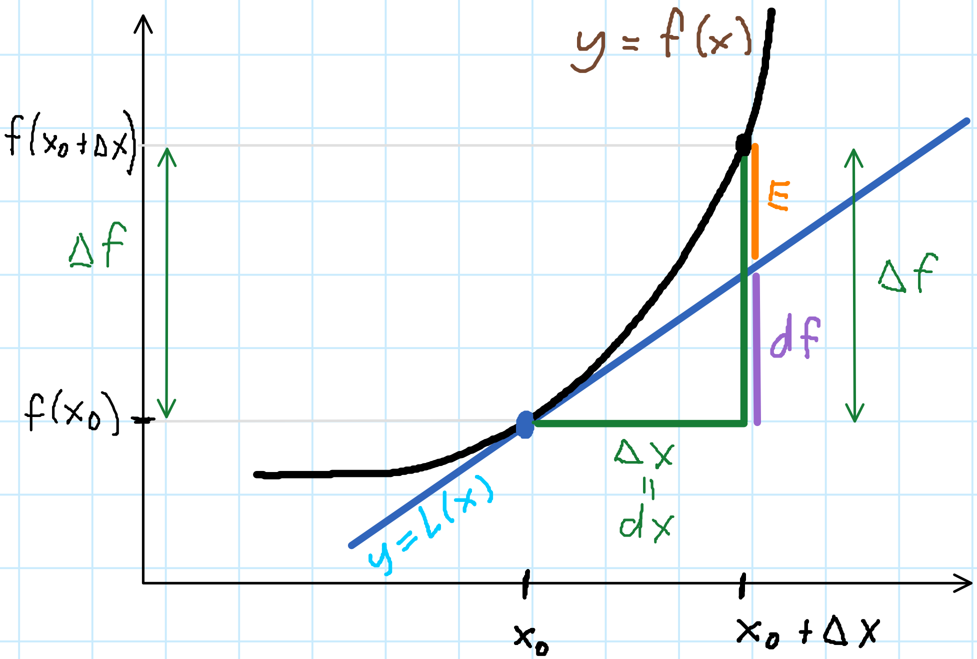
- Reviewed linear approximation and differentials in one variable.
- Analogously defined the linear (tangent plane) approximation in 2 variables and the associated idea of the differential (which is just the approximate change of the function using the tangent plane approximation). The formulas all look like the one variable case with just the addition of an extra term.
- Discussed "differentiablity". We saw the the definition of the derivative in one variable can be re-writen as $(\Delta f - \mathrm{d}f)/\Delta x \to 0$ as $\Delta x \to 0$. By analogy we gave the two variable definition $(\Delta f - \mathrm{d}f)/\sqrt{\Delta x^2 + \Delta y^2} \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Stated (without rigorous justification) the fact that differntiablity implies continuity. Also stated without justification the fact that continous partial derivatives implied differentiability.
- Pointed out that conceptual the "differentiablity" condition says that the linear (tangent plane) approximation "works" in the sense that the relative error goes to zero as the initial point is approached.
- As we discussed when we first introduced partial derivatives, we would like to know what the slope of a surface (or rate of change of a function) is in any given direction. Let f be the function and u be the unit vector specifying the direction. Then D_u f is the notation for the directional derivative.
- Gave the limit definition of D_u f and showed how D_u f can be computed using the partial derivatives as long as f is differentiable.
- The intuition for this formula is that the partial derivatives determine the tangent plane, and the directional derivative is just the slope of the plane in given direction.
- Briefly introduced the gradient vector.
- Briefly mentoined that there are many applications for which finding the direction of the maximum rate of the change is useful, and that we will be able to solve this using the gradient vector. (details next class)

Where to find this material

- Adams and Essex 4.9 (review), 12.6 12.7
- Corral, 2.4 (linear approximation is not discussed)
- Guichard, 6.4 (review) 14.5 (linear approximation is not discussed)
- Active Calculus. 10.4, 10.6

Linear approximation (1)

A quick review of the 1-variable case

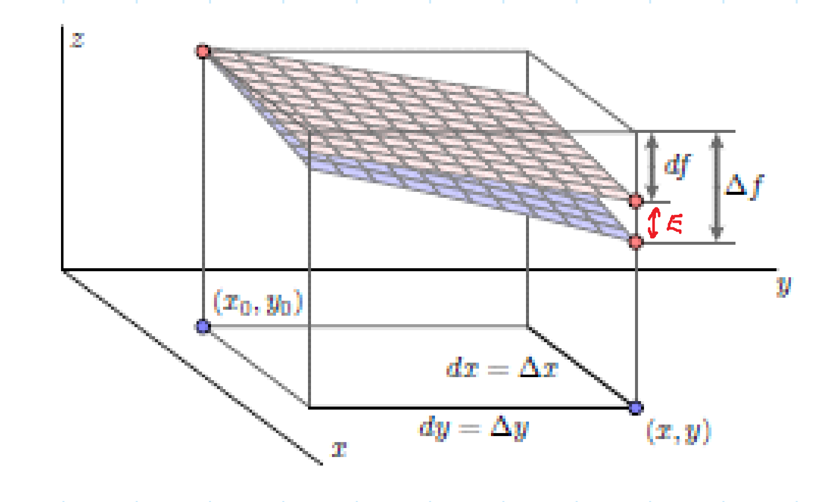


Object ≈	Linear approximation
Graph = curve	Tangent line
Function $f(x)$	$L(x) = f(x_0) + \overbrace{f'(x_0)(x - x_0)}^{df}$
Change in $f(x) = \Delta f$	differential = df

Error $= \mathrm{E} = \Delta f - \mathrm{d}f$

Note: dx and df are a numbers, not to be confused with the notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\int \cdots dx$ which are limits.

Two variable case - f(x,y)



(image from Active Calculus, page 134)

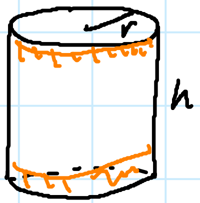
Object ≈	Linear approximation
Graph = surface	Tangent plane
Function $f(x, y)$	$L(x, y)$ $= f(x_0, y_0) + \underbrace{f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{df}$
Change in $f(x, y) = \Delta f$	differential = df

Error $= \mathrm{E} = \Delta f - \mathrm{d}f$

Linearization (2)

Example

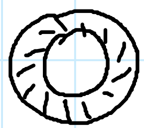
Estimate, using linear approximation, the amount of metal in a empty can



Inside dimensions: $r = 1$, $h = 5$

Thickness of the.

top/bottom $= 0.1$ sides $= 0.3$



Solution

Volume of a cylinder $= \pi r^2 h$

Exact volume of metal $= V_{OUTSIDE} - V_{INSIDE}$

$= \Delta V$

Linear approximation

$$\Delta V \approx dV$$

$V(r,h) = \pi r^2 h$, $r_0 = 1$, $\Delta r = 0.3$

$h_0 = 5$, $\Delta h = 0.2$

$\frac{\partial V}{\partial r} = 2\pi r h$, $\frac{\partial V}{\partial r}(1,5) = 10\pi$

$\frac{\partial V}{\partial h} = \pi r^2$, $\frac{\partial V}{\partial h}(1,5) = \pi$

Now, $dV = \frac{\partial V}{\partial r}(1,5)\,\Delta r + \frac{\partial V}{\partial h}\,\Delta h$

$= 10\pi * 0.3 + \pi * 0.2$

$= 3.2\pi$

The exact answer is $\Delta V = V(1.3, 5.2) - V(1,5)$

$= 3.79\pi$

The approximation is not very good because the Δr and Δh are relatively large.

change to $\Delta r = 0.03$, $\Delta h = 0.02$ then

$dV = 0.32\pi$ (immediately)

And $\Delta V = 0.326\pi$ BETTER

Differentiability

Version 1: A function of one variable $f(x)$is differentiable at a point $x=a$ if and only if $\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{h}$ exists.

To generalize this to two or more variables we need to rewrite this is a conceptually different way.

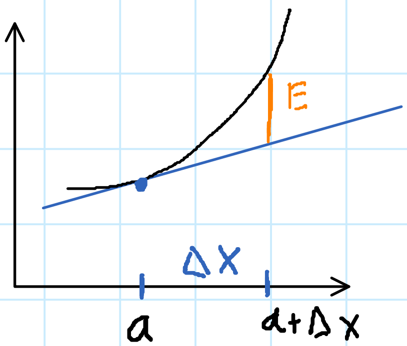
When the above limit exists we denote its value by $f'(a)$. Then

$$\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}=f'(a)$$

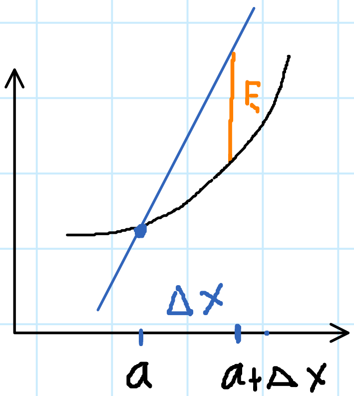
$$\Leftrightarrow \lim_{\Delta x\to 0}\frac{\overbrace{f(a+\Delta x)-f(a)}^{\Delta f}-\overbrace{f'(a)\Delta x}^{df}}{\Delta x}=0$$

$$\Leftrightarrow \lim_{\Delta x\to 0}\frac{\overbrace{\Delta f-df}^{E}}{\Delta x}=0$$

$\frac{E}{\Delta x}$ = relative error = error per distance



$E\to 0$ $\frac{E}{\Delta x}\to 0$



$E\to 0$ $\frac{E}{\Delta x}\not\to 0$

Version 2: A function of one variable $f(x)$is differentiable at a point $x=a$ if and only if there exists a number m such that $\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)-m\Delta x}{h}$ exists.

Note: $df=m\Delta x=f'(a)\Delta x$

2 variable case

Definition

A function $f(x,y)$ is diffentiable at (a,b) if and only if there exists numbers m and n such that

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\overbrace{f(a+\Delta x,b+\Delta y)-f(a,b)}^{\Delta f}-\overbrace{(m\Delta x+n\Delta y)}^{df}}{\sqrt{\Delta x^2+\Delta y^2}}=0$$

Or equivalently

(a,b) $(a+\Delta x, b+\Delta y)$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta f-df}{\text{distance from }(a,b)\text{ to }(a+\Delta x,b+\Delta y)}$$

$$=\lim_{\langle\Delta x,\Delta y\rangle\to\vec{0}}\frac{\Delta f-df}{\|\langle\Delta x,\Delta y\rangle\|}$$

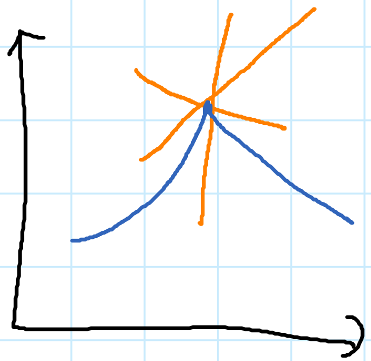
Differentiability (2)

Conceptual summary

- In 1 variable we say a function $f(x)$ is differentiable at a if and only if there is a line passing through $(a, f(a))$ which approximates the function ``well``. In this case we call the line the tangent line

- In 2 variables we say a function $f(x, y)$ is differentiable at (a, b) if and only if there is a plane passing through $(a, b, f(a, b))$ which approximates the function ``well``. In this case we call the plane the tangent plane.

In particular if a function is not differentiable at a point then it does not have a tangent line/plane at that point.



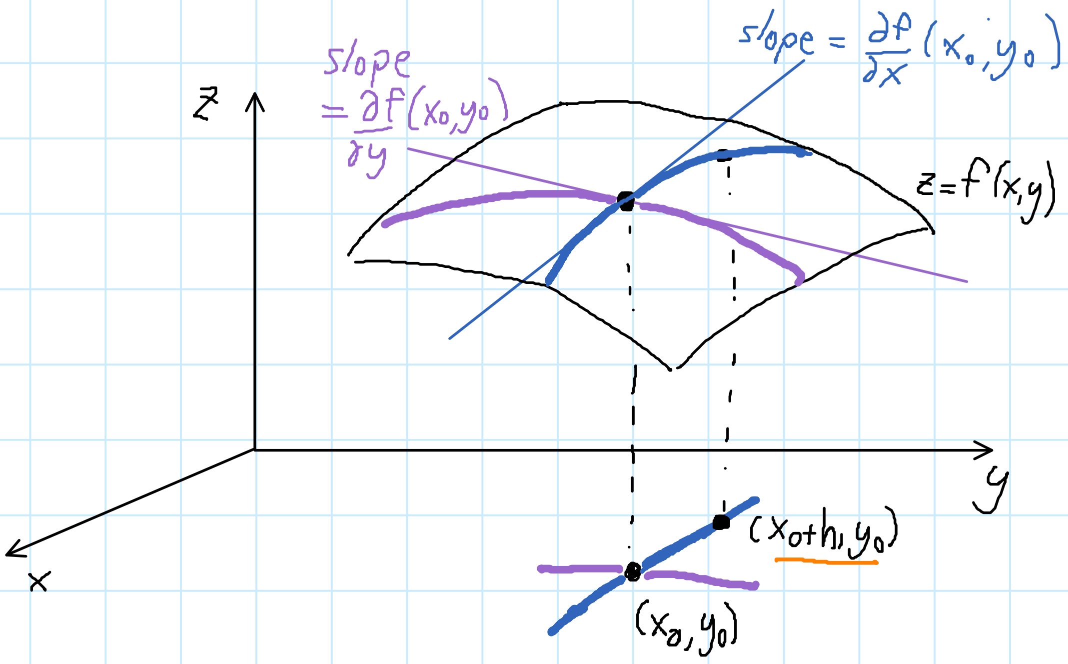
HW #3
2-variable
example

Here is a useful theorem that can be applied to most of the familiar functions.

Theorem:
If the partial derivatives of a function $f(x, y)$ exist and are continuous at a point (a, b), then $f(x, y)$ is differentiable at (a, b)

Directional derivative

Recall:



Slope in the x-direction

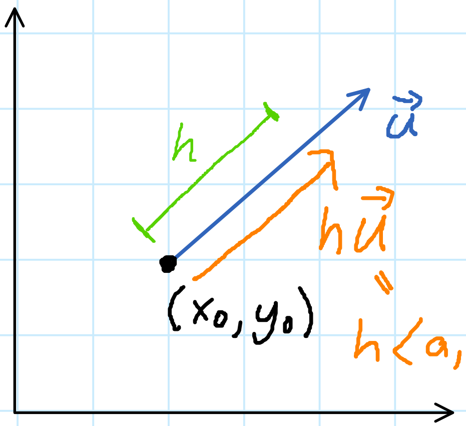
$$= \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Question: What is the slope in a given direction

$\|\vec{u}\| = \text{length} = 1$

Let's specify the direction by a $\boxed{\text{unit vector}}$

$\vec{u} = \langle a, b \rangle$



the vector of magnitude h in the direction of $\vec{u}$ is

$h\vec{u} = h\langle a, b \rangle = \langle ha, hb \rangle$

Definition The directional derivative of $f(x,y)$ in the direction of $\vec{u}$ at the point (x_0, y_0) is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0, y_0)}{h}$$

Directional derivative (2)

How to compute

Using algebra, and skipping technicalities, we can rewrite the directional derivative in familiar terms.

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ah,\, y_0+bh) - f(x_0,y_0)}{h}$$

[where $\vec{u} = \langle a,b \rangle$, $\|\vec{u}\| = 1$]

$$= \lim_{h \to 0} \Big[f(x_0+ah,\, y_0+bh) \overbrace{- f(x_0,\, y_0+bh) + f(x_0,\, y_0+bh)}^{=0} - f(x_0,y_0) \Big] / h$$

$$= \lim_{h \to 0} \frac{f(x_0+\overset{H}{ah},\, y_0+bh) - f(x_0,\, y_0+bh)}{\underset{H}{ah}} \cdot a \;+\; \lim_{h \to 0} \text{ '' } \quad \text{ '' }$$

Can be proved but it requires f to be differentiable

$$= a \frac{\partial f}{\partial x}(x_0,y_0) + b \frac{\partial f}{\partial y}(x_0,y_0)$$

Note: This looks like the chain rule "proof"

→ HW 4 - use the chain rule to get

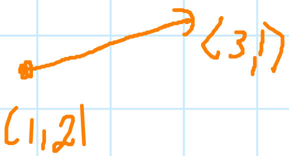
Directional derivative (2)

Example Compute the directional derivative of $f(x,y) = e^{x^2 y}$ at $(1,2)$ in the direction $\langle 3,1\rangle$

$$u = \langle 3,1\rangle / \|\langle 3,1\rangle\| = \frac{1}{\sqrt{10}}\langle 3,1\rangle = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle$$

$$\frac{\partial f}{\partial x} = 2xy\,e^{x^2 y}, \quad \frac{\partial f}{\partial x}(1,2) = 4e^2$$

$$\frac{\partial f}{\partial y} = x^2 e^{x^2 y}, \quad \frac{\partial f}{\partial y}(1,2) = e^2$$



$$D_{\vec{u}} f(1,2) = e^2/\sqrt{10}\,(3\cdot 4 + 1\cdot 1) = 13e^2/\sqrt{10}$$

Now let's look at the formula again

$$D_{\vec{u}} f = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} \quad \left(\begin{array}{l}\vec{u} = \langle a,b\rangle \\ \|\vec{u}\| = 1\end{array}\right)$$

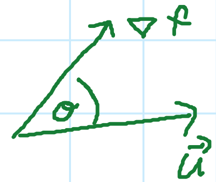
$$= \langle a,b\rangle \cdot \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle}_{\vec{\nabla} f \,=\, \text{Gradient of } f}$$

Everything from here on will be covered in the next lecture

$\boxed{\text{QUESTION}}$ In which direction does f increase most rapidly? (which $\vec{u}$ makes $D_{\vec{u}} f$ maximal)

$\boxed{\text{Answer}}$

$$D_{\vec{u}} f = \vec{u}\cdot\vec{\nabla} f$$
$$= \|\vec{u}\|\,\|\vec{\nabla} f\|\cos\theta$$
$$= \|\vec{\nabla} f\|\cos\theta$$



$\cos\theta$ has a max of 1 at $\theta = 0$

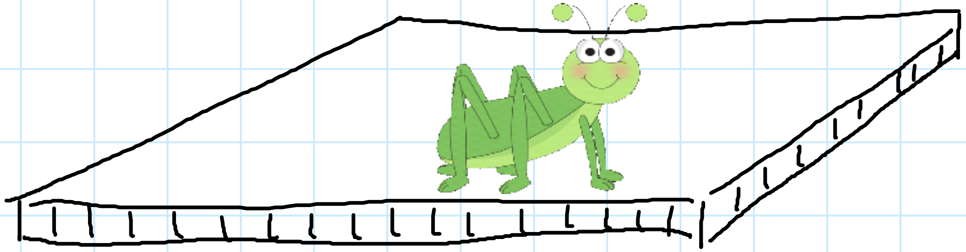
So, max of $D_{\vec{u}} f = \|\nabla f\|$
and this occurs when $\vec{u}$ is parallel to $\vec{\nabla} f$
(that is, $\vec{u} = \dfrac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$)

<u>Note</u> Minimum occurs when $\theta = \pi$.

So $\vec{u} = -\vec{\nabla} f / \|\vec{\nabla} f\|$ and

$$D_{\vec{u}} f = -\|\nabla f\|$$

Gradient vector example



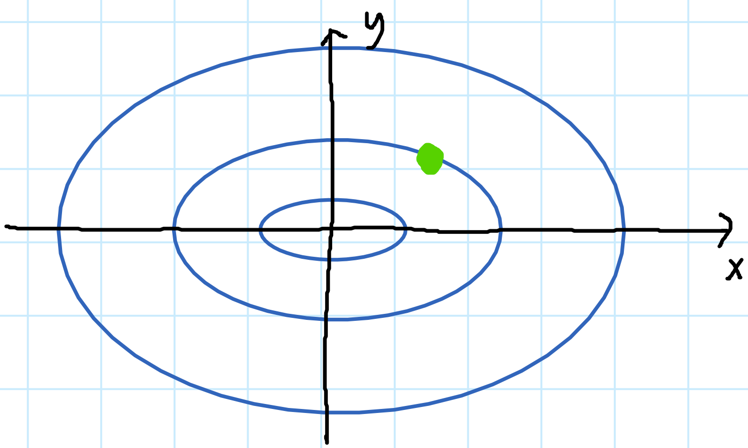
A happy bug accidentally lands on a hot grill plate with

surface temperature given by $T(x,y) = 5000e^{-(x^2+3y^2)}$

The bug has landed at the point $(1,1)$.

Question: In which direction should the bug start walking in order to cool its feet most rapidly.

__Intuition__ The level curves are ellipses



__Calculations__ (let $c = 5000$)

$\frac{\partial T}{\partial x} = -2xc\, e^{''\,''}$, $\frac{\partial T}{\partial x}(1,1) = -2ce^{''\,''}$

$\frac{\partial T}{\partial y} = -6yc\, e^{''\,''}$, $\frac{\partial T}{\partial y}(1,1) = -6ce^{''\,''}$

$$\nabla T(1,1) = ce^{-4}\langle -2,-6\rangle$$
$$= -2ce^{-4}\langle 1,3\rangle$$

Direction of max increase is $\langle -1,-3\rangle$

Direction of max decrease is $\langle 1,3\rangle$