# Nonlinear dynamics & chaos

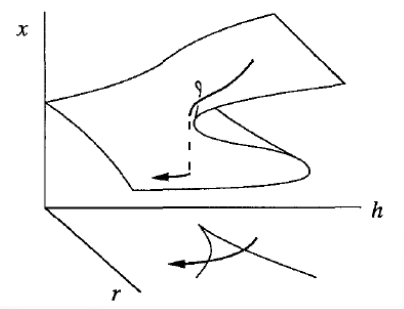
## Insect outbreak & Flows on the circle

**Lecture III**

# Recap

1D, three kinds of bifurcations: Normal form

1. Saddle-node: $\dot{x} = r + x^2$
2. Transcritical: $\dot{x} = rx - x^2$
3. Pitchfork:
   a) Supercritical $\dot{x} = rx - x^3$
   b) Subcritical $\dot{x} = rx + x^3$
4. Imperfect bifurcations
   for example $\dot{x} = h + rx - x^3$



# Insect outbreak

Spruce budworm: a pest in eastern Canada, where it attacks the leaves of the balsam fir tree

When outbreak occurs the budworm can kill most of the fir trees in the forest in about four years.

# Insect outbreak

Model by Ludwig et al. (1978)

**Time scale separation**: budworm population evolves on a *fast* time scale, trees grow and die on a *slow* time scale $\rightarrow$ for the purpose of budworm dynamics, forest variables may be treated as constants

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$

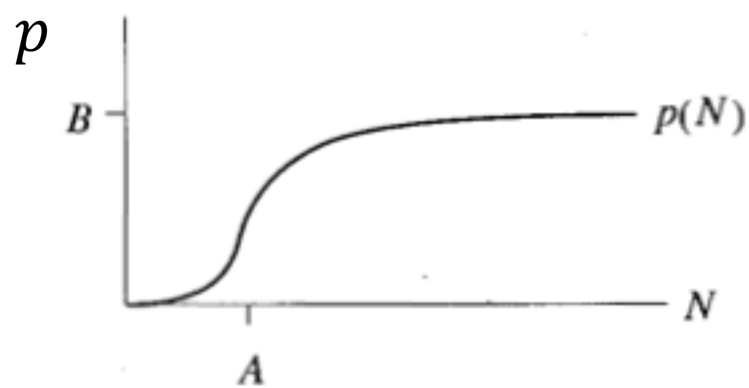
$N(t)$ = budworm population

$p(N)$ = death rate due to predation

In the absence of predation, $N(t)$ grows logistically, with growth rate $R$ and carrying capacity $K$ (foliage, slowly drifting).

# Insect outbreak

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$




$p(N)$ is small when $N$ is small (birds seek food elsewhere). $p(N)$ becomes relevant when budworm population exceeds some critical level $A$ and saturates (birds are eating as fast as they can).

# Insect outbreak

Ludwig et al. assumed the form ($A$, $B > 0$):

$$p(N) = \frac{BN^2}{A^2 + N^2} \quad \rightarrow \quad \dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$

Dimensionless formulation: one could use either $x = N/A$ or $x = N/K$. In order to push the **dimensionless groups** into the logistic part to ease the graphical analysis, we choose

$$x = \frac{N}{A} \quad \rightarrow \quad \frac{A}{B}\frac{dx}{dt} = \frac{R}{B}Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1 + x^2}$$

$$\tau = \frac{Bt}{A},\ r = \frac{RA}{B},\ k = \frac{K}{A} \quad \rightarrow \quad \frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

$r$ and $k$ are the dimensionless growth rate and carrying capacity

# Insect outbreak

Dimensionless form
$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

Fixed points

FP $x^* = 0$ is unstable for any choice of the parameters: $dx/d\tau = f(x)$; $f'(x=0) = r > 0$. Exponential growth of $x$ in the absence of predation.

Other FPs:
$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$$

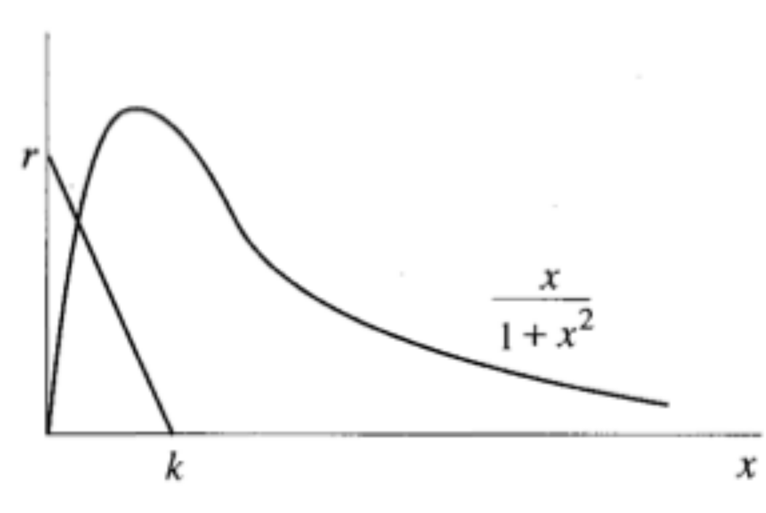
Graphical method
$$\begin{cases} y = & r\left(1 - \frac{x}{k}\right) \\ y = & \frac{x}{1+x^2} \end{cases}$$

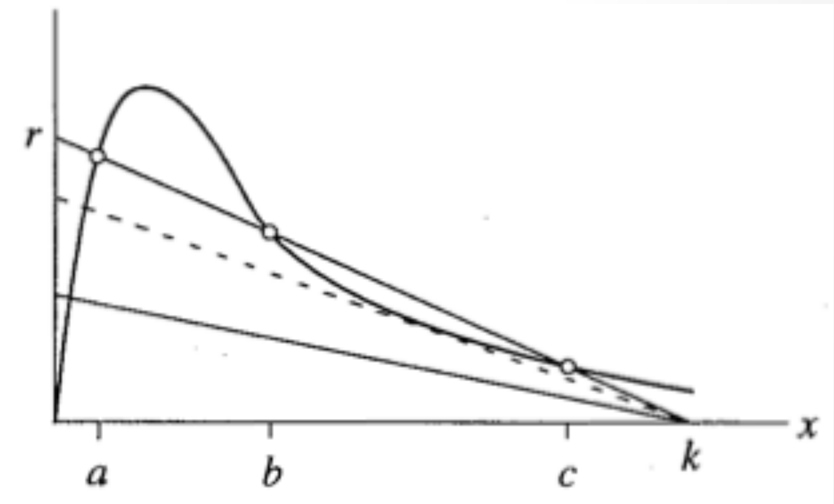
# Insect outbreak

Graphical method

$$\begin{cases} y = & r\left(1-\frac{x}{k}\right) \\ y = & \frac{x}{1+x^2} \end{cases}$$

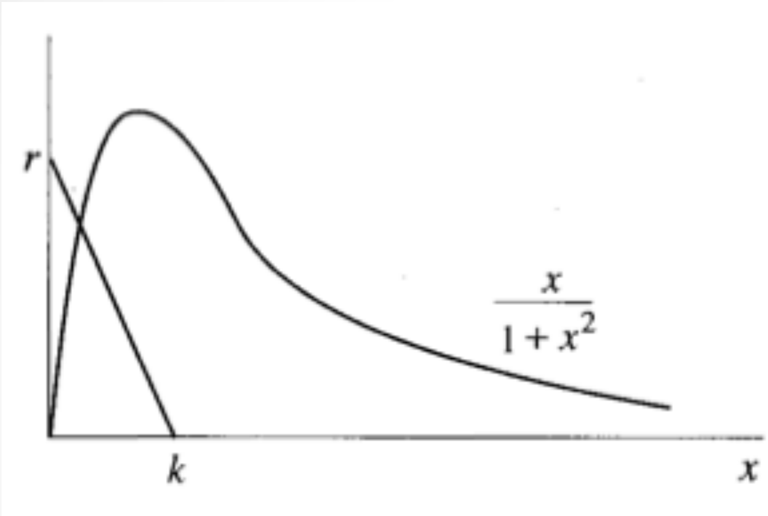
Thanks to the way we nondimensionalised, only the line moves as we vary parameters *r* and *k*.

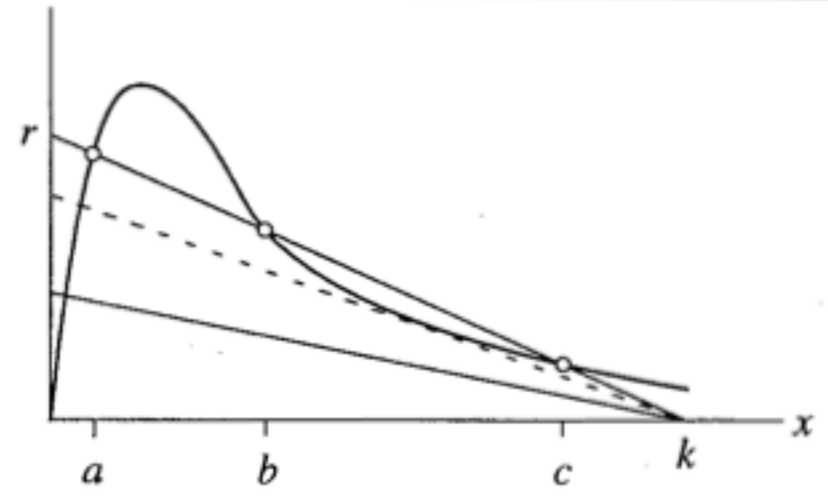



- For small *k* only one intersection for any r
- For large *k* one, two or three intersections are possible, depending on *r*

# Insect outbreak

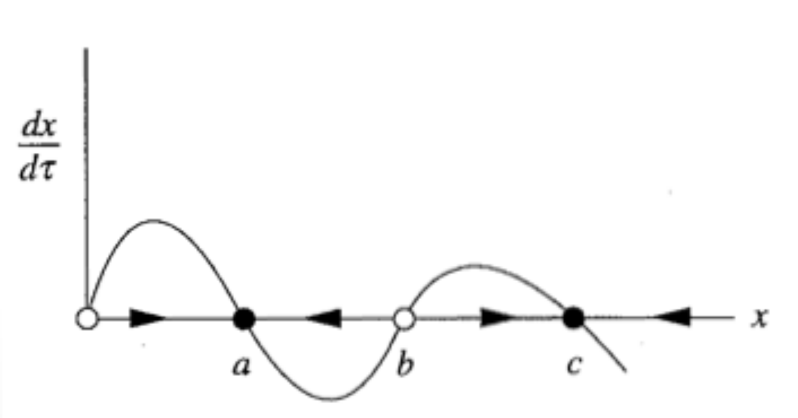



**Saddle-node bifurcation** when straight line is tangent to curve (*b* coincides with *c*; dashed line).

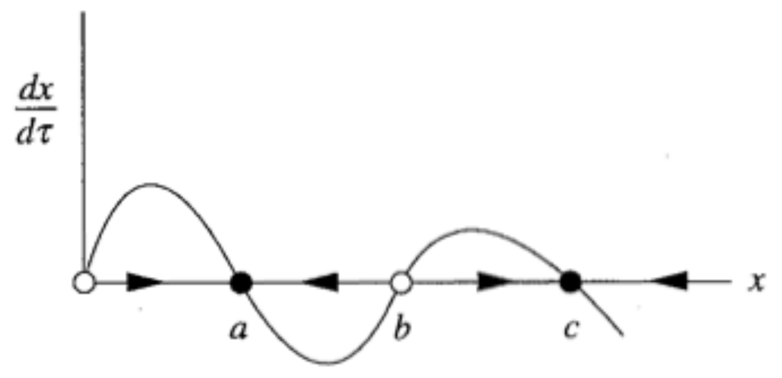
$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

Stability of fixed points



# Insect outbreak

Stability of fixed points




$a$ = refuge level of budworm population
$c$ = outbreak level of budworm population

For pest control one should keep the population near $a$ and away from $c$!

Outbreak occurs for the initial condition: $x_0 > b$ (threshold).

Outbreak can also be triggered by a saddle-node bifurcation.
$r$, $k$ grow large → $a$ disappears and $x$ jumps to $c$.

# Insect outbreak

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

Bifurcation curves will be in $(k, r)$ space.

Condition for bifurcation: straight line intersects curve tangentially

$$\begin{cases} \frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right] &= \frac{d}{dx}\left[\frac{x}{1+x^2}\right] \\ r\left(1-\frac{x}{k}\right) &= \frac{x}{1+x^2} \end{cases}$$

$$\begin{cases} -\frac{r}{k} &= \frac{1-x^2}{(1+x^2)^2} \\ r\left(1-\frac{x}{k}\right) &= \frac{x}{1+x^2} \end{cases}$$

# Insect outbreak

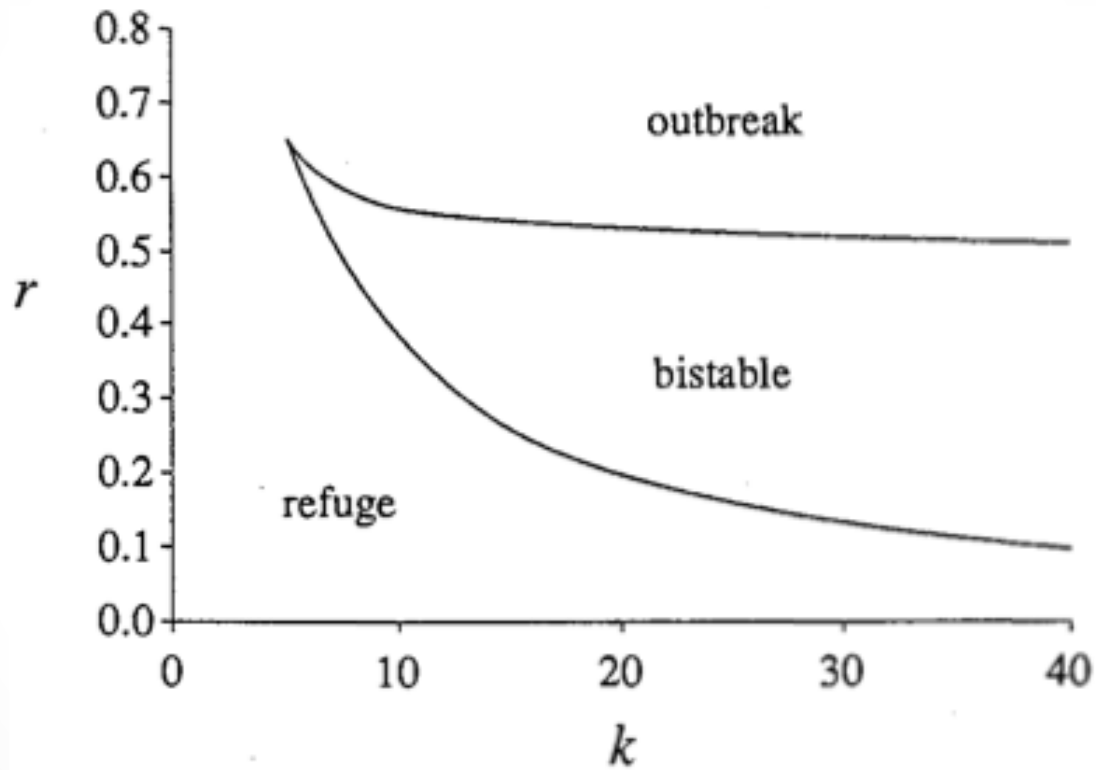
Bifurcation curves

$$\begin{cases} r &= \frac{2x^3}{(1+x^2)^2} \\ k &= \frac{2x^3}{x^2-1} \end{cases}$$

Parametric equations to derive the bifurcation relation between $r$ and $k$: for a given value of $x$, we compute $r(x)$ and $k(x)$ and plot the points on the $(k, r)$ plane. (Note: since $k > 0$, $x > 1$.)

# Insect outbreak

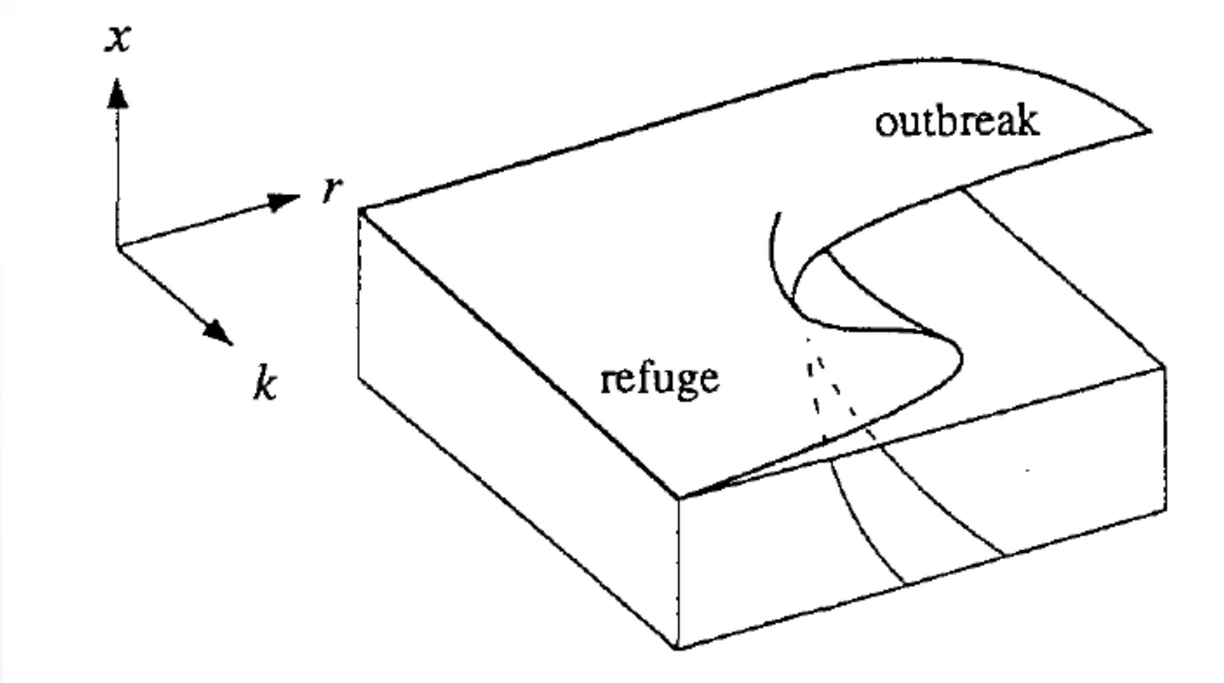
The regions in the *stability diagram* labelled by the existing **stable** fixed points.




- The refuge level is the only stable state for low $r$
- The outbreak level is the only stable state for large $r$
- For intermediate $r$ both stable states exist

# Insect outbreak

The stability diagram is a projection of the *cusp catastrophe surface*.



# Insect outbreak

## Comparison with observation

Determine biologically plausible values of dimensionless variables $r = RA/B$ and $k = K/A$ .

$r$ increases as the forest grows, $k$ remains fixed (Ludwig et al., 1978).

$S$ = average size of the trees $\rightarrow$ total surface area of the branches in a stand. Carrying capacity and half-saturation proportional to $S$. For birds the relevant quantity $A'$ is in dimensions of budworms per unit area:

$$K = K'S, \qquad A = A'S \quad \rightarrow \quad r = \frac{RA'}{B}S, \qquad k = \frac{K'}{A'}$$

Experimental observations: $k \approx 300$, $r < 1/2$ (bistable region)

As the forest grows $S$ increases $\rightarrow$ $r$ increases (danger of outbreak).

# Flows on the circle

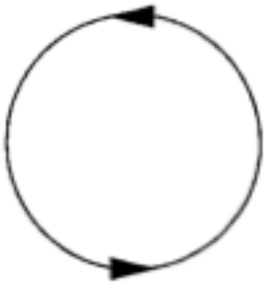
Vector field on the circle

$$\dot{\theta} = f(\theta)$$

$\theta = \text{point on the circle}$

$\dot{\theta} = \text{angular velocity at that point}$

Flows on the circle are like flows on the line with a new important property: by flowing in one direction a particle can eventually return to its starting place. → periodicity



The most basic model of systems that can oscillate.

# Example I

Sketch the vector field on the circle corresponding to

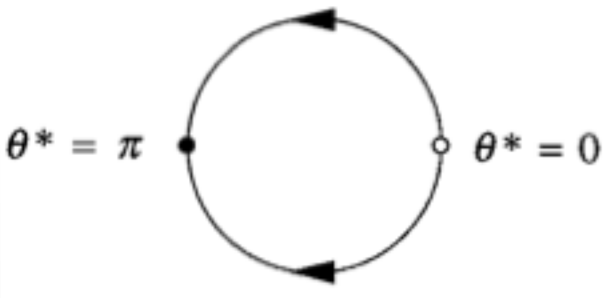
$$\dot{\theta} = \sin\theta$$

Fixed points

$$\sin\theta^* = 0 \quad \rightarrow \quad \theta^*_{1,2} = 0, \pi$$

Linear stability

$$f'(\theta^*) = \cos\theta^* \quad \rightarrow \quad \begin{array}{lcll} \cos\theta_1^* & = & +1 > 0 & \text{unstable point} \\ \cos\theta_2^* & = & -1 > 0 & \text{stable point} \end{array}$$



# Example II

Question: Can the linear system

$$\dot{\theta} = \theta$$

be regarded as a vector field on the circle, if $-\infty < \theta < +\infty$?

Answer: No, velocity is not uniquely defined, $\theta = 0, 2\pi$ coincide on the circle, but velocities are different!

Q: What happens if $-\pi < \theta < +\pi$ ?

A: No cigar. The ends of the range correspond to the same point on the circle, so there is a discontinuity in the velocity at that point, i.e. the vector field is not smooth. (No problem if the vector field is on the line.)

Definition: A vector field on the circle is a rule that assigns a unique velocity vector to each point on the circle $\rightarrow f(\theta)$ must be a periodic function with period $2\pi$.

# Uniform oscillator

The angle (or phase) changes uniformly

$$\dot{\theta} = \omega$$

Solution:

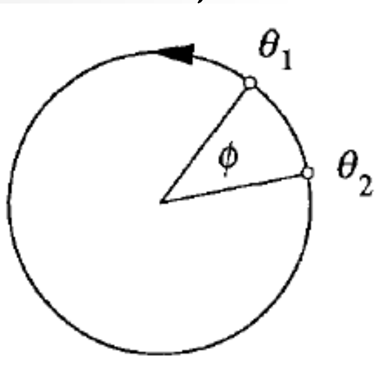
$$\theta(t) = \omega t + \theta_0$$

$\omega$ is the angular frequency.

$T = 2\pi/\omega$ is the period of the oscillation.
**Note**: There is no amplitude (or the amplitude is constant). If the amplitude is changed, the phase space would be two dimensional (phase plane).

# Example

Two joggers, Speedy and Pokey, are running at a steady pace around a circular track. It takes Speedy $T_1$ seconds to run once around the track, whereas it takes pokey $T_2 > T_1$ seconds. How long does it take for Speedy to lap Pokey once, assuming that they start together?




$$\dot{\theta}_1 = \omega_1, \qquad \omega_1 = 2\pi/T_1$$
$$\dot{\theta}_2 = \omega_2, \qquad \omega_2 = 2\pi/T_2$$

Phase difference:

$$\phi = \theta_1 - \theta_2 \quad \rightarrow \quad \dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$$

Time for $\phi$ to increase by $2\pi$

$$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}$$

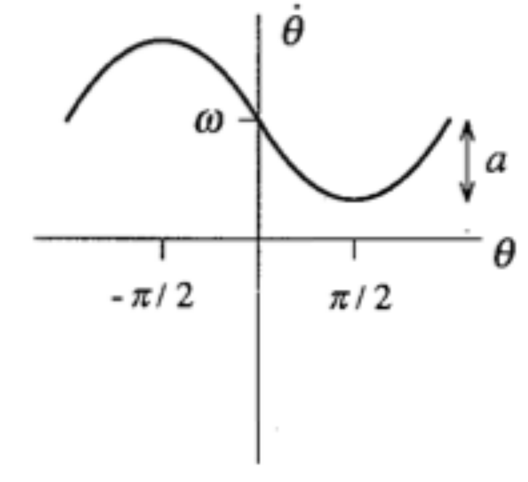
# Nonuniform oscillator

$$\dot{\theta} = \omega - a\sin\theta$$

It is very common, for example, in:

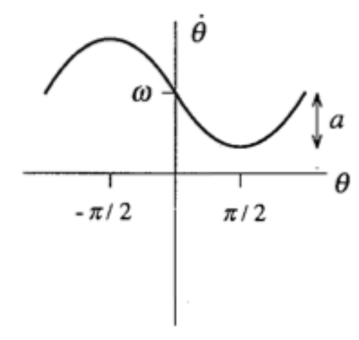
1) *Electronics* (phase-locked loops)
2) *Biology* (oscillating neurons, firefly flashing rhythm, human sleep-wake cycle)
3) *Condensed-matter physics* (Josephson junction, charge-density waves)
4) *Mechanics* (Overdamped pendulum driven by a constant torque)

Assume: $\omega > 0, a \geq 0$ (results for negative values are similar).



# Nonuniform oscillator

$$\dot{\theta} = \omega - a \sin\theta$$



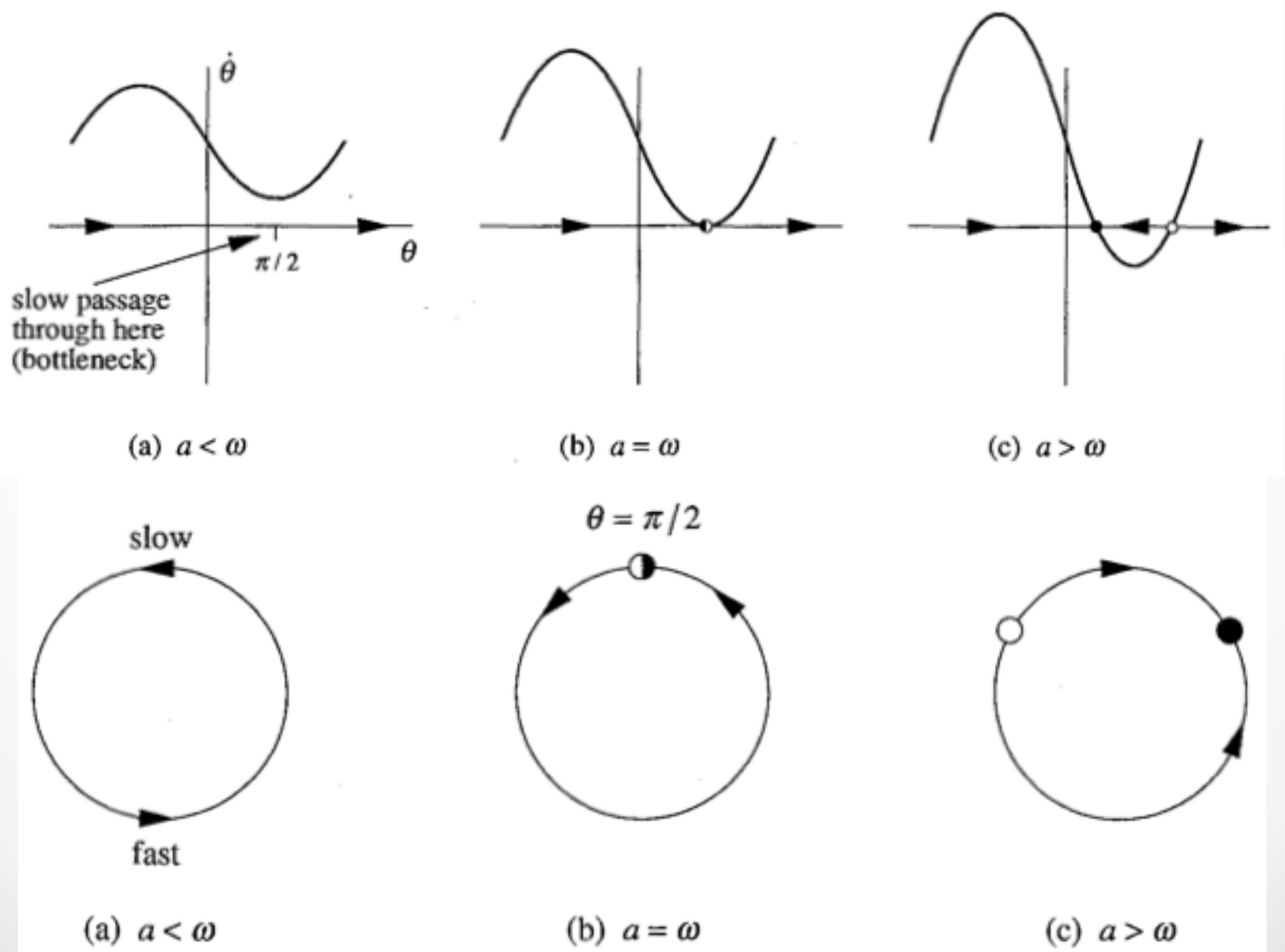

$a = 0$: uniform oscillator

$a > 0$: flow is not uniform: fastest at $\theta = -\pi/2$, slowest at $\theta = \pi/2$

- If $a$ is slightly less than $\omega$, it takes a long time for the phase point to pass through the bottleneck near $\theta = \pi/2$.
- If $a = \omega$, the system stops oscillating: a half-stable fixed point has been born in a saddle-node bifurcation at $\theta = \pi/2$.
- If $a > \omega$, a pair of fixed points appears (one stable, the other unstable): all orbits are attracted by the stable fixed point as $t \to \infty$.

# Nonuniform oscillator

$$\dot{\theta} = \omega - a\sin\theta$$



# Nonuniform oscillator

$$\dot{\theta} = \omega - a \sin \theta$$

Fixed points for $a > \omega$

$$\sin \theta^* = \omega / a \quad \rightarrow \quad \cos \theta^* = \pm \sqrt{1 - (\omega / a)^2}$$

Linear stability

$$f'(\theta^*) = -a \cos \theta^* = \mp a \sqrt{1 - (\omega / a)^2}$$

$$\cos \theta_1^* = +\sqrt{1 - (\omega / a)^2} \quad \rightarrow \quad \text{stable point}$$

$$\cos \theta_2^* = -\sqrt{1 - (\omega / a)^2} \quad \rightarrow \quad \text{unstable point}$$

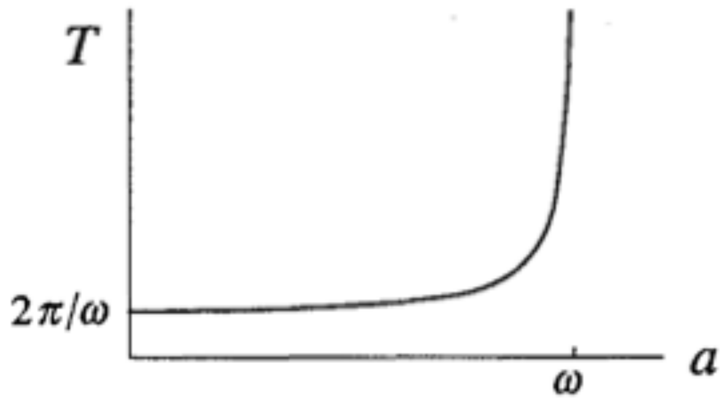
# Nonuniform oscillator

$$\dot{\theta} = \omega - a\sin\theta$$

Oscillations for $a < \omega$: the period?

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

T versus a




- When $a = 0$, $T = 2\pi/\omega$ (uniform oscillator)
- When $a = \omega$, $T$ diverges

# Nonuniform oscillator

$$\dot{\theta} = \omega - a\sin\theta$$

Order of divergence of period $T$

Square-root scaling law

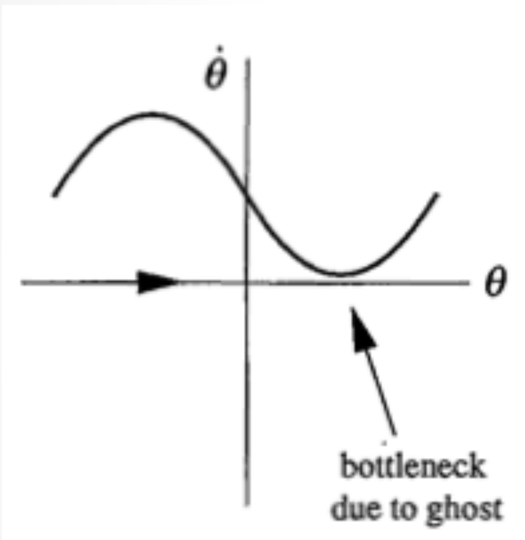
$$\sqrt{\omega^2 - a^2} = \sqrt{\omega + a}\sqrt{\omega - a} \approx \sqrt{2\omega}\sqrt{\omega - a} \quad \rightarrow \quad \lim_{a\to\omega^-} T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right)\frac{1}{\sqrt{\omega - a}}$$

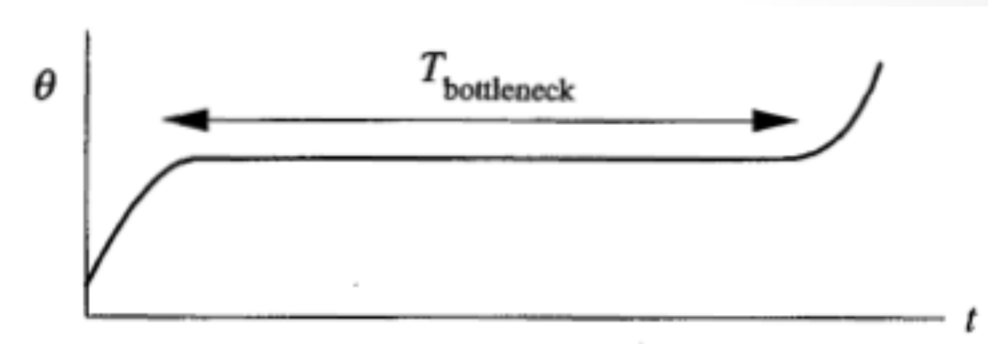
as $a \rightarrow \omega^-$

Very general feature of systems close to a saddle-node bifurcation: after the fixed points collide they disappear, however there is a saddle-node remnant (ghost) leading to slow passage through a bottleneck.

# Nonuniform oscillator

$$\dot{\theta} = \omega - a \sin \theta$$




Trajectory spends practically all its time getting through the bottleneck.

# Nonuniform oscillator

General scaling law for time to get through the bottleneck

Two observations:

1) What counts is the behavior of the velocity field $f(\theta)$ near its minimum, since the time spent there dominates over all other time scales of the problem
2) $f(\theta)$ looks parabolic near its minimum

Normal form for a saddle-node bifurcation:

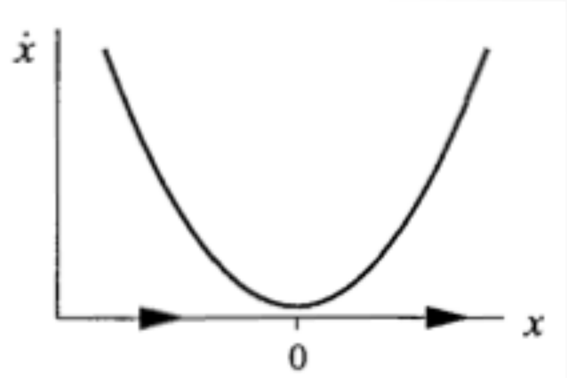
$$\dot{x} = r + x^2, \qquad 0 < r \ll 1$$

($r$ is the distance from the bifurcation)

# Nonuniform oscillator

$$\dot{x} = r + x^2, \qquad 0 < r \ll 1$$

$$\arctan z = \int_0^z \frac{dt}{1+t^2}$$




$$t = \frac{x}{\sqrt{r}}$$

(An unfortunate choice for denoting the change of variable, sorry. *t* here is just an arbitrary variable to do the integral.)

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{\infty} \frac{dx}{\dot{x}} = \int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{1}{\sqrt{r}} \int_{-\infty}^{\infty} \frac{dt}{1+t^2}$$

$$= \frac{\arctan(\infty) - \arctan(-\infty)}{\sqrt{r}} = \frac{\pi}{\sqrt{r}}$$

# Example

Estimate the period of

$$\dot{\theta} = \omega - a\sin\theta$$

in the limit $a \to \omega^-$ **using the normal form method**.

The period is essentially the time required to go through the bottleneck.

Taylor expansion about $\theta = \pi/2$, where the bottleneck occurs

$$\phi = \theta - \pi/2 \quad \to \quad \dot{\phi} = \omega - a\sin\left(\phi + \frac{\pi}{2}\right) = \omega - a\cos\phi = \omega - a + \frac{1}{2}a\phi^2 + \cdots$$

$$x = \left(\frac{a}{2}\right)^{1/2}\phi, \quad r = \omega - a \quad \to \quad \left(\frac{2}{a}\right)^{1/2}\dot{x} \approx r + x^2$$

# Example

$$\left(\frac{2}{a}\right)^{1/2} \dot{x} \approx r + x^2$$

Separate the variables to get

$$T \approx \sqrt{\frac{2}{a}} \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \sqrt{\frac{2}{a}} \frac{\pi}{\sqrt{r}}$$

Close to the saddle-node ghost:

$$r = \omega - a, \quad a \to \omega^{-} \quad \to \quad \frac{2}{a}\Big|_{a \to \omega^{-}} = \frac{2}{\omega}$$

$$\lim_{a \to \omega^{-}} T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right) \frac{1}{\sqrt{\omega - a}}$$

# Overdamped pendulum

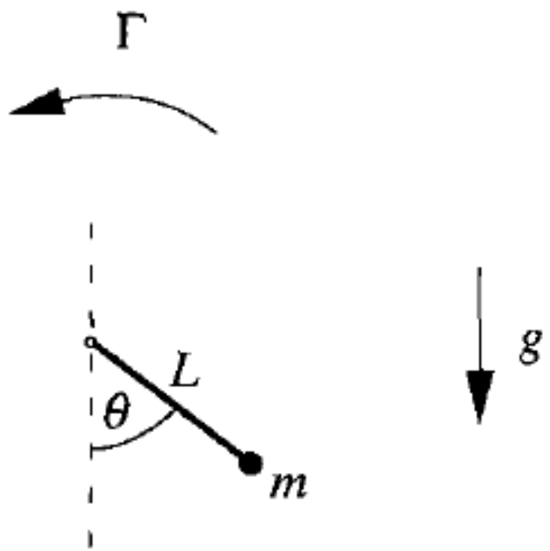


**Figure 4.4.1**

Newton's law:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

Overdamped limit:

$$b\dot{\theta} + mgL\sin\theta = \Gamma$$

Nondimensionalise:

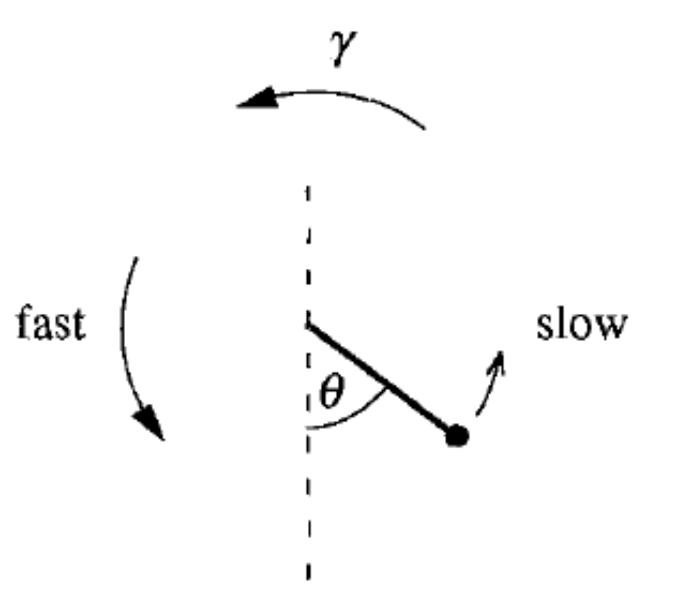
$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin\theta$$

Let $\tau = \frac{mgL}{b}t,\ \gamma = \frac{\Gamma}{mgL}$ $\leftarrow$ dimensionless groups

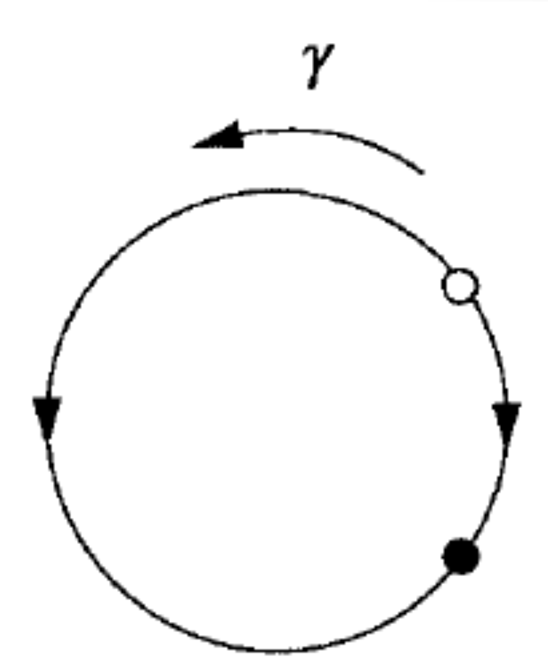
$$\Rightarrow \theta' = \gamma - \sin\theta, \text{where } \theta' = \frac{d\theta}{d\tau}.$$

# Overdamped pendulum

$$\theta' = \gamma - \sin\theta$$




Continual overturning when $\gamma > 1$. When $\gamma \to 1^+$, a FP appears at $\theta^* = \pi/2$. This splits into two when $\gamma < 1$.




Saddle-node bifurcation.

# Fireflies

Thousands of male fireflies gather in trees and flash on and off to attract females flying overhead. The males **synchronise**.

$\theta(t)$ is the phase of the flashing rhythm; $\theta = 0$ corresponds to the instant when a flash is emitted.

No stimuli: $\dot{\theta} = \omega$

# Fireflies

A periodic stimulus: $\dot{\Theta} = \Omega$

Model: $\dot{\theta} = \omega + A \sin(\Theta - \theta)$, where $A > 0$.

(=resetting strength)

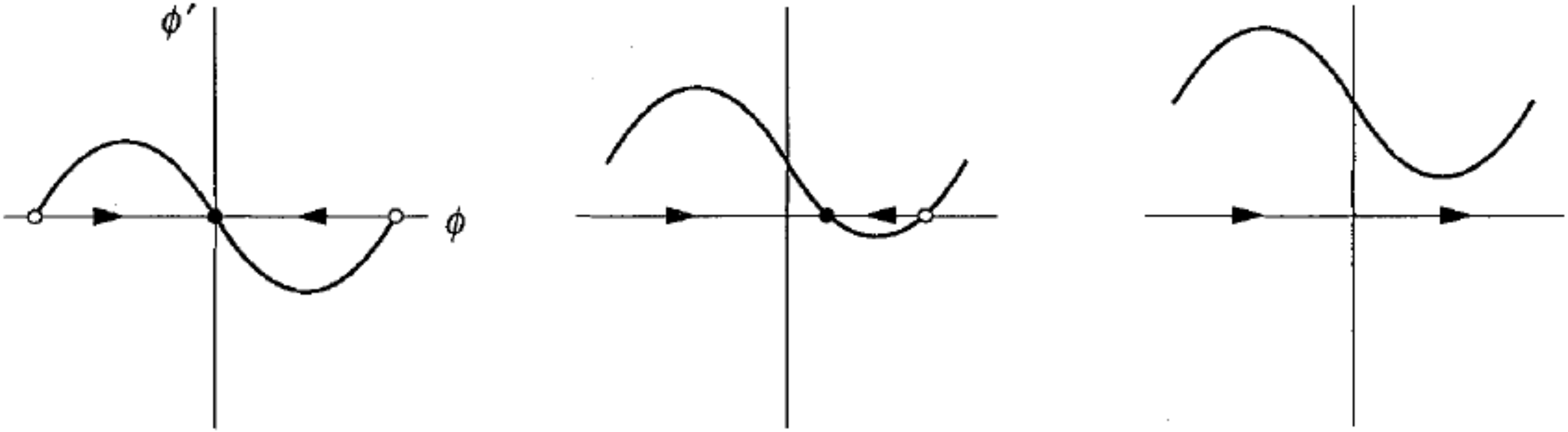
Dynamics of the phase difference:

$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A \sin \phi$ (nonuniform oscillator)

Nondimensionalizing: $\tau = At, \ \mu = \dfrac{\Omega - \omega}{A}$

$\Rightarrow \phi' = \mu - \sin \phi \qquad (\phi' = d\phi/d\tau)$

# Fireflies




(a) $\mu = 0$ (b) $0 < \mu < 1$ (c) $\mu > 1$

$$\tau = At,\ \ \mu = \frac{\Omega - \omega}{A}$$

(a) Simultaneous flashing, (b) phase-locking to the stimulus, (c) phase drift.

The range of entrainment: $\omega - A \leq \Omega \leq \omega + A$

# Fireflies

FP gives the phase difference during entrainment

$$\sin\phi^* = \frac{\Omega - \omega}{A}$$

Period of the **phase drift:**

$$T_{\text{drift}} = \int dt = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A\sin\phi}$$

Comparing with the previous solution for nonuniform oscillator, we get

$$T_{\text{drift}} = \frac{2\pi}{\sqrt{(\Omega - \omega)^2 - A^2}}$$

# Next time: 2D