Nonlinear dynamics & chaos

Linear systems & intro to phase plane

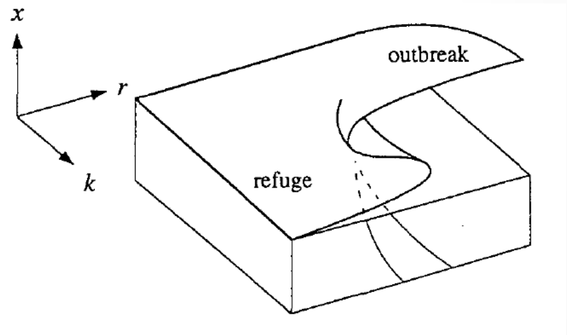
Lecture IV

Recap

Flows on the line. Analysis of growth model having logistic growth part and predation. Insect outbreak.

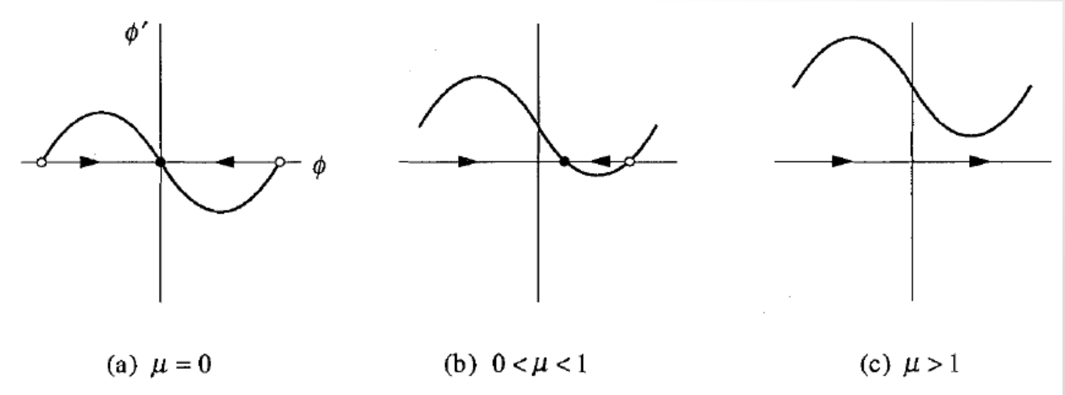
$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$

Emergence of a cusp catastrophe:



Flows on the circle to describe one-dimensional periodic systems.

Attemped phase-locking of fireflies:



Part II

Two-Dimensional Flows

Linear systems

In one-dimensional spaces flow is confined: trajectories are forced to move monotonically or remain constant.

In higher-dimensional spaces there are many possibilities.

First step: linear systems in two dimensions.

Interesting in their own right, but particularly important for the classification of fixed points of nonlinear systems.

Linear systems

Two-dimensional linear system

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

a, *b*, *c*, and *d* are parameters.

Matrix form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Linear systems

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

Linear system $\rightarrow$ if $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions, any linear combination $c_1\mathbf{x_1}+ c_2\mathbf{x_2}$ is also solution

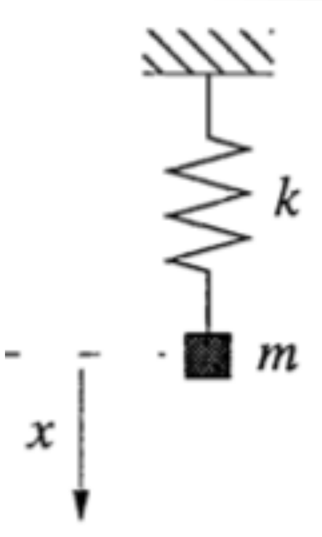
$\dot{\mathbf{x}} = 0$ when $\mathbf{x} = 0 \quad \rightarrow \quad \mathbf{x}^* = 0$ always fixed point, $\forall A$

Example I

Simple harmonic oscillator

$$m\ddot{x} + kx = 0$$

m = mass, k = spring constant, x = displacement of mass from equilibrium



Analytical solution in terms of sines and cosines.

Phase plane analysis

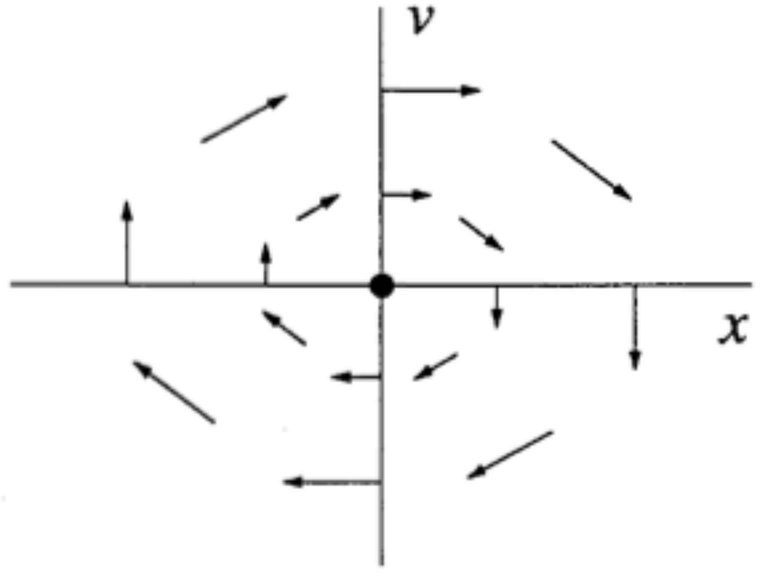
$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x \end{aligned}$$

Example I

Simple harmonic oscillator

$$\omega^2 = \frac{k}{m} \quad \rightarrow \quad \begin{array}{rcl} \dot{x} & = & v \\ \dot{v} & = & -\omega^2 x \end{array}$$

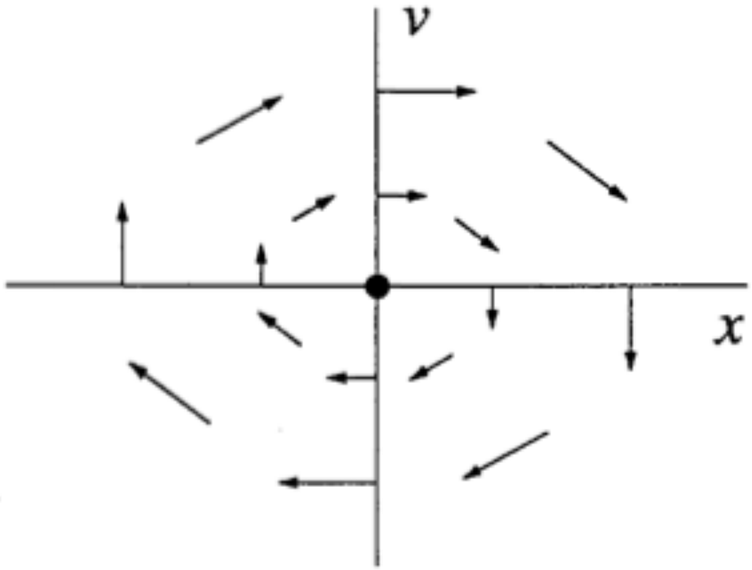
Vector field



The origin is a fixed point.

Example I

Simple harmonic oscillator

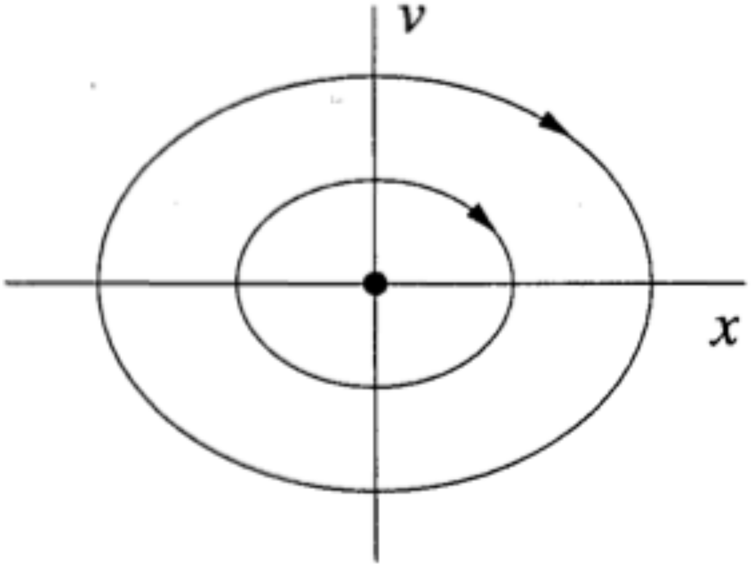


Phase point initiating anywhere (except the origin) would circulate around the origin and return to its starting point.

Example I

Simple harmonic oscillator

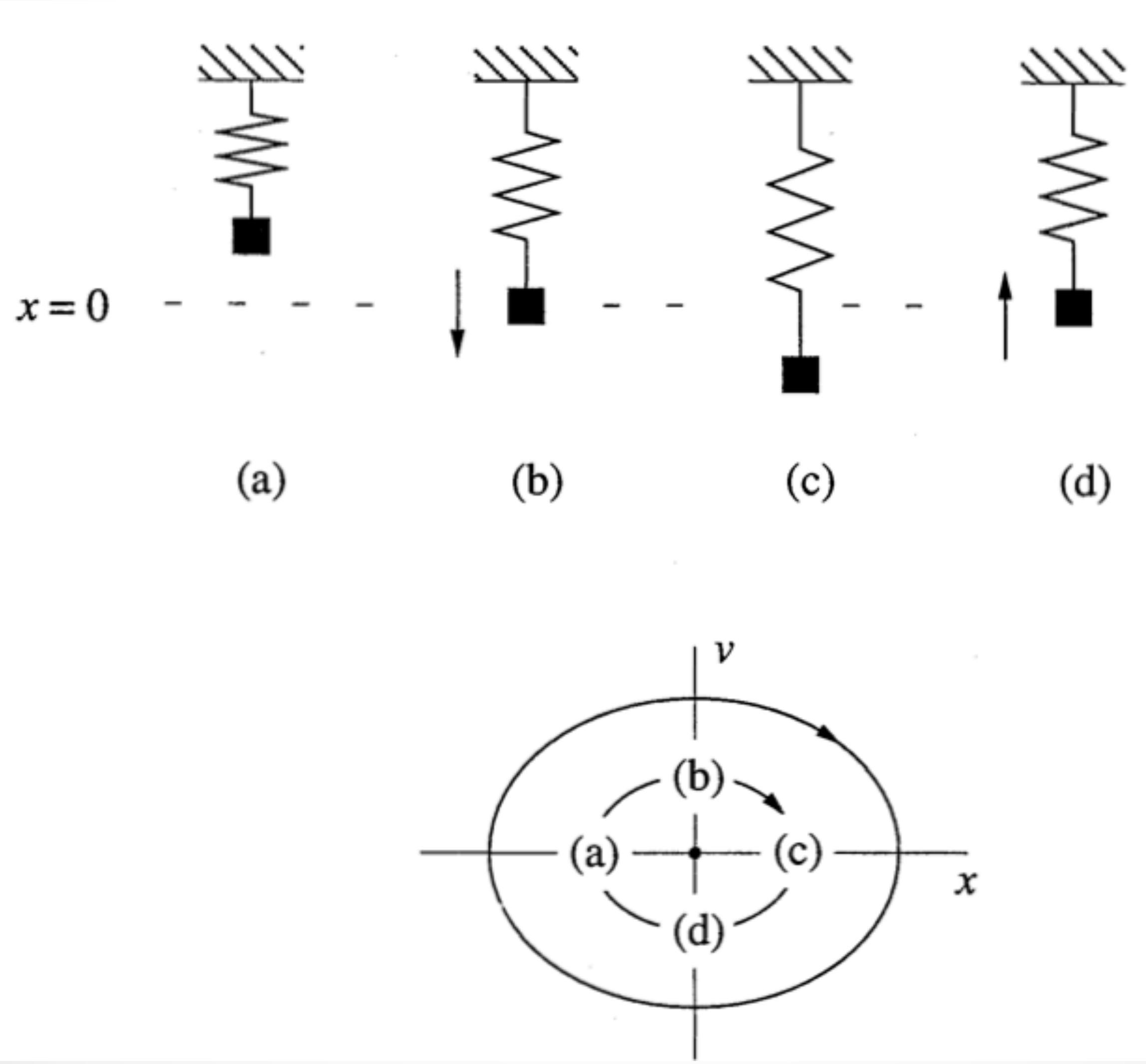
Phase portrait



Closed orbits correspond to periodic oscillations of the mass.

Example I

Simple harmonic oscillator



Example II

Solve the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \quad \rightarrow \quad \mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned}$$

Equations are decoupled: they can be solved individually.

Example II

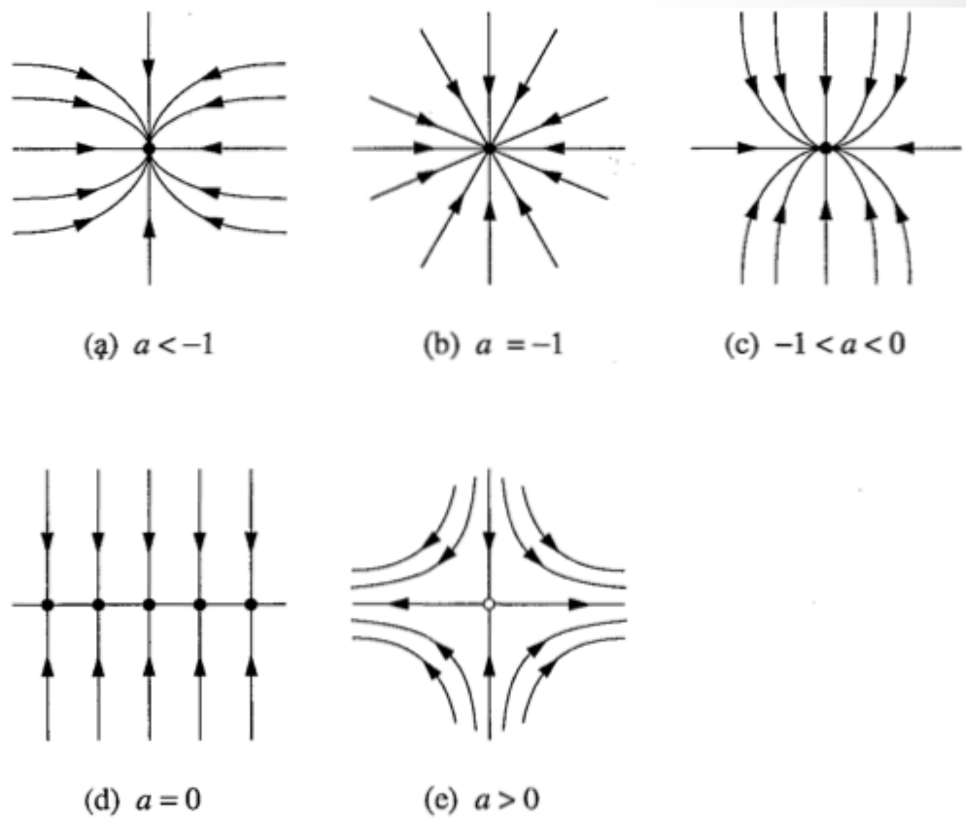
$$\begin{aligned}\dot{x} &= ax \\ \dot{y} &= -y\end{aligned}$$

- $a < -1 \rightarrow x(t)$ decays faster than $y(t)$ ($\boldsymbol{x}^*=0$ stable node)
- $a = -1 \rightarrow$ straight lines (ratio $x(t)/y(t)$ is constant, $\boldsymbol{x}^*=0$ star)
- $-1 < a < 0 \rightarrow y(t)$ decays faster than $x(t)$ ($\boldsymbol{x}^*=0$ stable node)
- $a = 0 \rightarrow x(t)$ is constant, trajectories are vertical (x-axis is line of fixed points)
- $a > 0 \rightarrow \boldsymbol{x}^*=0$ unstable $\rightarrow$ saddle point

Solution

$$\begin{aligned}x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}\end{aligned}$$

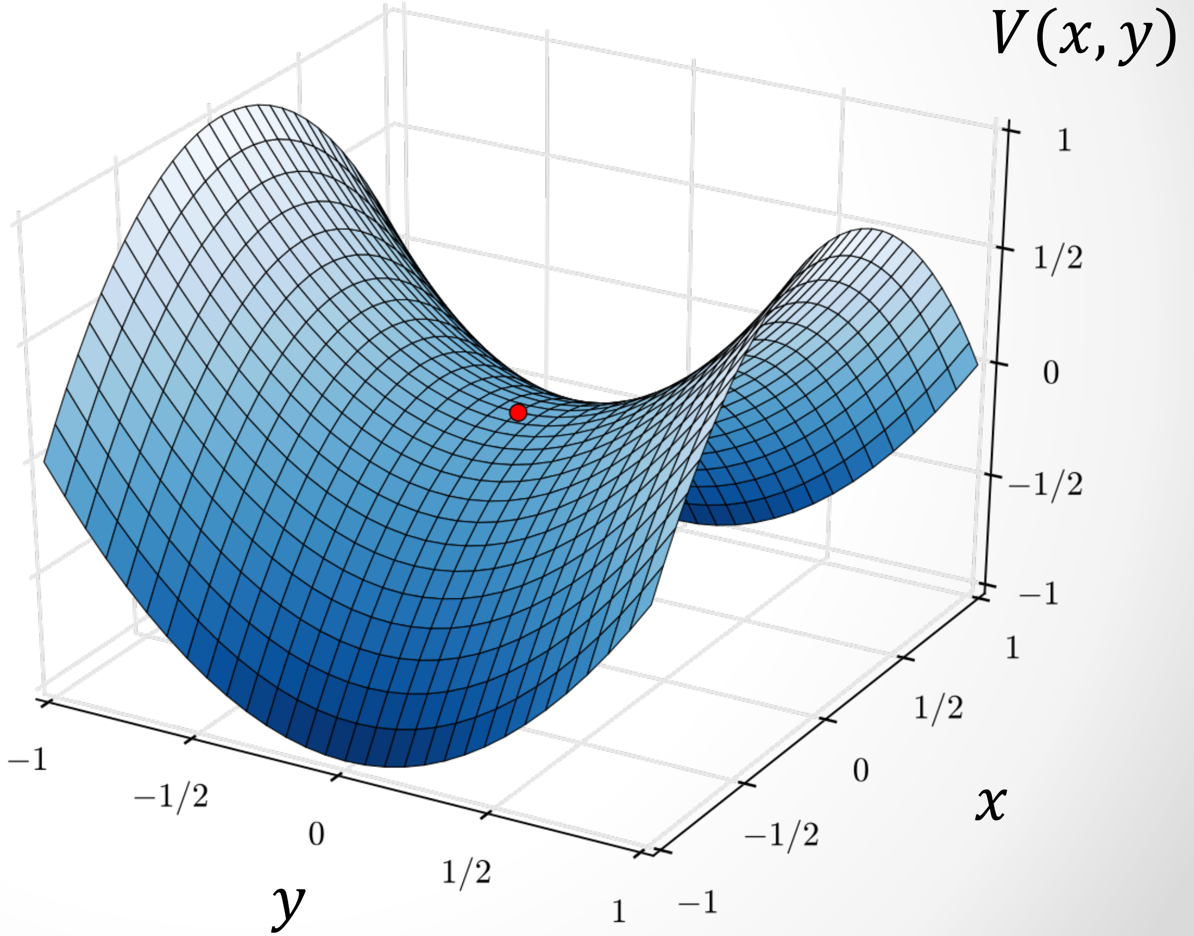
Phase portraits



Example II

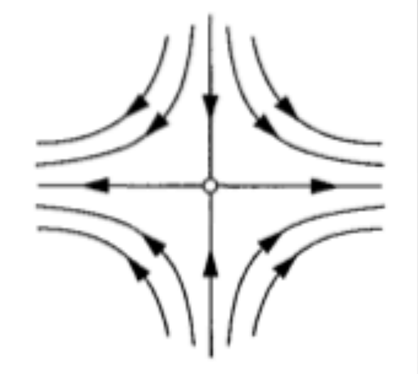
Saddle points

The Man known for his work on saddle points.



Example II

Saddle points

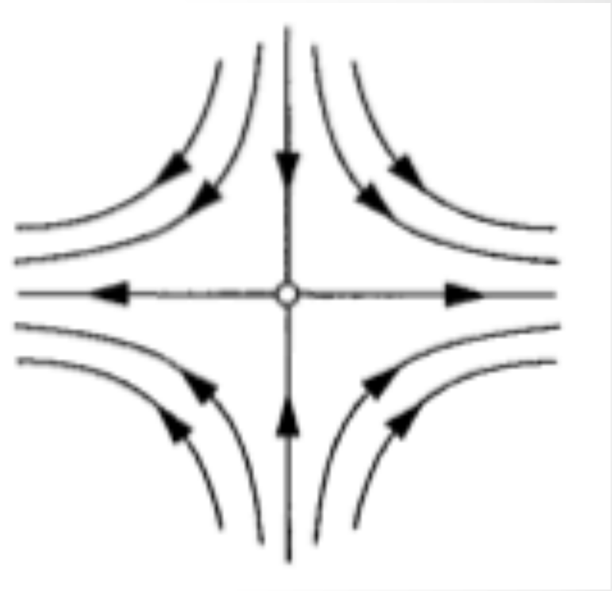


- Trajectories veer away from $\boldsymbol{x}^*$ and head out to infinity
- If a trajectory starts on the y-axis, the system's state converges to $\boldsymbol{x}^*$: the y-axis is the stable manifold of the saddle point (i.e. initial conditions bringing the system to $\boldsymbol{x}^*$)
- Stable manifold: set where one ends up when starting at $\boldsymbol{x}^*$ and runs the dynamics backward in time ($t \rightarrow -\infty$) ; here the y-axis
- Trajectories starting off the y-axis converge asymptotically ($t \rightarrow \infty$) to the x-axis that is the unstable manifold
- Unstable manifold: set of initial conditions leading to $\boldsymbol{x}^*$ when dynamics runs backward in time ($t \rightarrow -\infty$) ; here the x-axis

Example II

Saddle points

Manifold: A topological space, which is *homeomorphic* to Euclidian space $\mathbb{R}^m$ *locally*. (homeomorpism: there is continuos function f between spaces, and f^{-1} exists)



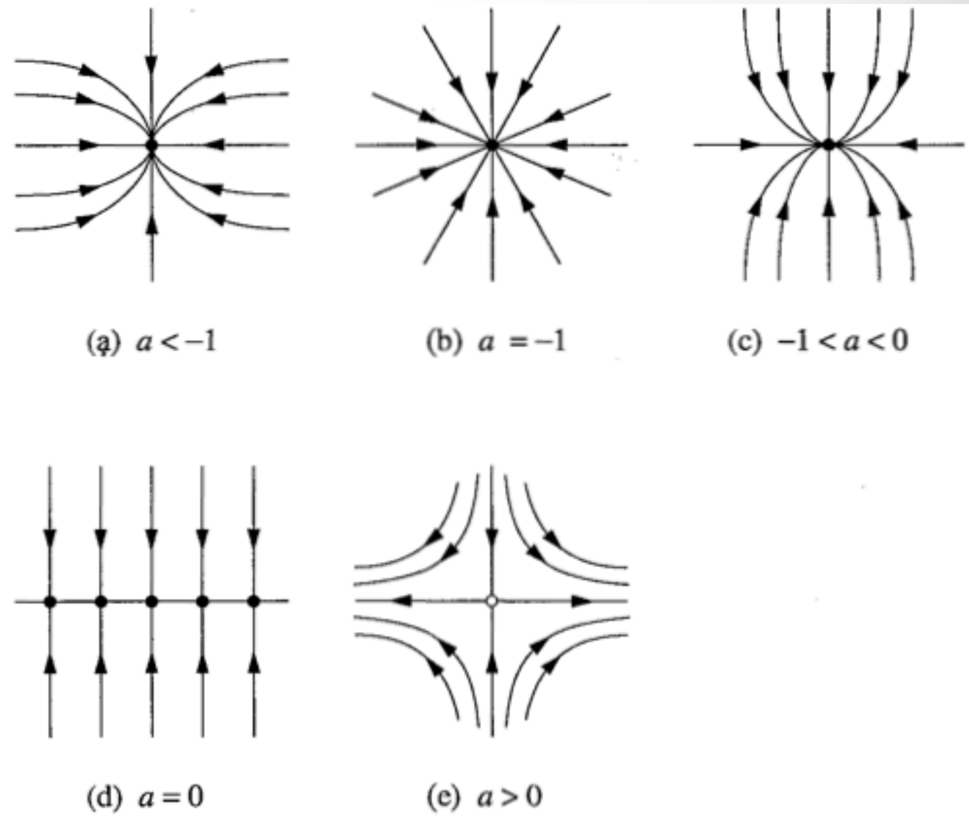
(Recall: the half-stable fixed point is the saddle point in 1D.)

Stability language

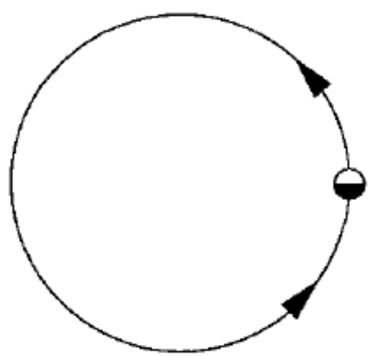
- x^* is an attracting fixed point when all trajectories starting near x^* approach it asymptotically: if all trajectories are attracted, the point is called globally attracting
- x^* is Lyapunov stable if all trajectories that start sufficiently close to x^* remain close to it at any time
- x^* is neutrally stable if it is Lyapunov stable but not attracting: nearby trajectories are neither attracted nor repelled from the point (common in mechanical systems without friction: e.g. simple harmonic oscillator)
- x^* is stable (or asymptotically stable) if it is both Lyapunov stable and attracting
- x^* is unstable if it is neither Lyapunov stable nor attracting

Stability language

- $x^*=0$ is (globally) attracting (a-c)
- $x^* = 0$ is Lyapunov stable (a-d)
- $x^*= 0$ is neutrally stable (d)
- $x^*= 0$ is unstable (e)



Typically, a globally attracting FP is also Lyapunov stable, but there are some rare counter examples:



Classification of linear systems

Goal: to classify all possible phase portraits that can occur.

From the previous example: Any straight line trajectories? More generally:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

- exponential motion along the line of vector **v**
- λ = growth rate

What does **v** stand for?

$$\begin{array}{lcl} \mathbf{x}(t) & = & e^{\lambda t}\mathbf{v} \\ \dot{\mathbf{x}} & = & A\mathbf{x} \end{array} \quad \rightarrow \quad \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v} \quad \rightarrow \quad A\mathbf{v} = \lambda\mathbf{v}$$

v is an eigenvector of A with eigenvalue λ

Classification of linear systems

Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda \mathbf{v}$$

Characteristic equation

$$\det(A - \lambda I) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

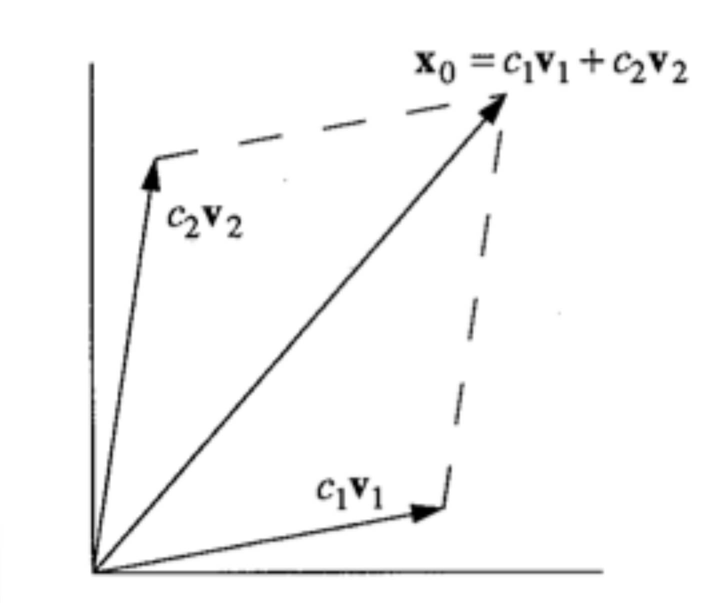
$$\begin{aligned} \tau &= \operatorname{trace}(A) = a + d \\ \Delta &= \det(A) = ad - bc \end{aligned}$$

Classification of linear systems

Solution(s)

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Typical situation: $\lambda_1 \neq \lambda_2 \rightarrow$ the corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent, so they span the entire plane.



If $\mathbf{x_0}$ is any initial condition:

$$\mathbf{x_0} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$$

Classification of linear systems

General solution for $x(t)$

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}$$

- It is a solution, because it is a linear combination of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$
- For $t = 0 \rightarrow \mathbf{x}(0) = \mathbf{x}_0$.
- Due to the uniqueness theorem, for this $\mathbf{x}(0)$ it must be the only solution.

Example I

Solve the initial value problem

$$\begin{aligned}\dot{x} &= x + y \\ \dot{y} &= 4x - 2y\end{aligned}$$

subject to the initial condition $(x_0, y_0) = (2, -3)$

$$\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 1 \\ 4 & -2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$$

$$\tau = -1, \;\; \Delta = -6 \;\; \rightarrow \;\; \lambda^2 + \lambda - 6 = 0 \;\; \rightarrow \;\; \lambda_1 = 2, \;\; \lambda_2 = -3$$

Example I

Solve the initial value problem: first the eigenvectors

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 = 2 \quad \rightarrow \quad \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3 \quad \rightarrow \quad \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Example I

General solution:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

Initial condition:

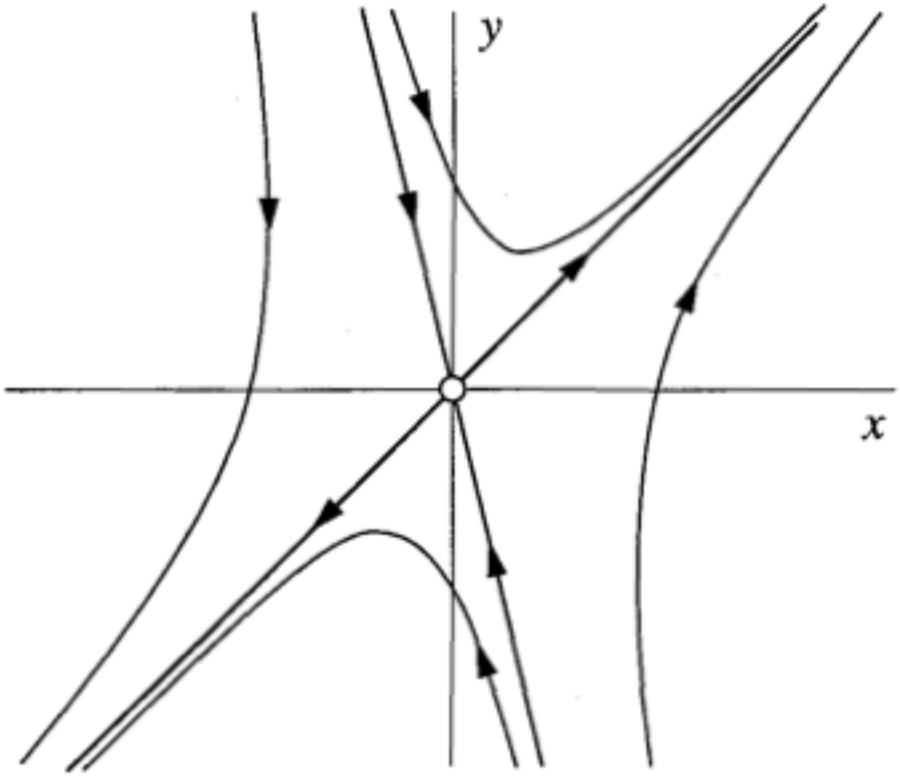
$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} 2 &= c_1 + c_2 \\ -3 &= c_1 - 4c_2 \end{aligned} \quad \rightarrow \quad c_1 = c_2 = 1$$

$$\begin{aligned} x(t) &= e^{2t} + e^{-3t} \\ y(t) &= e^{2t} - 4e^{-3t} \end{aligned}$$

Example I

Phase portrait



$$\lambda_1 = 2 \quad \rightarrow \quad \text{eigensolution grows exponentially}$$

$$\lambda_2 = -3 \quad \rightarrow \quad \text{eigensolution decays exponentially}$$

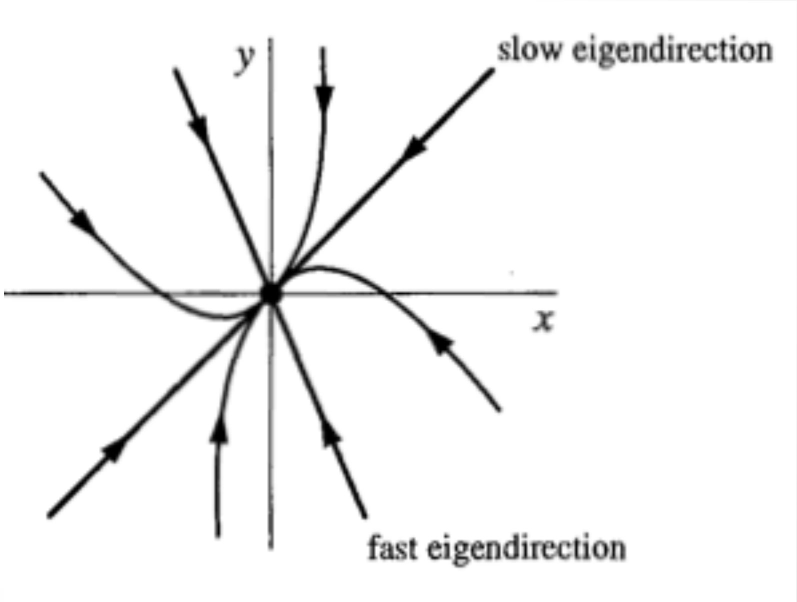
The origin is a saddle point.

Example II

$\lambda_2 < \lambda_1 < 0 \rightarrow$ phase portrait?

Both eigenvalues are negative, so the eigensolutions are both decaying exponentially towards the origin, which is stable.

Trajectories approach origin tangent to the slow eigendirection (the direction of the eigenvector with smaller $|\lambda|$).

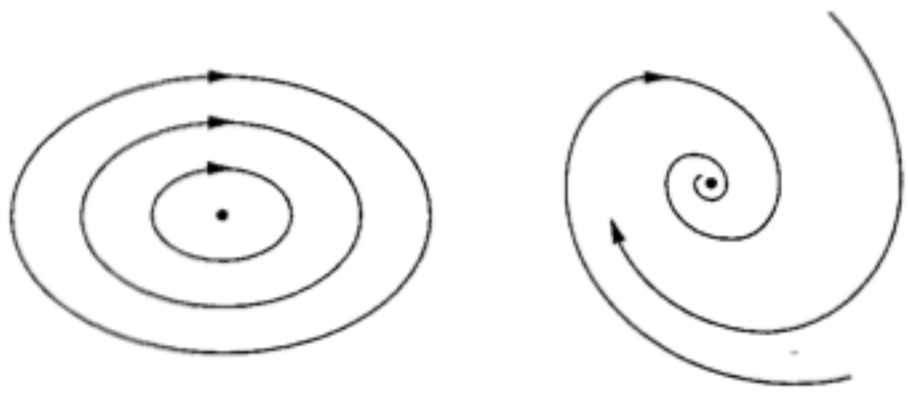


Example III

What happens if the eigenvalues are complex numbers?

Two possibilities:

- The fixed point is a center
- The fixed point is a spiral



(a) center (b) spiral

- Orbits around a center are closed → a center is neutrally stable.
- Orbits around a spiral are not closed → they may converge towards the fixed point or go away from it.

Example III

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

$$\text{if} \quad \tau^2 - 4\Delta < 0 \quad \rightarrow \quad \text{complex solutions}$$

$$\lambda_{1,2} = \alpha \pm i\omega \qquad \left(\alpha = \frac{\tau}{2}, \omega = \frac{\sqrt{4\Delta - \tau^2}}{2} \right)$$

If $\omega \neq 0$, the eigenvalues are distinct.

$$\mathbf{x}(t) = c_1 e^{(\alpha + i\omega)t} \mathbf{v_1} + c_2 e^{(\alpha - i\omega)t} \mathbf{v_2}$$

Example III

$$\mathbf{x}(t) = c_1 e^{(\alpha + i\omega)t}\mathbf{v_1} + c_2 e^{(\alpha - i\omega)t}\mathbf{v_2}$$

$$e^{(\alpha \pm i\omega)t} = e^{\alpha t}(\cos \omega t \pm i \sin \omega t)$$

- Exponentially decaying oscillations for $\alpha < 0$ (stable spiral).
- Exponentially growing oscillations for $\alpha > 0$ (unstable spiral).

If $\alpha = 0$ (purely imaginary eigenvalues), the solutions are periodic with period $T = 2\pi/\omega \rightarrow$ the fixed point is a center (= neutrally stable).

Example IV

What happens if $\lambda_1 = \lambda_2 = \lambda$?

Two possibilities:

- There are two independent eigenvectors corresponding to λ
- There is only one eigenvector corresponding to λ

If there are two independent eigenvectors, their linear combinations are also eigenvectors and span the whole space, so every vector is an eigenvector with the same eigenvalue λ . An arbitrary vector $\mathbf{x}_0$ can be written as

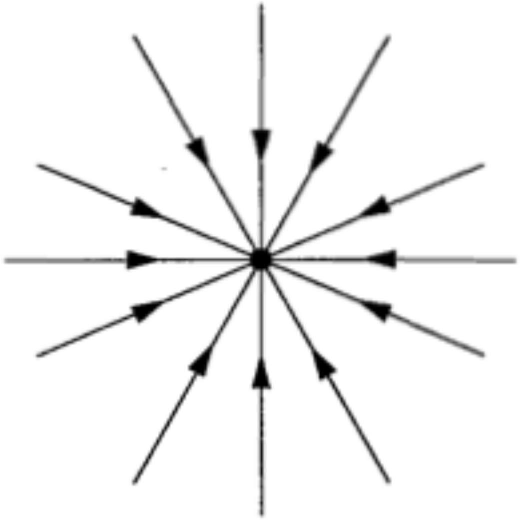
$$\mathbf{x_0} = c_1\mathbf{v_1} + c_2\mathbf{v_2} \quad \rightarrow \quad A\mathbf{x_0} = A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1\lambda\mathbf{v_1} + c_2\lambda\mathbf{v_2} = \lambda\mathbf{x_0}$$

So the arbitrary vector $\mathbf{x}_0$ is an eigenvector.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \rightarrow \quad \mathbf{x}(t) = e^{\lambda t}\mathbf{x_0}$$

Example IV

All trajectories are straight lines through the origin, which is a star node.



If $\lambda = 0$, the whole plane is filled with fixed points (trivial system $\dot{\mathbf{x}} = \mathbf{0}$).

Example IV

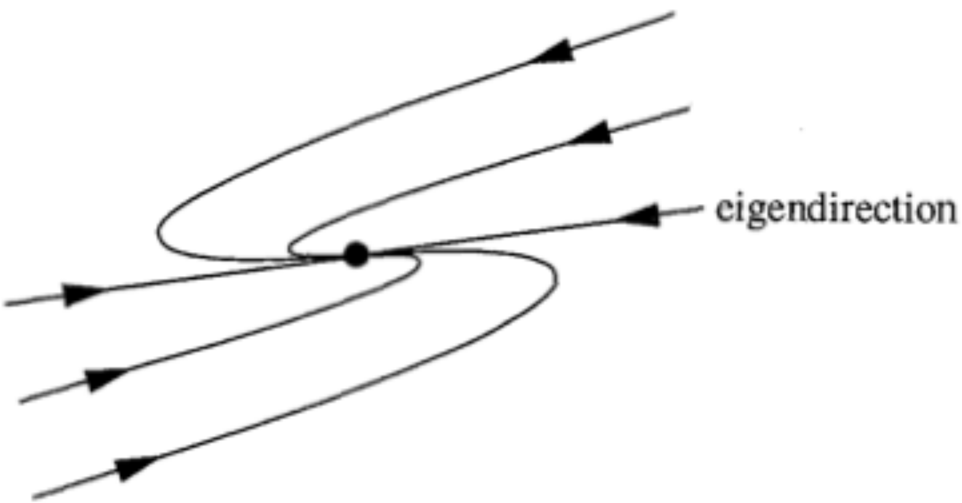
On the other hand, if there is **only one eigenvector** → the eigenspace corresponding to λ is one-dimensional

Example:

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

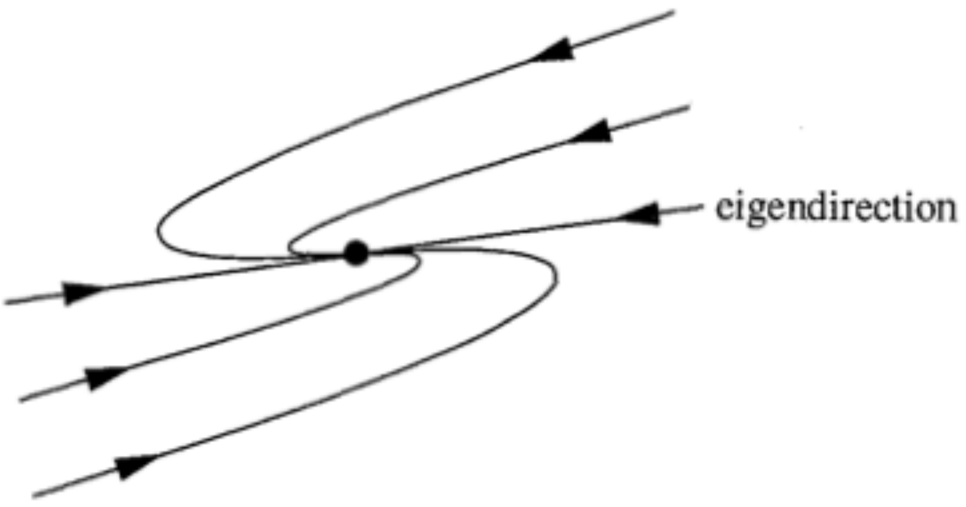
with $b \neq 0$

In this case the fixed point is a degenerate node.

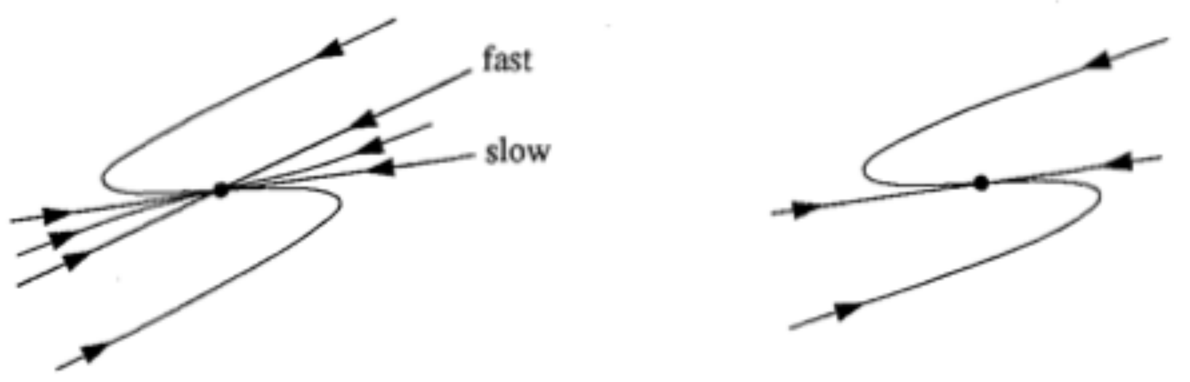


Example IV

As t ⟶ + ∞ or t ⟶ -∞ the trajectories become all parallel to the only available eigendirection.



A degenerate node can be viewed as the limit of an ordinary node when eigendirections converge.



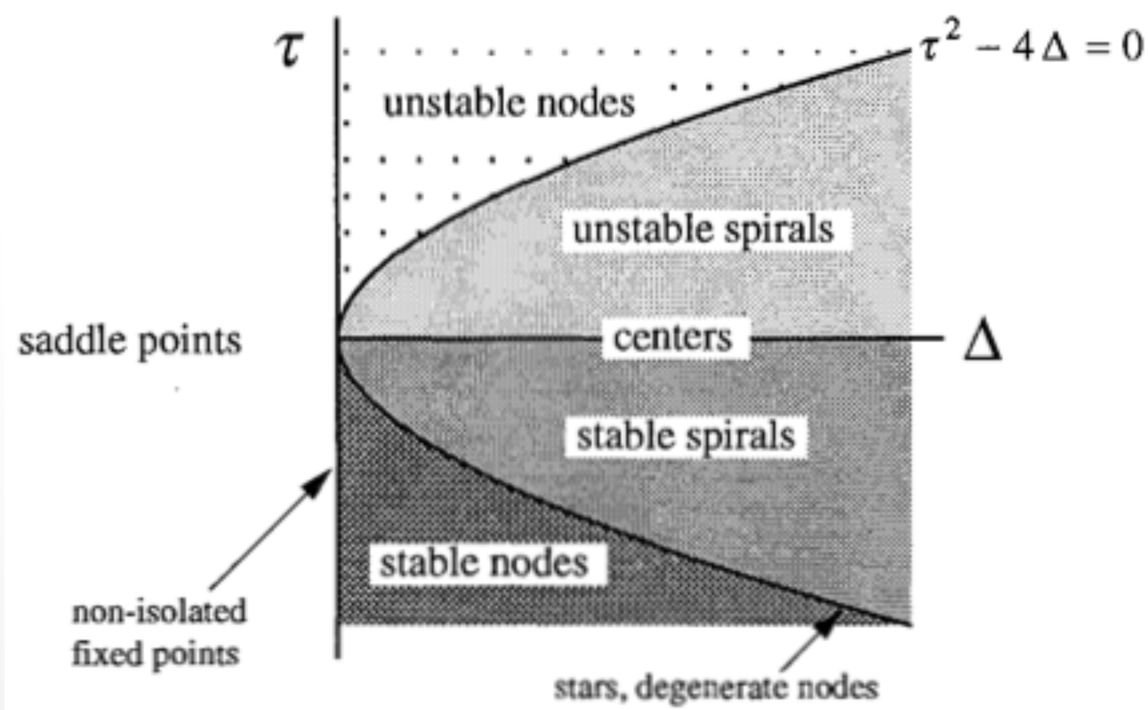
(a) node (b) degenerate node

Classification of fixed points

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2$$

Δ and τ are solved from

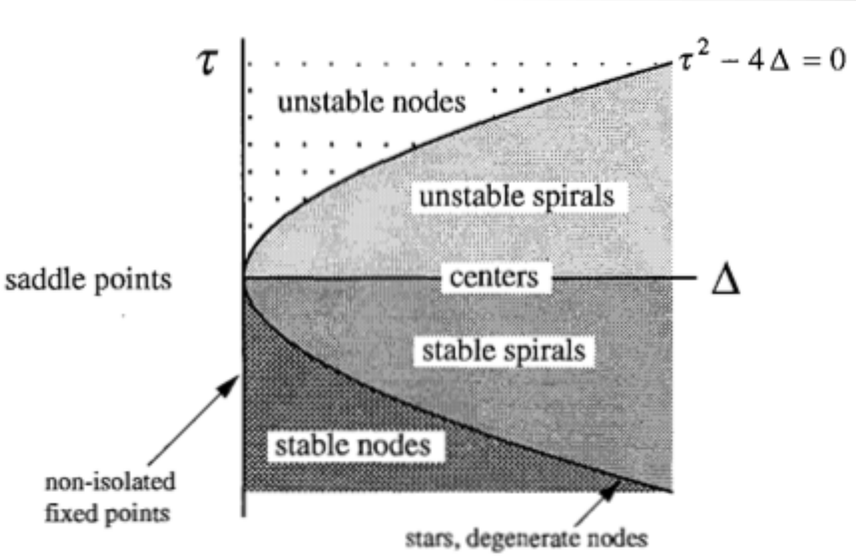
$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - \tau\lambda + \Delta = 0$$



Classification of fixed points

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2$$

- $\Delta < 0 \rightarrow$ eigenvalues are real and have opposite signs $\rightarrow$ origin is a saddle point
- $\Delta > 0 \rightarrow$ eigenvalues are real with the same sign (nodes), if τ^2-4Δ > 0. Stable (unstable) node if $\tau < 0$ ($\tau > 0$)
- $\Delta > 0 \rightarrow$ eigenvalues are complex conjugate (spirals and centers), if τ^2-4Δ < 0. Stable (unstable) spiral if $\tau < 0$ ($\tau > 0$). Center if $\tau = 0$
- $\Delta > 0 \rightarrow$ the origin is a star node, or degenerate node, if τ^2-4$\Delta = 0$
- $\Delta = 0 \rightarrow$ at least one eigenvalue is zero $\rightarrow$ the origin is not an isolated fixed point: there is either a whole line or a plane of fixed points



Examples

1) Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system

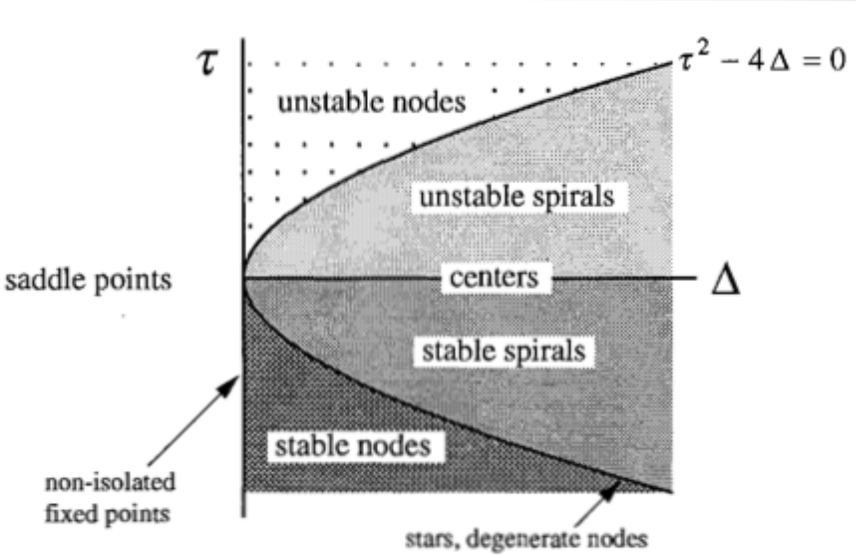
$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$\Delta = -2 \rightarrow$ saddle point

2) Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$\Delta = 5$, $\tau = 6$, $\tau^2 - 4\Delta = 16 > 0$
$\rightarrow$ unstable node

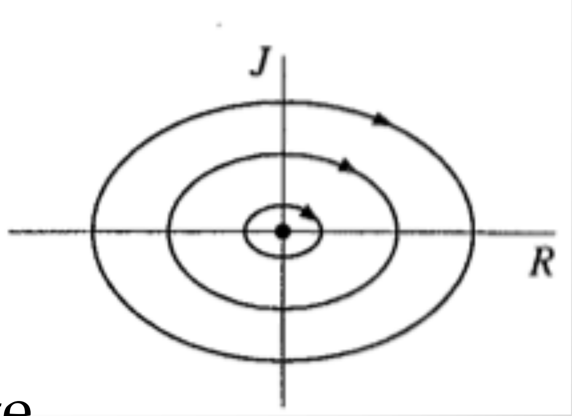


Love affairs

Strogatz is no Shakespeare:
1) The more Romeo loves Juliet, the more she wants to withdraw
2) When Romeo backs off, Juliet finds him attractive
3) Romeo loves Juliet the more, the more she loves him

$R(t)$ = Romeo's love/hate for Juliet at time t
$J(t)$ = Juliet's love/hate for Romeo at time t

$$\begin{aligned} \dot{R} &= aJ \\ \dot{J} &= -bR \end{aligned}$$



$a, b > 0$

Result: a never-ending cycle of love and hate.

Love affairs

Forecast by the general linear equation

$$\begin{aligned}\dot{R} &= aR + bJ \\ \dot{J} &= cR + dJ\end{aligned}$$

a, b, c, d can be of all possible signs.

In this model

$a > 0,\ b > 0 \rightarrow$ Romeo is spurred on by both his own and Juliet's love

OR

$a < 0,\ b > 0 \rightarrow$ Romeo is a cautious lover

Love affairs

Special case: identically cautious lovers

$$\begin{aligned}\dot{R} &= aR + bJ \\ \dot{J} &= bR + aJ\end{aligned}$$

$a < 0$, $b > 0$.

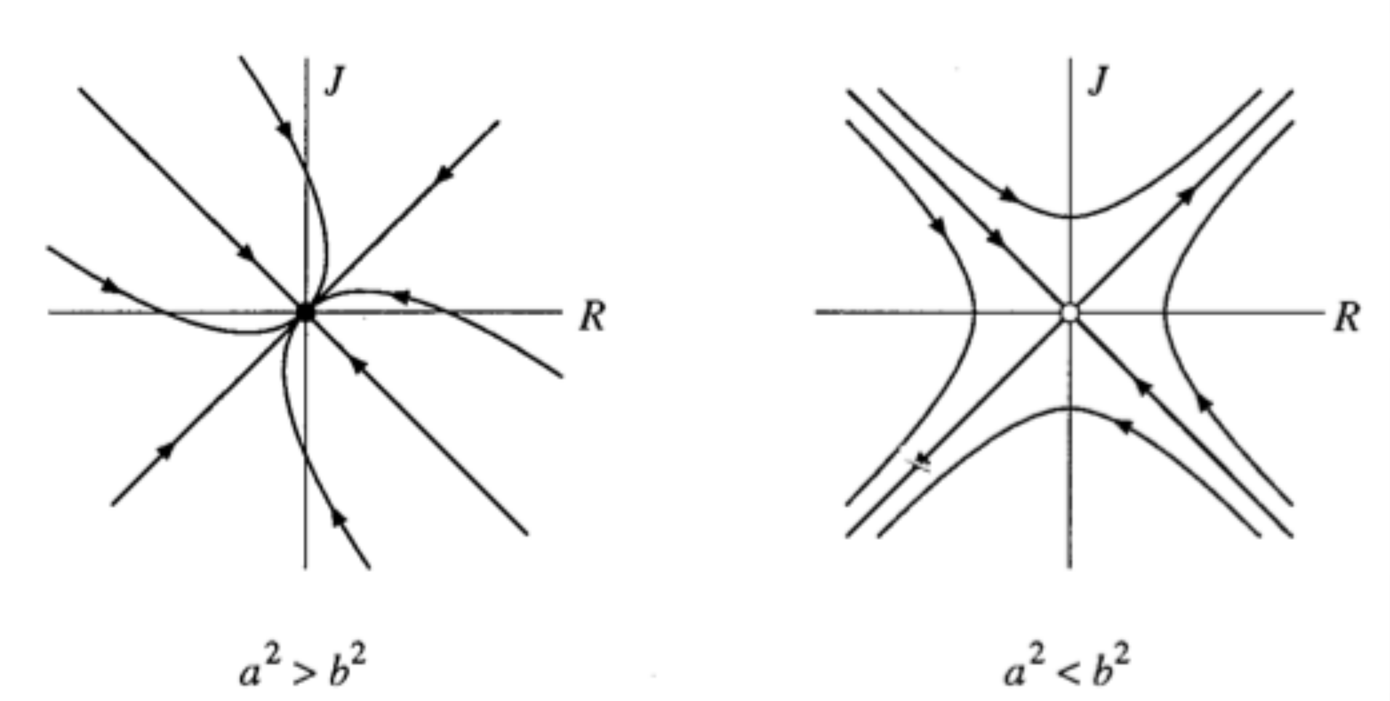
a = measure of cautiousness (tendency not to throw oneself at the other)
b = measure of responsiveness (tendency to get excited by the other's advances)

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \rightarrow \quad \tau = 2a < 0, \;\; \Delta = a^2 - b^2, \;\; \tau^2 - 4\Delta = 4b^2 > 0$$

Fixed point $(R, J) = (0,0)$ is a saddle point if $a^2 < b^2$ and a stable node if $a^2 > b^2$.

Love affairs

$$\lambda_1 = a + b, \quad \mathbf{v_1} = (1,1), \quad \lambda_2 = a - b, \quad \mathbf{v_2} = (1,-1)$$



- If $a^2 > b^2$ relationship evolves towards indifference ($R = 0$, $J = 0$)
- If $a^2 < b^2$ explosive relationship (love fest or war): asymptotically $R = J$ (mutual feelings)

Phase Plane

And now, after all the introductory drill, we finally start learning on

two-dimensional nonlinear systems.

Phase portraits

The general form of a vector field on the phase plane:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

In vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

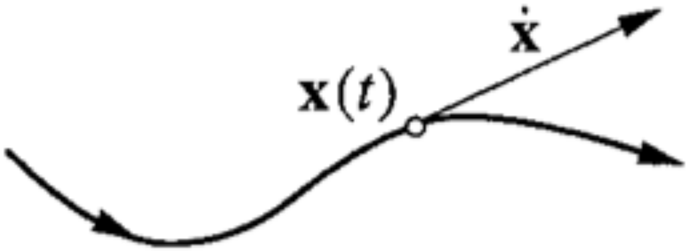
$$[\mathbf{x} = (x_1, x_2), \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))]$$

$$\mathbf{x} = \text{point in phase plane}$$

$$\dot{\mathbf{x}} = \text{velocity at that point}$$

Phase portraits

Solution **x**(t) describes a trajectory on the phase plane



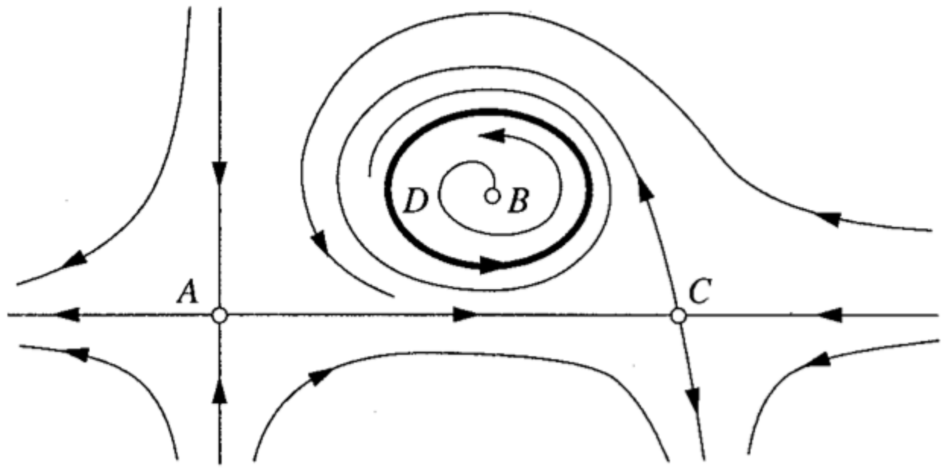
The whole plane is filled with (non-intersecting) trajectories starting from different phase points.

For nonlinear systems there is no hope to find trajectories analytically + the analytical solutions would not provide much insight.

Our approach: determine the qualitative behavior of the solutions via phase portraits.

Phase portraits

There's a zoo of possible phase portraits



Salient features:

1) *Fixed points* (*A*, *B*, *C*): **f(x*)=0**, steady states or equilibria of the system.
2) *Closed orbits* (*D*): periodic solutions, $\mathbf{x}(t+T)=\mathbf{x}(t)$.
3) *Arrangement of trajectories* near fixed points and closed orbits.
4) *Stability or instability* of the fixed points and closed orbits.

Numerical computation of phase portraits

Runge-Kutta method in the vector form.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x_0}$$

$$\begin{aligned} \mathbf{k_1} &= \mathbf{f}(\mathbf{x_n})\Delta t \\ \mathbf{k_2} &= \mathbf{f}(\mathbf{x_n} + \tfrac{1}{2}\mathbf{k_1})\Delta t \\ \mathbf{k_3} &= \mathbf{f}(\mathbf{x_n} + \tfrac{1}{2}\mathbf{k_2})\Delta t \\ \mathbf{k_4} &= \mathbf{f}(\mathbf{x_n} + \mathbf{k_3})\Delta t \end{aligned}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{6}(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4})$$

A stepsize Δt =0.1 usually provides sufficient accuracy.

See e.g. Press, Teukolsky, et al., Numerical Recipes In C/C++/Fortran. The book and free open source codes available online.

Example I

$$
\begin{aligned}
\dot{x} &= x + e^{-y} \\
\dot{y} &= -y
\end{aligned}
$$

Procedure: First find out analytically/graphically qualitative features of the phase portrait, then solve numerically for a **direction field**.

Fixed points

$$
\begin{aligned}
\dot{x} &= x + e^{-y} &= 0 \\
\dot{y} &= -y &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x^* &= -1 \\
y^* &= 0
\end{aligned}
$$

Stability

$$\text{for } t \to \infty, \qquad y(t) \sim e^{-t} \to 0$$

So, as $t \to \infty: \qquad e^{-y} \to 1 \qquad \to \qquad \dot{x} \approx x + 1$

Exponentially growing solutions: the fixed point is unstable (in the x-direction).

Example I

$$\begin{aligned} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{aligned}$$

Phase portrait: plot the nullclines.

The **nullclines** are the curves where

$$\dot{x} = 0 \qquad \text{or} \qquad \dot{y} = 0$$

On the nullclines the flow is either purely horizontal or purely vertical

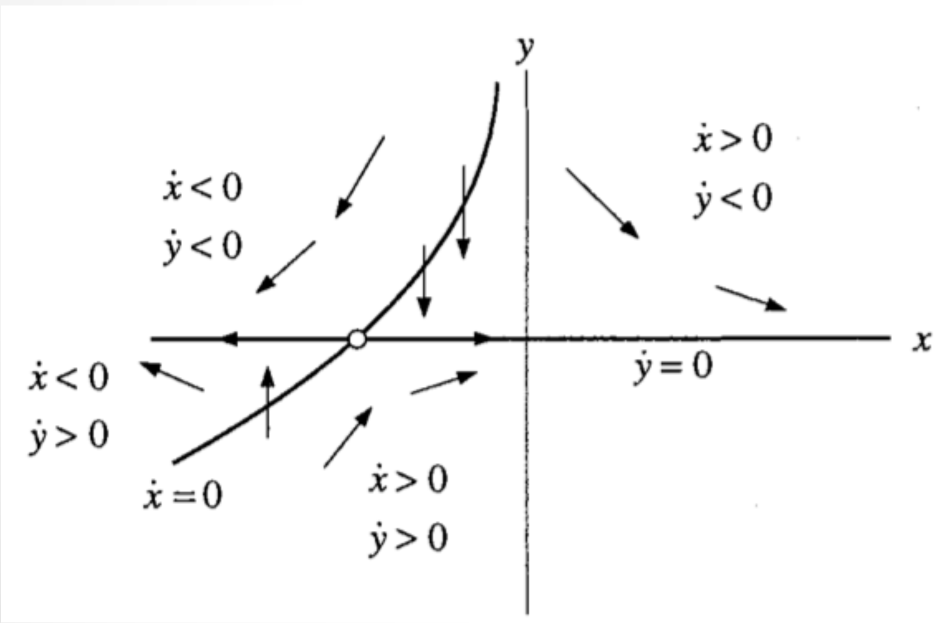
$$\begin{aligned} x + e^{-y} &= 0 \\ y &= 0 \end{aligned}$$

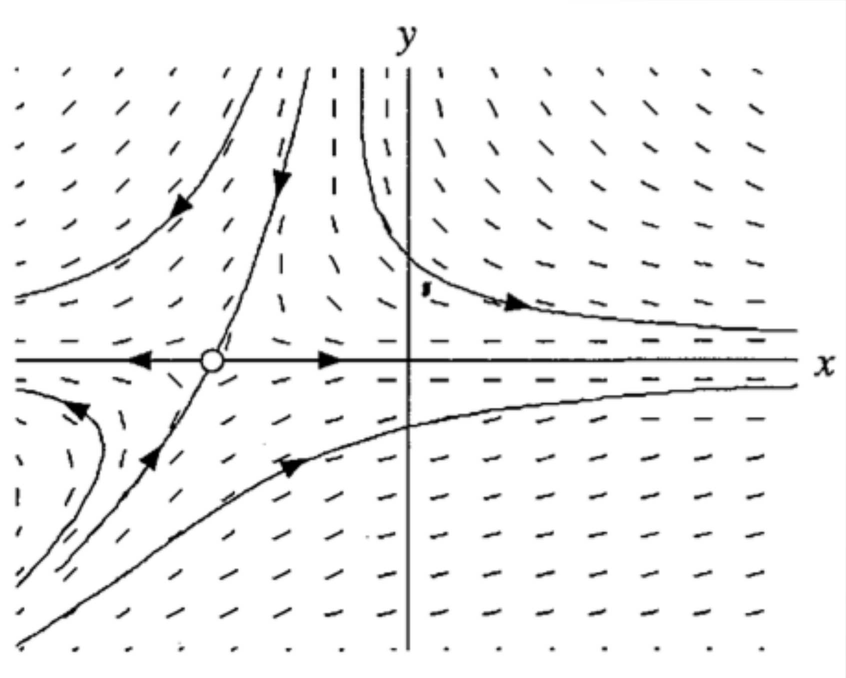
Example I

$$\begin{aligned} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{aligned}$$

Analysis:

Numerical solution:





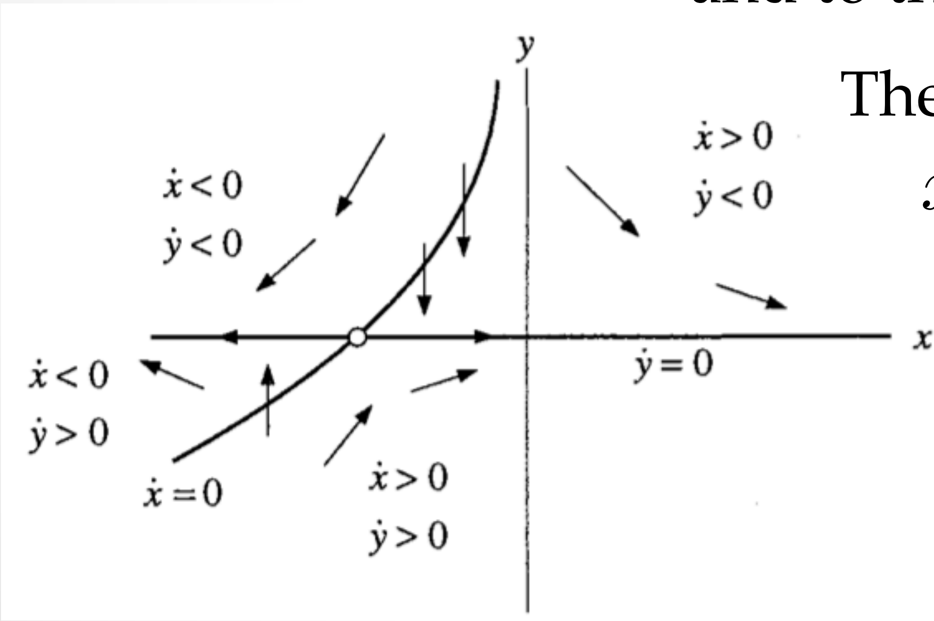
Example I

$$\begin{aligned}\dot{x} &= x + e^{-y} \\ \dot{y} &= -y\end{aligned}$$

The flow is horizontal on the nullcline x-axis and to the right when

$$\dot{x} = x + e^{-y} = x + 1 > 0 \Leftrightarrow x > -1$$

and to the left when $x < -1$.



The flow is vertical on the nullcline

$$\dot{x} = x + e^{-y} = 0.$$

Due to $\dot{y} = -y$
the flow is downwards,
when $y > 0$, and upwards,
when $y < 0$.

Directions of flow in the different regions can now be deduced from the directions on the nullclines.

Existence, uniqueness and topological consequences

The existence and uniqueness theorem given previously for 1 D can be generalized to *n* dimensions.

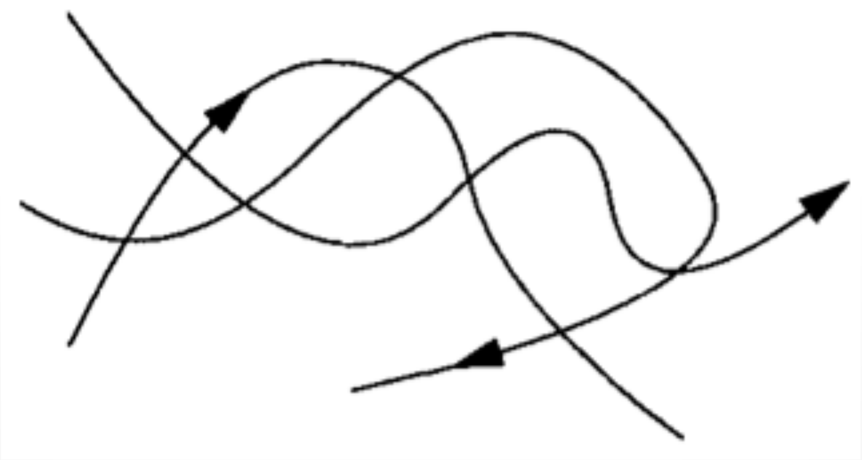
Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x}(t_0) = \mathbf{x}_0.$ Let $\mathbf{f}$ and all its partial derivatives $\partial f_i/\partial x_j,\ \ i, j = 1, \cdots, n$ be continuous for **x** in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x_0} \in D$ the initial value problem has a solution **x**(*t*) on some time interval (-τ, τ) about *t* = 0, and the solution is unique.

Existence, uniqueness and topological consequences

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x_0}$$

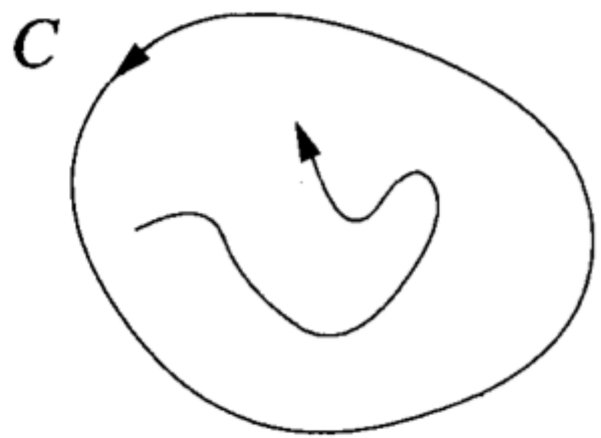
Corollary: different trajectories never intersect!

If two trajectories did intersect there would be two solutions starting from the same point (the crossing point).



Existence, uniqueness and topological consequences

Consequence in two dimensions: any trajectory starting from inside a closed orbit will be trapped inside it forever!



What will happen in the limit $t \to \infty$? The trajectory will either converge to a fixed point or to a closed (periodic) orbit! (The last part: **Poincaré-Bendixson theorem**.)