Order, topology, and sigma-algebras on the extended real line

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The *extended real line* is a set $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, where $\mathbb{R}$ denotes the set of real numbers and $-\infty$ and ∞ are two distinct elements not in $\mathbb{R}$. We also denote $\bar{\mathbb{R}} = [-\infty, \infty]$.

1 Order

We define a relation $\leq$ on $\bar{\mathbb{R}}$ by saying that $x \leq y$ if either $x = -\infty$, $y = \infty$, or $x, y \in (-\infty, \infty)$ and $x \leq y$ in the usual ordering on the real line. We denote $x < y$ whenever $x \leq y$ and $x \neq y$. Then $(\bar{\mathbb{R}}, \leq)$ is a totally ordered set, and a complete lattice in the sense that $\inf(A), \sup(A) \in \bar{\mathbb{R}}$ for every nonempty $A \subset \bar{\mathbb{R}}$. We denote intervals with endpoints $a, b \in \bar{\mathbb{R}}$ by (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ as usual.

2 Topology

Sets of the form $(a, b) = \{x : a < x < b\}$ are called *open intervals* in $\bar{\mathbb{R}}$. Sets of the form $[-\infty, a) = \{x : x < a\}$ and $(a, +\infty) = \{x : x > a\}$ *open rays* in $\bar{\mathbb{R}}$. A set $A \subset \bar{\mathbb{R}}$ is called *open* if it can be expressed as a union of open intervals and open rays in $\bar{\mathbb{R}}$. The collection of all open sets is denoted $\mathcal{T}(\bar{\mathbb{R}})$, and called the *topology* of $\bar{\mathbb{R}}$. Hence the open sets of $\bar{\mathbb{R}}$ are the sets in $\mathcal{T}(\bar{\mathbb{R}})$, and the closed sets of $\mathbb{R}$ are the complements of the sets in $\mathcal{T}(\bar{\mathbb{R}})$. The collection of open intervals and open rays forms a basis of the topology. Examples of open sets are the intervals $(-\infty, 0)$, $[-\infty, 0)$, $[-\infty, \infty)$, $[-\infty, \infty]$. Examples of closed sets include the singleton sets $\{a\}$ with $a \in \bar{\mathbb{R}}$ and the sets $[-\infty, 0]$ and $[-\infty, \infty]$. This type of topology can be defined for any totally ordered space — in general such topologies are called *order topologies*.

2.1 Mapping to the unit interval

Define a function logit : $[0,1] \to [-\infty,+\infty]$ by

$$\operatorname{logit}(x) = \begin{cases} -\infty, & x = 0, \\ \log \frac{x}{1-x}, & x \in (0,1), \\ \infty, & x = 1. \end{cases}$$

Then one can verify that logit is an increasing bijection having an increasing continuous inverse expit : $[-\infty,+\infty] \to [0,1]$ defined by

$$\operatorname{expit}(x) = \begin{cases} 0, & x = -\infty, \\ \frac{1}{1+e^{-x}}, & x \in (-\infty,\infty), \\ 1, & x = \infty. \end{cases}$$

Remark 2.1. The function expit is known with many names, such as (standard) logistic function and logistic sigmoid function.

Lemma 2.2. *The logit and the expit functions are continuous.*

Proof. We will show that the preimage $\operatorname{logit}^{-1}(A) = \{x : \operatorname{logit}(x) \in A\}$ is open for any open set A in $\bar{\mathbb{R}}$. Assume that A is open in $\bar{\mathbb{R}}$. Then we may express A as a union $A = \cup_{i \in I} B_i$ in which each B_i is either an open interval or an open ray in $\bar{\mathbb{R}}$. Then

$$\operatorname{logit}^{-1}(A) = \bigcup_{i \in I} \operatorname{logit}^{-1}(B_i). \tag{2.1}$$

A little contemplation confirms that preimages of open intervals and open rays by the logit function can be expressed as

$$\begin{aligned} \operatorname{logit}^{-1}((a,b)) &= (\operatorname{expit}(a), \operatorname{expit}(b)), \\ \operatorname{logit}^{-1}([-\infty,a)) &= [0, \operatorname{expit}(a)), \\ \operatorname{logit}^{-1}((a,+\infty]) &= (\operatorname{expit}(a), 1]. \end{aligned}$$

Because all intervals on the right side above are open subsets of $[0,1]$, and because unions of open sets are open, we conclude with the help of (2.1) that $\operatorname{logit}^{-1}(A)$ is open.

The proof that expit : $\bar{\mathbb{R}} \to [0,1]$ is continuous can be done in a similar way, after noting that every open set in $[0,1]$ be expressed as a union of intervals of the form (a, b), $[0, a)$, and $(a, 1]$, and observing that preimages of such intervals by the expit function can be written as

$$\begin{aligned}
\mathrm{expit}^{-1}((a,b)) &= (\mathrm{logit}(a), \mathrm{logit}(b)), \\
\mathrm{expit}^{-1}([0,a)) &= [-\infty, \mathrm{logit}(a)), \\
\mathrm{expit}^{-1}((a,1]) &= (\mathrm{logit}(a), +\infty].
\end{aligned}$$

□

Remark 2.3. Lemma 2.2 shows that logit serves as a homeomorphism and an order isomorphism between $[0, 1]$ and $[-\infty, +\infty]$, and hence these sets share the same topological and order-theoretic properties. Especially, we find that $[-\infty, +\infty]$ is a compact and connected topological space. We can also express the topology of the extended real line as $\mathcal{T}(\bar{\mathbb{R}}) = \mathrm{expit}^{-1}(\mathcal{T}([0, 1])) = \mathrm{logit}(\mathcal{T}([0, 1])$.

Remark 2.4. Instead of the function logit, one may extend the function tan from its natural domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ to a function $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-\infty, +\infty]$ which then yields a homeomorphism between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\bar{\mathbb{R}}$ with inverse arctan, as in [Kytölä 2020, Probability Theory].

3 Borel sets

We define the Borel sigma-algebra of $\bar{\mathbb{R}}$ by $\mathcal{B}(\bar{\mathbb{R}}) = \sigma(\mathcal{T}(\bar{\mathbb{R}}))$, the smallest sigma-algebra containing the open sets of $\bar{\mathbb{R}}$.

Lemma 3.1. $\mathcal{I}(\bar{\mathbb{R}}) = \{[-\infty, x] : x \in \bar{\mathbb{R}}\}$ *is a* π*-system on* $\bar{\mathbb{R}}$ *which generates the Borel sigma-algebra* $\mathcal{B}(\bar{\mathbb{R}})$*.*

Proof. The fact that $\mathcal{I}(\bar{\mathbb{R}})$ is a π-system follows immediately by noting that $[-\infty, x] \cap [-\infty, y] = [-\infty, x \wedge y]$ for all x, y. To finish the proof, it suffices to verify that $\mathcal{I}(\bar{\mathbb{R}}) \subset \sigma(\mathcal{T}(\bar{\mathbb{R}}))$ and $\mathcal{T}(\bar{\mathbb{R}}) \subset \sigma(\mathcal{I}(\bar{\mathbb{R}}))$, because these imply that $\sigma(\mathcal{I}(\bar{\mathbb{R}})) = \sigma(\mathcal{T}(\bar{\mathbb{R}}))$. The first inclusion is easy to verify because every set of the form $[-\infty, x] = (x, \infty]^c$ is a complement of an open ray in $\bar{\mathbb{R}}$. To verify the second inclusion, we proceed in three steps.

(i) First we observe that $(a, b] \in \sigma(\mathcal{I}(\bar{\mathbb{R}}))$ for all $a, b \in \bar{\mathbb{R}}$, because $(a, b] = [-\infty, b] \cap [-\infty, a]^c$.

(ii) By applying (i), we see that $(a, b) \in \sigma(\mathcal{I}(\bar{\mathbb{R}}))$ for all $a, b \in \bar{\mathbb{R}}$, because

$$(a,b) \;=\; \begin{cases} \cup_{n\in\mathbb{N}}(a, b - \frac{1}{n}], & b < +\infty, \\ \cup_{n\in\mathbb{N}}(a, n], & b = +\infty. \end{cases}$$

(iii) By applying (ii), we see that $[a, b) \in \sigma(\mathcal{I}(\bar{\mathbb{R}}))$ for all $a, b \in \bar{\mathbb{R}}$, because

$$\begin{aligned}
[a, b) &= [-\infty, b) \cap [-\infty, a)^c \\
&= \Big([-\infty, -\infty] \cup (-\infty, b)\Big) \cap \Big([-\infty, -\infty] \cup (-\infty, a)\Big)^c.
\end{aligned}$$

The claim $\mathcal{T}(\bar{\mathbb{R}}) \subset \sigma(\mathcal{I}(\bar{\mathbb{R}}))$ follows from the above observations, because every open set in $\mathcal{T}(\bar{\mathbb{R}})$ can be expressed as a countable union of intervals of the form (a, b) and $[-\infty, a)$ and $(a, +\infty]$ with $a, b \in \bar{\mathbb{R}}$. □

3.1 Random variables

An $\bar{\mathbb{R}}$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \bar{\mathbb{R}}$ which is $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$-measurable in the sense that $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\bar{\mathbb{R}})$.