Computational Algebraic Geometry

Elimination theory

Kaie Kubjas

kaie.kubjas@aalto.fi

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Last time

Definition

Given $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ the l-th **elimination ideal** I_l is the ideal of $k[x_{l+1}, \ldots, x_n]$ defined by

$$I_l = I \cap k[x_{l+1}, \ldots, x_n].$$

Theorem (The Elimination Theorem)

Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let G be a Groebner basis of I wrt to lex order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 \leq l \leq n$, the set

$$G_l = G \cap k[x_{l+1}, \ldots, x_n]$$

is a Groebner basis of the l-th elimination ideal I_l.

Last time

Theorem (The Extension Theorem)

Let $I = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$ and let I_1 be the first elimination ideal of I. For each $1 \leq i \leq s$, write f_i in the form

$$f_i = g_i(x_2, \ldots, x_n)x_1^{N_i} + \text{terms in which } x_1 \text{ has degree } < N_i,$$

where $N_i \geq 0$ and $g_i \in \mathbb{C}[x_2, \ldots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \ldots, a_n) \in V(I_1)$. If $(a_2, \ldots, a_n) \notin V(g_1, \ldots, g_s)$, then there exists $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in V(I)$.

Quiz

Let $I = \langle xy - 1 \rangle \subseteq \mathbb{C}[x, y]$. Fix the lex order with $x > y$.

1. What is the first elimination ideal I_1?
2. What is the set of partial solutions $\mathbb{V}(I_1)$?
3. Which partial solutions in $\mathbb{V}(I_1)$ extend to a complete solution in $\mathbb{V}(I)$?

Overview

Today:

- The geometry of elimination
- Implicitization

The geometry of elimination

Variety of the elimination ideal

Let π_l be the projection map

$$\begin{aligned} \pi_l : \quad \mathbb{C}^n &\to \mathbb{C}^{n-l}, \\ (a_1, \ldots, a_n) &\mapsto (a_{l+1}, \ldots, a_n). \end{aligned}$$

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Lemma

Let $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{C}^n$ and let $I_l = \langle f_1, \ldots, f_s \rangle \cap \mathbb{C}[x_{l+1}, \ldots, x_n]$ be the l-th elimination ideal of $\langle f_1, \ldots, f_s \rangle$. Then, in $\mathbb{C}^{n-l}$ we have

$$\pi_l(V) \subseteq \mathbb{V}(I_l).$$

Proof: Let $f \in I_e$. Let $(a_1, \ldots, a_n) \in V$. Then

$$f(a_1, \ldots, a_n) = 0, \quad \text{because} \quad f \in \langle f_1, \ldots, f_s \rangle.$$

Since f includes only the variable $x_{e+1}, \ldots, x_n$, then

$$f(a_1, \ldots, a_n) = f(\pi_e(a_1, \ldots, a_n)) = 0.$$

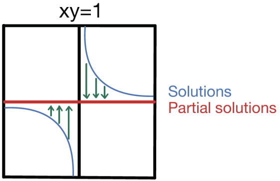
Hence f vanishes on every point of $\pi_e(V)$. $\blacksquare$

Variety of the elimination ideal

- Recall: The points of $\mathbb{V}(I_l)$ are called partial solutions.
- By Lemma, we can write $\pi_l(V)$ as

$$\pi_l(V) = \{(a_{l+1}, \ldots, a_n) \in \mathbb{V}(I_l) : \quad \exists a_1, \ldots, a_l \in \mathbb{C} \text{ with } (a_1, \ldots, a_l, a_{l+1}, \ldots, a_n) \in V\}.$$

- $\pi_l(V)$ consists precisely of the partial solutions that extend to complete solutions.



$$\pi_1(V) = \{a \in \mathbb{C} : a \neq 0\} \Rightarrow \pi_1(V) \text{ is not an affine variety!}$$

The Geometric Extension Theorem

Theorem (The Geometric Extension Theorem)

Let $V = \mathbb{V}(f_1, \dots, f_s) \subseteq \mathbb{C}^n$, let g_i be as in the Extension Theorem. If I_1 is the first elimination ideal of $\langle f_1, \dots, f_s \rangle$, then we have the equality in $\mathbb{C}^{n-1}$

$$\mathbb{V}(I_1) = \pi_1(V) \cup (\mathbb{V}(g_1, \dots, g_s) \cap \mathbb{V}(I_1)) .$$

- $\pi_1(V)$ fills up $\mathbb{V}(I_1)$ besides possibly a part that lies in $(\mathbb{V}(g_1, \dots, g_s) \cap \mathbb{V}(I_1))$

Proof: This theorem follows from the Extension Theorem and the previous lemma. $\boxtimes$

The Geometric Extension Theorem

It is not clear how big the missing part is and it can be unnaturally large:

- $(y-z)x^2 + xy - 1$ and $(y-z)x^2 + xz - 1$ generate the same ideal as $yx - 1$ and $zx - 1$
- $I_1 = \langle y - z \rangle$
- The partial solutions are $\{(a, a) : a \in \mathbb{C}\}$

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- $I_1 = \langle y - z \rangle$
- The partial solutions are $\{(a, a) : a \in \mathbb{C}\}$
- The first set of generators: $g_1 = g_2 = (y - z)$ and hence the Geometric Extension Theorem says nothing about the size of $\pi_1(V)$
- The second set of generators: $g_1 = y$ and $g_2 = z$ and hence all partial solutions besides $(0, 0)$ extend to a complete solution

The Closure Theorem

Theorem (The Closure Theorem)

Let $V = \mathbb{V}(f_1, \dots, f_s) \subseteq \mathbb{C}^n$ *and let* I_l *be the* l*-th elimination ideal of* $\langle f_1, \dots, f_s \rangle$. *Then*

1. $\mathbb{V}(I_l)$ *is the smallest affine variety containing* $\pi_l(V) \subseteq \mathbb{C}^{n-l}$.
2. *When* $V \neq \emptyset$, *there is an affine variety* $W \subsetneq \mathbb{V}(I_l)$ *such that* $\mathbb{V}(I_l) - W \subseteq \pi_l(V)$.

Proof: 1) We will postpone the proof of the first part.
2) We prove it for the special case $l=1$.
By the Geometric Extension Theorem

$$\mathbb{V}(I_1) = \pi_1(V) \cup \underbrace{\left(\mathbb{V}(g_1,\dots,g_s) \cap \mathbb{V}(I_1)\right)}_{W}.$$

Hence $\mathbb{V}(I_1) - W \subseteq \pi_1(V)$ and we are done if $W \neq \mathbb{V}(I_1)$.

If $W = \mathbb{V}(I_1)$, we need to modify eqns defining V so that W becomes smaller.

OBSERVATION: If $W = \mathbb{V}(I_1)$, then

$$V = \mathbb{V}(f_1,\dots,f_s,\, g_1,\dots,g_s).$$

Proof of obs: "$\supseteq$" is clear. "$\subseteq$" Let $(a_1,\dots,a_n) \in V$. Then $(a_2,\dots,a_n) \in \pi_1(V) \subseteq \mathbb{V}(I_1) = W$. Hence g_i's vanish on $(a_2,\dots,a_n)$ and $(a_1,\dots,a_n) \in \mathbb{V}(f_1,\dots,f_s,g_1,\dots,g_s)$. ☒

Let $\tilde{I} = \langle f_1,\dots,f_s,g_1,\dots,g_s\rangle$. Thus I and $\tilde{I}$ might be different and hence I_1 and $\tilde{I}_1$ might be different. However, $\mathbb{V}(I_1) = \mathbb{V}(\tilde{I}_1)$ because they are both the smallest varieties containing $\pi_1(V)$.

Next we define a better basis for $\tilde{I}$. Let

$\tilde{f}_i = f_i - g_i x_1^{N_i}$. Then $\tilde{I} = \langle \tilde{f}_1, \dots, \tilde{f}_s, g_1, \dots, g_s \rangle$.
The Geom. Ext. Thm for $V = \mathbb{V}(\tilde{f}_1, \dots, \tilde{f}_s, g_1, \dots, g_s)$
gives

$$\mathbb{V}(I_1) = \mathbb{V}(\tilde{I}_1) = \pi_1(V) \cup \tilde{W}$$

where $\tilde{W}$ consists of those partial solutions where the leading coefficients of $\tilde{f}_1, \dots, \tilde{f}_s, g_1, \dots, g_s$ vanish.

If $\tilde{W} \subsetneq W$, then we are done. It is not guaranteed that $\tilde{W}$ is strictly smaller. In this case we can repeat the process. Each time the degrees of x_1 drop (or remain zero), so that eventually the generators will have degree 0 in x_1. This means that V can be defined by the vanishing of pol's in $\mathbb{C}[x_2, \dots, x_n]$ and every partial solutions $(a_2, \dots, a_n)$ extends to a solution $(a_1, \dots, a_n)$ for any $a_1 \in \mathbb{C}$. In this case we can choose $W = \emptyset$. $\blacksquare$

The Closure Theorem

Theorem (The Closure Theorem)

Let $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{C}^n$ and let I_l be the l-th elimination ideal of $\langle f_1, \ldots, f_s \rangle$. Then

1. *$\mathbb{V}(I_l)$ is the smallest affine variety containing $\pi_l(V) \subseteq \mathbb{C}^{n-l}$.*
2. *When $V \neq \emptyset$, there is an affine variety $W \subsetneq \mathbb{V}(I_l)$ such that $\mathbb{V}(I_l) - W \subseteq \pi_l(V)$.*

Corollary

Let $V \subseteq \mathbb{C}^n$ be an affine variety. Then there are affine varieties $Z_i \subset W_i \subseteq \mathbb{C}^{n-1}$ for $i \leq 1 \leq p$ such that

$$\pi_l(V) = \bigcup_{i=1}^{p} (W_i - Z_i).$$

Sets of this form are called constructible.

Corollary

Let $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{C}^n$, *and assume that for some* i, f_i *can be written as*

$$f_i = c_i x_1^N + \textit{terms in which } x_1 \textit{ has degree } < N,$$

where $c \in \mathbb{C}$ *is nonzero and* $N > 0$. *If* I_1 *is the first elimination ideal, then*

$$\pi_1(V) = \mathbb{V}(I_1).$$

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$$\pi_1(V) = \mathbb{V}(I_1).$$

The Extension and Closure Theorems hold over any algebraically closed field.

Implicitization

Rational parametrizations

Definition

A **rational function** in $t_1, \dots, t_m$ with coefficients in k is a quotient f/g of two polynomials $f, g \in k[t_1, \dots, t_m]$, where g is not the zero polynomial. The set of all rational functions is denoted $k(t_1, \dots, t_m)$.

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- rational parametric representation of V consists of $r_1, \dots, r_n \in k(t_1, \dots, t_m)$ such that

$$x_1 = r_1(t_1, \dots, t_m), \cdots, x_n = r_n(t_1, \dots, t_m)$$

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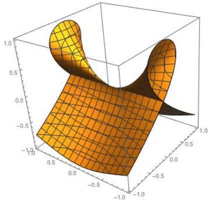
  lie in V
- require that V is the smallest variety containing these points
- if $r_1, \ldots, r_n$ are polynomials, then polynomial parametric representation

Parametric vs implicit form

- original defining equations $f_1 = \ldots = f_s = 0$ are called an implicit representation
- it is easy to draw a parametric description of a curve on a computer



- plotted not using $x^2 - y^2z^2 + z^3 = 0$ but
$$x = t(u^2 - t^2), y = u, z = u^2 - t^2.$$
- if we want to know whether the point $(1, 2, -1)$ is on the above surface, then implicit representation is useful: $1^2 - 2^2(-1)^2 + (-1)^3 = 1 - 4 - 1 = -4$

Parametric vs implicit form

Desirability of having both representations leads to the questions

- (Parametrization) Does every affine variety have a rational parametric description?
- (Implicitization) Given a parametric representation of an affine variety, can we find the defining equations (i.e. can we find an implicit representation)?
- The answer to the first question is no. Those that can be parametrized are called unirational.
- It is difficult to tell whether a given variety is unirational or not.
- We will learn that the answer to the second question is always yes.

Implicitization

- The parametrization might not fill up all of the variety.
- Implicitization asks for the defining equations of the smallest variety containing the parametrization.
- How to find missing points?

Twisted cubic

- the twisted cubic has parametrization

$$x = t, y = t^2, z = t^3$$

- the tangent vector to the curve at a point is $(1, 2t, 3t^2)$
- the tangent line is parametrized

$$(t, t^2, t^3) + u(1, 2t, 3t^2) = (t + u, t^2 + 2tu, t^3 + 3t^2 u)$$

- a parametrization of the entire surface is

$$x = t + u, y = t^2 + 2tu, z = t^3 + 3t^2 u$$

- the tangent surface lies on the variety V defined by

$$-4x^3 z + 3x^2 y^2 - 4y^3 + 6xyz - z^2 = 0$$

- Is V the smallest variety containing the tangent surface?
- If yes, does the tangent surface fill up V completely?

Polynomial parametrization

- We consider the polynomial parametrization

$$x_1 = f_1(t_1, \ldots, t_m), \cdots, x_n = f_n(t_1, \ldots, t_m),$$

  where $f_1, \ldots, f_n \in k[t_1, \ldots, t_m]$.
- We can think of it as the map F

$$k^m \to k^n,$$
$$(t_1, \ldots, t_m) \mapsto (f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m))$$

- Then $F(k^m) \subseteq k^n$ is equal to the subset of k^n parametrized by the polynomials $f_1, \ldots, f_n$.
- The solution to the implicitization problem finds the smallest algebraic variety containing $F(k^m)$

Polynomial parametrization

Next we want to connect the implicitization and elimination. The polynomial equations

$$x_1 = f_1(t_1, \dots, t_m), \cdots, x_n = f_n(t_1, \dots, t_m),$$

define a variety

$$V = \mathbb{V}(x_1 - f_1, \dots, x_n - f_n) \subseteq k^{n+m}.$$

The points of V can be written as

$$(t_1, \dots, t_m, f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m)).$$

Hence V is the graph of the map F.

Polynomial parametrization

We also have the inclusion $i : k^m \to k^{n+m}$ defined by

$$i(t_1, \dots, t_m) \mapsto (t_1, \dots, t_m, f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m))$$

and the projection $\pi_m : k^{n+m} \to k^n$ defined by

$$\pi_m(t_1, \dots, t_m, x_1, \dots, x_n) = (x_1, \dots, x_n).$$

$$\begin{tikzcd} & k^{n+m} \arrow[dr, "\pi_m"] & \\ k^m \arrow[ur, "i"] \arrow[rr, "F"] & & k^n \end{tikzcd}$$

$$i(k^m) = V \text{ and } F(k^m) = \pi_m(i(k^m)) = \pi_m(V)$$

Polynomial Implicitization

Theorem (Polynomial Implicitization)

If k is an infinite field, let $F : k^m \to k^n$ be the map defined by the polynomial parametrization

$$x_1 = f_1(t_1, \dots, t_m), \cdots, x_n = f_n(t_1, \dots, t_m).$$

Let $I = \langle x_1 - f_1, \dots, x_n - f_n \rangle \subseteq k[t_1, \dots, t_m, x_1, \dots, x_n]$ and let $I_m = I \cap k[x_1, \dots, x_n]$ be the m-th elimination ideal. Then $\mathbb{V}(I_m)$ is the smallest variety in k^n containing $F(k^m)$.

Proof: Let $V = \mathbb{V}(I) \subseteq k^{n+m}$, the graph of F. If $k = \mathbb{C}$, then $F(\mathbb{C}^m) = \pi_m(V)$ and by the Closure Theorem $\mathbb{V}(I_m)$ is the smallest variety containing $\pi_m(V)$.

Suppose k is an infinite subfield of $\mathbb{C}$. Let $\mathbb{V}_k(I_m)$ be the variety in k^n and $\mathbb{V}_{\mathbb{C}}(I_m)$ be the variety in $\mathbb{C}^n$. We have $F(k^m) = \pi_m(V_k) \subseteq \mathbb{V}_k(I_m)$.

Let $Z_k = \mathbb{V}_k(g_1, \dots, g_s) \subseteq k^n$ be any variety containing $F(k^m)$. We must show that $\mathbb{V}_k(I_m) \subseteq Z_k$. Since $g_i = 0$ on Z_k, also $g_i = 0$ on $F(k^m)$. Hence $g_i \circ F$ vanishes on k^m, i.e. it is the zero function on k^m.

Since k is an infinite field, then $g_i \circ F$ is a zero function in $k[t_1, \dots, t_m]$. Hence also $g_i \circ F$ vanishes on $\mathbb{C}^m$ and thus g_i's vanish on $F(\mathbb{C}^m)$. Hence $Z_{\mathbb{C}} = \mathbb{V}_{\mathbb{C}}(g_1, \dots, g_s)$ is a variety containing $F(\mathbb{C}^m)$. Since the theorem true for $\mathbb{C}$, it follows that $\mathbb{V}_{\mathbb{C}}(I_m) \subset Z_{\mathbb{C}}$ in $\mathbb{C}^n$. It follows $\mathbb{V}_k(I_m) \subset Z_k$. ∎

Implicitization algorithm

- Let
$$x_1 = f_1(t_1, \dots, t_m), \cdots, x_n = f_n(t_1, \dots, t_m).$$
for polynomials $f_1, \dots, f_n \in k[t_1, \dots, t_m]$.
- Consider the ideal $I = \langle x_1 - f_1, \dots, x_n - f_n \rangle$.
- Compute the Groebner basis of I with respect to a lexicographic order where every t_i is greater than every x_j.
- The elements of the Groebner basis not involving $t_1, \dots, t_m$ define the smallest variety in k^n containing the parametrization.

Implicitization algorithm

Tangent surface of the twisted cubic:

- $I = \langle x - t - u, y - t^2 - 2tu, z - t^3 - 3t^2u \rangle \subseteq \mathbb{R}[t, u, x, y, z]$
- Fix the lex order with $t > u > x > y > z$
- A Groebner basis is given by

$$
\begin{aligned}
g_1 &= t + u - x, \\
g_2 &= u^2 - x^2 + y, \\
g_3 &= ux^2 - uy - x^3 + (3/2)xy - (1/2)z, \\
g_4 &= uxy - uz - x^2y - xz + 2y^2, \\
g_5 &= uxz - uy^2 + x^2z - (1/2)xy^2 - (1/2)yz, \\
g_6 &= uy^2 - uz^2 - 2x^2yz + (1/2)xy^3 - xz^2 + (5/2)y^2z, \\
g_7 &= x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)z^2.
\end{aligned}
$$

- Since g_7 is the only Groebner basis element consisting of variables x, y, z only, then $\mathbb{V}(g_7)$ solves the implicitization problem.

Tangent surface of the twisted cubic

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Quiz: Which partial solutions $(x, y, z) \in \mathbb{V}(I_2) = \mathbb{V}(g_7) \subseteq \mathbb{C}^3$ extend to a solution of $\mathbb{V}(I) \subseteq \mathbb{C}^5$?

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- $I_1 = \langle g_2, \dots, g_7 \rangle$ is the first elimination ideal of I_2 and the coefficient of u^2 in g_2 is 1 $\Rightarrow$ all partial solutions in $\mathbb{V}(I_2)$ extend to a solution in $\mathbb{V}(I_1)$

Tangent surface of the twisted cubic

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Quiz: Which partial solutions $(x, y, z) \in \mathbb{V}(I_2) = \mathbb{V}(g_7) \subseteq \mathbb{C}^3$ extend to a solution of $\mathbb{V}(I) \subseteq \mathbb{C}^5$?

- $I_1 = \langle g_2, \dots, g_7 \rangle$ is the first elimination ideal of I_2 and the coefficient of u^2 in g_2 is 1 $\Rightarrow$ all partial solutions in $\mathbb{V}(I_2)$ extend to a solution in $\mathbb{V}(I_1)$
- the coefficient of t in g_1 is 1 $\Rightarrow$ all partial solutions in $\mathbb{V}(I_1)$ extend to a solution in $\mathbb{V}(I)$

Tangent surface of the twisted cubic

- Since all partial solutions in $\mathbb{V}(I_2) = \mathbb{V}(g_2)$ extend to a complete solution in $\mathbb{V}(I)$, then the tangent surface of the twisted cubic fills $\mathbb{V}(g_2)$.

Tangent surface of the twisted cubic

- Since all partial solutions in $\mathbb{V}(I_2) = \mathbb{V}(g_2)$ extend to a complete solution in $\mathbb{V}(I)$, then the tangent surface of the twisted cubic fills $\mathbb{V}(g_2)$.
- It can be shown that in this example every real partial solution $(x, y, z) \in \mathbb{V}(I_2)$ extents to a real complete solution in $\mathbb{V}(I)$.

Tangent surface of the twisted cubic

- Since all partial solutions in $\mathbb{V}(I_2) = \mathbb{V}(g_2)$ extend to a complete solution in $\mathbb{V}(I)$, then the tangent surface of the twisted cubic fills $\mathbb{V}(g_2)$.
- It can be shown that in this example every real partial solution $(x, y, z) \in \mathbb{V}(I_2)$ extents to a real complete solution in $\mathbb{V}(I)$.
- In general there is no easy answer to the question whether a parametrization fills the minimal variety containing it and each case has to be analyzed separately.

Rational parametrization

$$x = \frac{u^2}{v}, y = \frac{v^2}{u}, z = u$$

- (x, y, z) always lies on the surface $x^2 y = z^3$
- clearing the denominators in the above parametrization gives the ideal

$$I = \langle vx - u^2, uy - v^2, z - u \rangle$$

- the second elimination ideal is $I_2 = \langle z(x^2 y - z^3) \rangle$
- $\mathbb{V}(I_2) = \mathbb{V}(x^2 y - z^3) \cup \mathbb{V}(z)$
- hence $\mathbb{V}(I_2)$ is not the smallest variety containing the parametrization

Rational parametrization

- We consider a rational parametrization

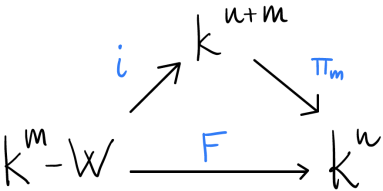
$$x_1 = \frac{f_1(t_1,\dots,t_m)}{g_1(t_1,\dots,t_m)}, \cdots, x_n = \frac{f_n(t_1,\dots,t_m)}{g_n(t_1,\dots,t_m)},$$

  where $f_1, g_1, \dots, f_n, g_n \in k[t_1,\dots,t_m]$.
- The map $F : k^m \to k^n$ given by

$$F(t_1,\dots,t_m) = \left(\frac{f_1(t_1,\dots,t_m)}{g_1(t_1,\dots,t_m)}, \cdots, \frac{f_n(t_1,\dots,t_m)}{g_n(t_1,\dots,t_m)}\right)$$

  might not be defined everywhere because of the denominators.
- Let $W = \mathbb{V}(g_1 g_2 \cdots g_n) \subset k^m$.
- Then $F : k^m - W \to k^n$ is well-defined.

Rational parametrization



- $i(k^m - W) \subseteq \mathbb{V}(I)$ where $I = \langle g_1x_1 - f_1, \ldots, g_nx_n - f_n \rangle$
- $\mathbb{V}(I)$ might not be the smallest variety containing $i(k^m - W)$
- add an extra variable to control the denominators:

$$J = \langle g_1x_1 - f_1, \ldots, g_nx_n - f_n, 1 - gy \rangle \subseteq k[y, t_1, \ldots, t_m, x_1, \ldots, x_n],$$

where $g = g_1 \cdot g_2 \cdots g_n$

Rational parametrization

$$\begin{tikzcd} & k^{n+m+1} \arrow[dr, "\pi_{m+1}"] & \\ k^m - W \arrow[ur, "j"] \arrow[rr, "F"] & & k^n \end{tikzcd}$$

The map $j : k^m - W \to k^{n+m+1}$ is defined by

$$j(t_1, \dots, t_m) = \left(\frac{1}{g(t_1, \dots, t_m)}, t_1, \dots, t_m, \frac{f_1(t_1, \dots, t_m)}{g_1(t_1, \dots, t_m)}, \cdots, \frac{f_n(t_1, \dots, t_m)}{g_n(t_1, \dots, t_m)} \right).$$

We have $j(k^m - W) = \mathbb{V}(J)$ and
$F(k^m - W) = \pi_{m+1}(j(k^m - W)) = \pi_{m+1}(\mathbb{V}(J))$.

Rational Implicitization

Theorem (Rational Implicitization)

If k is an infinite field, let $F : k^m - W \to k^n$ be the function defined by the rational parametrization

$$x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \cdots, x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}.$$

Let
$J = \langle g_1x_1 - f_1, \ldots, g_nx_n - f_n, 1 - gy \rangle \subseteq k[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$, where $g = g_1 \cdot g_2 \cdots g_n$, and let $J_{m+1} = J \cap k[x_1, \ldots, x_n]$ be the $(m+1)$-st elimination ideal. Then $\mathbb{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^m - W)$.

Theorem (Rational Implicitization)

If k is an infinite field, let $F : k^m - W \to k^n$ be the function defined by the rational parametrization

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Let $J = \langle g_1x_1 - f_1, \ldots, g_nx_n - f_n, 1 - gy\rangle \subseteq k[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$, *where* $g = g_1 \cdot g_2 \cdots g_n$, *and let* $J_{m+1} = J \cap k[x_1, \ldots, x_n]$ *be the* $(m+1)$*-st elimination ideal. Then* $\mathbb{V}(J_{m+1})$ *is the smallest variety in* k^n *containing* $F(k^m - W)$.

This theorem gives an implicitization algorithm for rational parametrizations similarly to the polynomial parametrizations case.

Implicitization algorithm

- $x = \frac{u^2}{v}, y = \frac{v^2}{u}, z = u$
- $J = \langle vx - u^2, uy - v^2, z - u, 1 - uvw \rangle$
- the third elimination ideal is $J_3 = \langle x^2y - z^3 \rangle$

Guiding questions

We started the Groebner bases chapter with four guiding questions:

- The ideal description problem: Does every ideal $I \subset k[x_1, \ldots, x_n]$ have a finite generating set?
- The ideal membership problem: Given $f \in k[x_1, \ldots, x_n]$ and ideal $I = \langle f_1, \ldots, f_s \rangle$, determine if $f \in I$.
- The problem of solving polynomial equations: Find all common solutions in k^n of a system of polynomial equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0.$$

- The implicitization problem: If V is given by a rational parametric representation, find a system of polynomial equations that defines V.

Solving systems of polynomial equations

Not all systems of polynomial equations have nice solutions:

$$xy = 4, y^2 = x^3 - 1$$

The Groebner basis for the lex order with $x > y$ is given by

$$g_1 = 16x - y^2 - y^4, g_2 = y^5 + y^3 - 64$$

The polynomial g_2 has no rational roots. One can obtain numerically

$$y = 2.21363, -1.78719 \pm 1.3984i, 0.680372 \pm 2.26969i.$$

These solutions can be substituted into g_1 to find the values of x. We will return to the topic when we talk about numerical algebraic geometry.

Conclusion

Today:

- The geometry of elimination
- Implicitization

Next time:

- Hilbert's Nullstellensatz
- Rational ideals
- Ideal-variety correspondence