Aalto University
School of Electrical
Engineering

Lecture 9: Elementary Single-Phase Machines and Lossless Magnetic Field

ELEC-E8402 Control of Electric Drives and Power Converters

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Single-Phase Machines

- Single-phase machines are seldom used in real applications
- Why should we study them?
- To get more thorough understanding of fundamental concepts
  - Flux linkages
  - Conservative magnetic field systems
  - Selection of state variables
  - Modeling concepts introduced are very general and powerful
- 2-pole single-phase machine with a field winding is used as an example

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3-phase machines will be considered in the next lecture. They are actually simpler to model, so don't worry!

Single-Phase Machine With a Field Winding

- Stator voltage

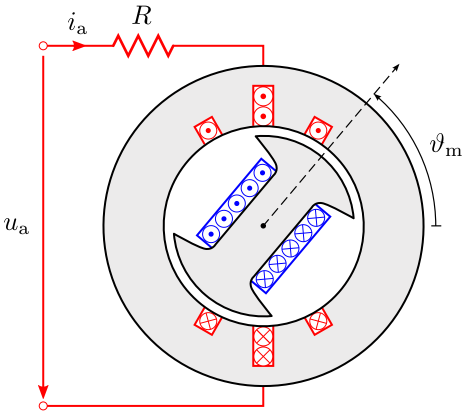
$$u_\mathrm{a} = Ri_\mathrm{a} + \frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t}$$

- Stator flux linkage

$$\psi_\mathrm{a} = L_\mathrm{a}(\vartheta_\mathrm{m})i_\mathrm{a} + L_\mathrm{af}(\vartheta_\mathrm{m})i_\mathrm{f}$$

  where L_a is the self-inductance and L_af is the mutual inductance

- How to model the inductances?
- How to calculate the produced torque?

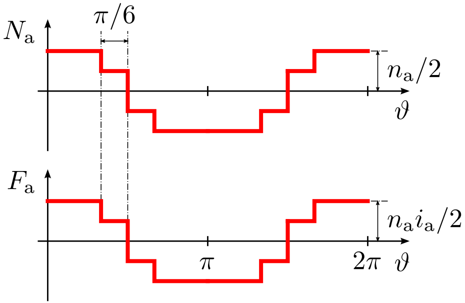


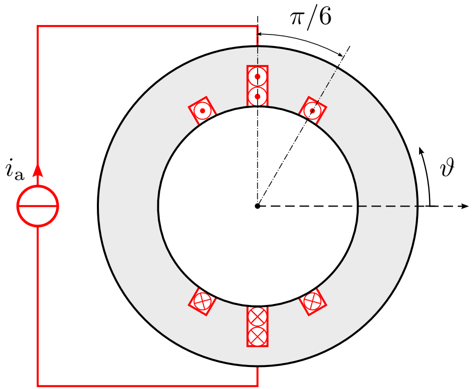
For constant field-winding current i_f, the flux linkage $\psi_\mathrm{af}(\vartheta_\mathrm{m}) = L_\mathrm{af}(\vartheta_\mathrm{m})i_\mathrm{f}$ due to the field winding depends only on the rotor position, just like in permanent-magnet machines.

Stator Winding

- Winding function N_a tells how many times the flux links with the winding at ϑ
- Magnetomotive force (MMF) distribution

$$F_\mathrm{a}(\vartheta) = N_\mathrm{a}(\vartheta) i_\mathrm{a}$$



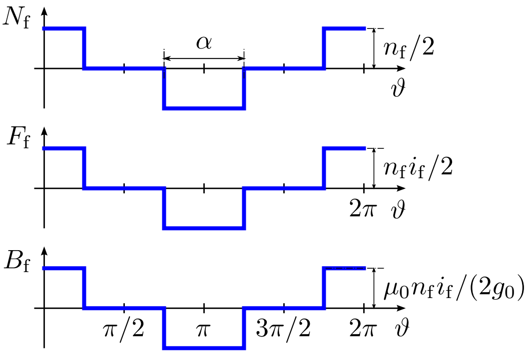


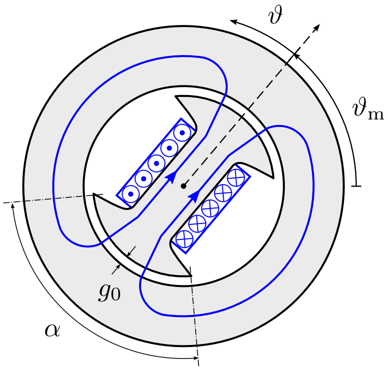
Example stator winding with n_a turns[1]

[1] Slemon, *Electric Machines and Drives*. Addison Wesley, 1992.

Rotor Field Winding

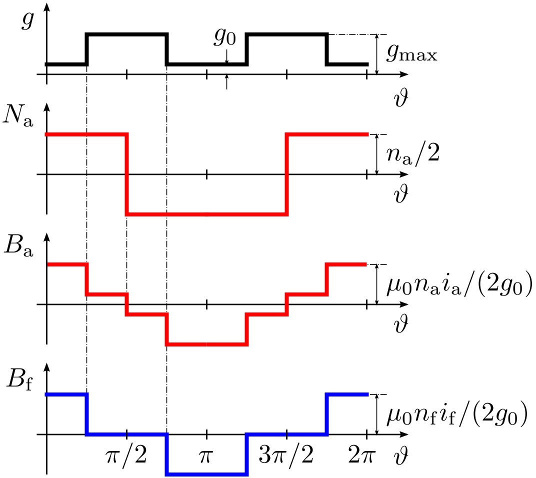
- Field winding produces the flux density distribution B_f in the airgap

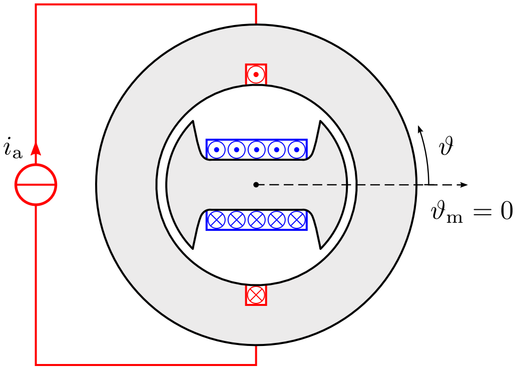




Example geometry with $\alpha = \pi/2$

Flux Density Space Waveforms at $\vartheta_{\mathrm{m}} = 0$

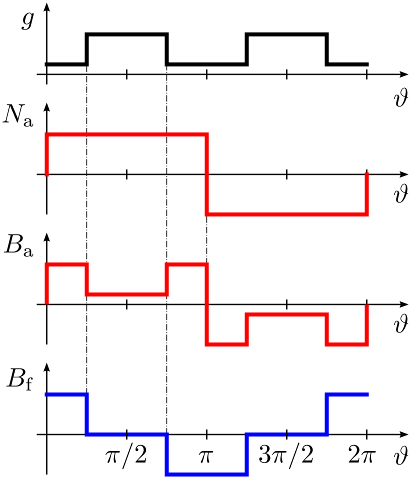




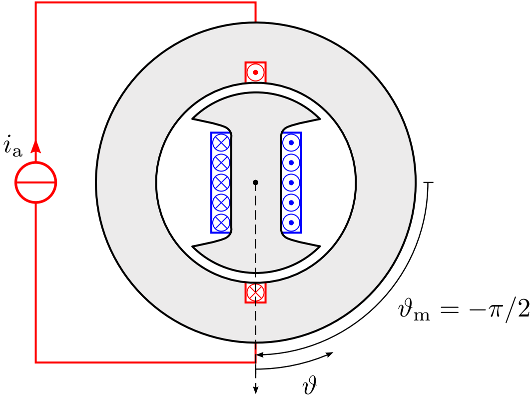
Total airgap flux density $B_{\mathrm{g}} = B_{\mathrm{a}} + B_{\mathrm{f}}$

Waveforms assume $g_{\max} = 4g_0$

Flux Density Space Waveforms at $\vartheta_m = -\pi/2$



Total airgap flux density $B_g = B_a + B_f$



Waveforms assume $g_{max} = 4g_0$

Flux Linkage and Inductances

- Total airgap flux density

$$B_{\mathrm{g}}(\vartheta) = B_{\mathrm{a}}(\vartheta) + B_{\mathrm{f}}(\vartheta)$$

- Stator flux linkage

$$\psi_{\mathrm{a}} = r\ell \int_0^{2\pi} N_{\mathrm{a}}(\vartheta) B_{\mathrm{g}}(\vartheta) \mathrm{d}\vartheta$$

  where r is the airgap radius and ℓ is the effective rotor length
- Inductances[2]

$$L_{\mathrm{a}} = \mu_0 r\ell \int_0^{2\pi} \frac{N_{\mathrm{a}}^2(\vartheta)}{g(\vartheta)} \mathrm{d}\vartheta \qquad L_{\mathrm{af}} = \mu_0 r\ell \int_0^{2\pi} \frac{N_{\mathrm{a}}(\vartheta) N_{\mathrm{f}}(\vartheta)}{g(\vartheta)} \mathrm{d}\vartheta$$

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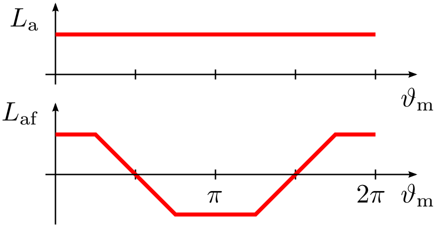
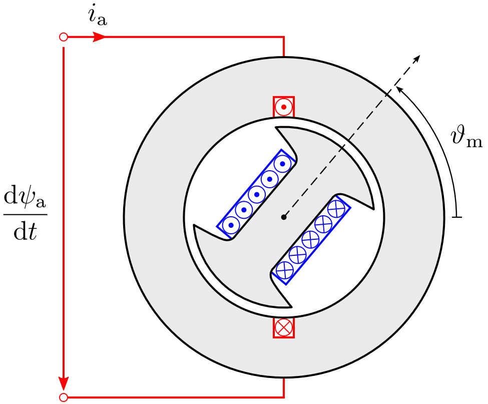
[2] Lipo, *Analysis of Synchronous Machines*, 2nd. CRC Press, 2012.

Inductances

- Stator flux linkage

$$\psi_{\mathrm{a}} = L_{\mathrm{a}}(\vartheta_{\mathrm{m}})i_{\mathrm{a}} + L_{\mathrm{af}}(\vartheta_{\mathrm{m}})i_{\mathrm{f}}$$





The self-inductance L_{a} of the ideal full-pitch coil is constant, independent of the rotor position ϑ_{m}. If the effect of the stator slots on the airgap function were taken into account, L_{a} would depend on ϑ_{m} (but not much).

Voltage Induced by the Field Winding

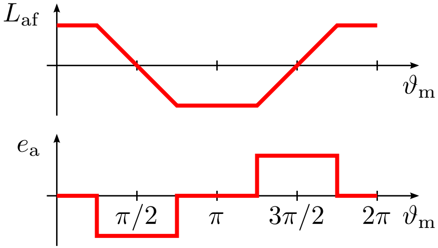
- Stator voltage can be expressed as

$$u_\mathrm{a} = Ri_\mathrm{a} + L_\mathrm{a}(\vartheta_\mathrm{m})\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} + i_\mathrm{a}\frac{\mathrm{d}L_\mathrm{a}(\vartheta_\mathrm{m})}{\mathrm{d}\vartheta_\mathrm{m}}\omega_\mathrm{m} + e_\mathrm{a}$$

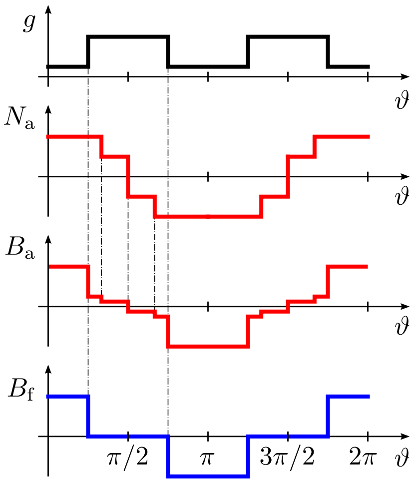
- Voltage induced by the field winding

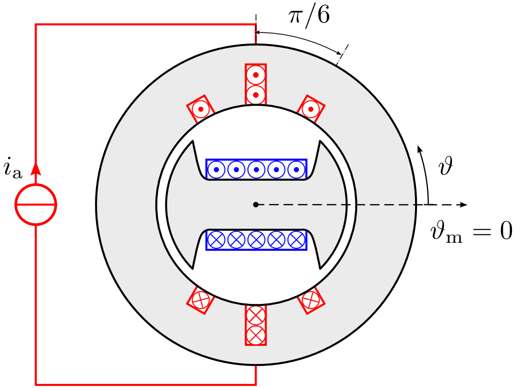
$$e_\mathrm{a} = i_\mathrm{f}\frac{\mathrm{d}L_\mathrm{af}(\vartheta_\mathrm{m})}{\mathrm{d}t} = i_\mathrm{f}\frac{\mathrm{d}L_\mathrm{af}(\vartheta_\mathrm{m})}{\mathrm{d}\vartheta_\mathrm{m}}\omega_\mathrm{m}$$

where constant i_f is assumed

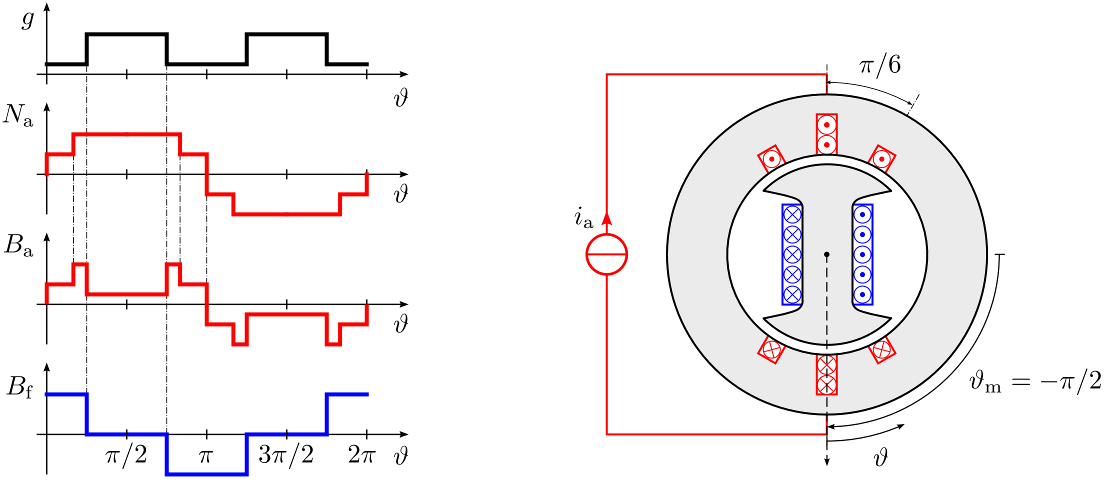


Flux Density Space Waveforms at $\vartheta_{\mathrm{m}} = 0$



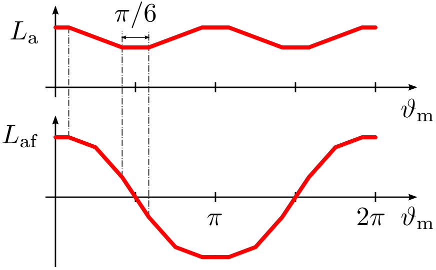


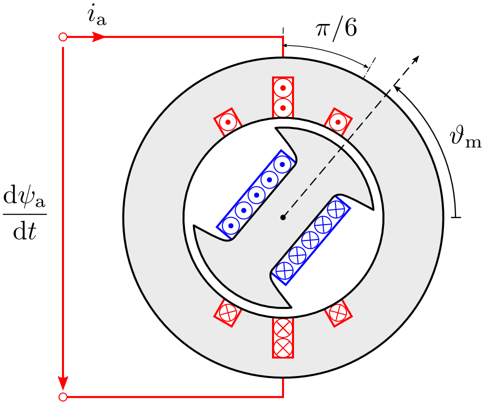
Flux Density Space Waveforms at $\vartheta_\mathrm{m} = -\pi/2$



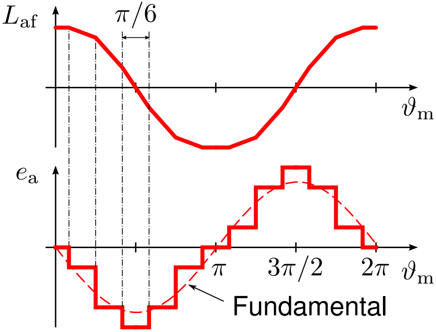
Inductances

$$\psi_{\mathrm{a}} = L_{\mathrm{a}}(\vartheta_{\mathrm{m}})i_{\mathrm{a}} + L_{\mathrm{af}}(\vartheta_{\mathrm{m}})i_{\mathrm{f}}$$

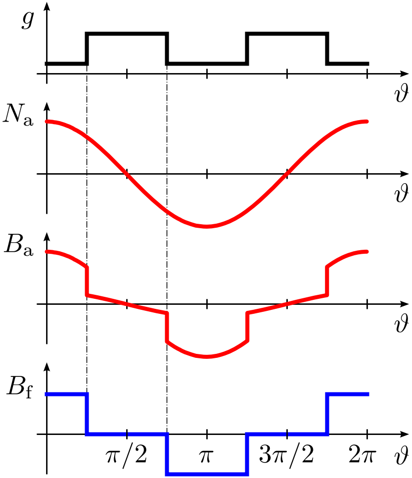




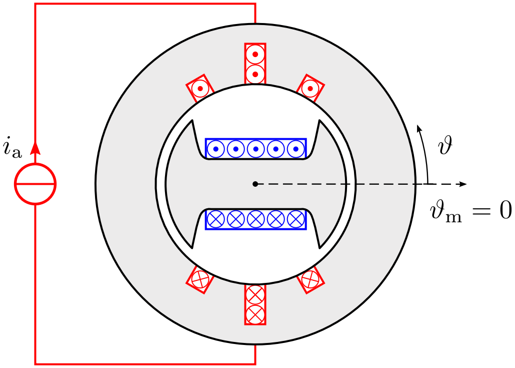
Voltage Induced by the Field Winding



Flux Density Space Waveforms at $\vartheta_\mathrm{m} = 0$

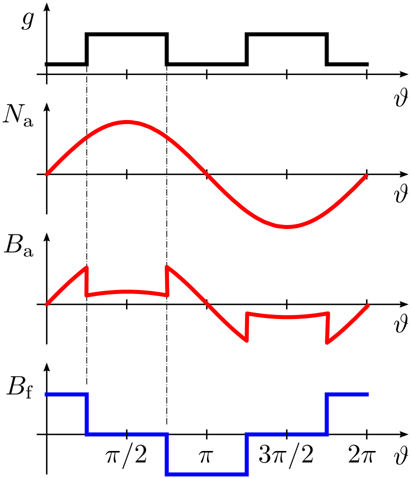


Ideal sinusoidal winding distribution

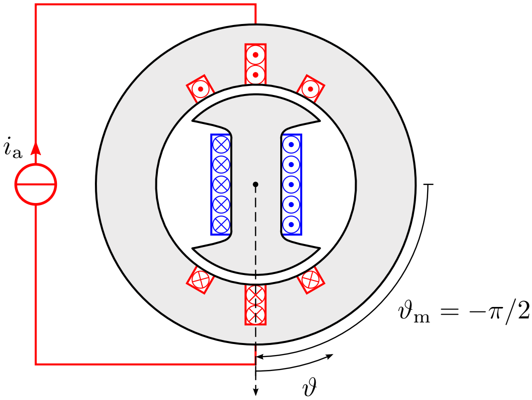


It is worth noticing that only the fundamental space component of flux density produces a net flux linkage in the sinusoidally distributed stator winding. This fact can be realized based on the flux linkage expression given earlier.

Flux Density Space Waveforms at $\vartheta_\mathrm{m} = -\pi/2$

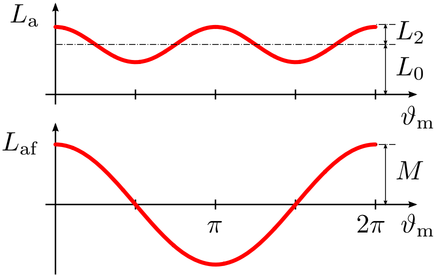


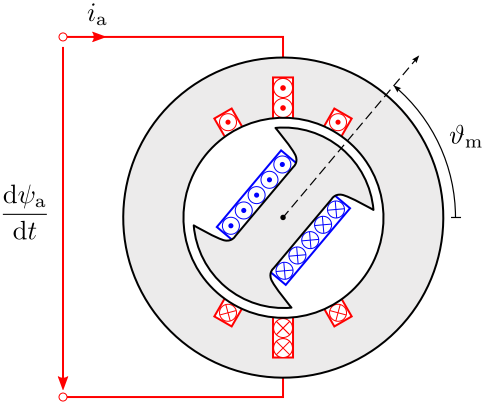
Ideal sinusoidal winding distribution



Inductances

$$\psi_{\mathrm{a}} = L_{\mathrm{a}}(\vartheta_{\mathrm{m}})i_{\mathrm{a}} + L_{\mathrm{af}}(\vartheta_{\mathrm{m}})i_{\mathrm{f}} = [L_0 + L_2\cos(2\vartheta_{\mathrm{m}})]i_{\mathrm{a}} + M\cos(\vartheta_{\mathrm{m}})i_{\mathrm{f}}$$





It can be noticed that the sinusoidal distribution of the winding increases the variation of the self-inductance L_{a}. The induced voltage e_{a} is not shown, but, naturally, it becomes sinusoidal as well.

Lossless Magnetic Field

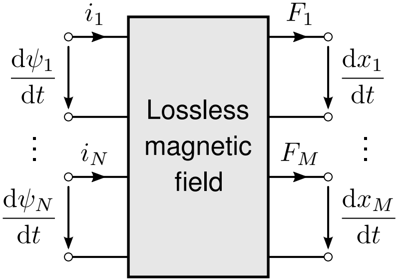
Understanding lossless magnetic field systems often helps in developing machine models for control purposes

- Very general and powerful concept
- Only assumption is that the magnetic field is lossless (conservative)
- Forces and torques in complex electromechanical systems can be determined
- Independent of machine type, number of terminals, number of poles, etc.
- Most lumped-parameter electric machine models are based on it
- Magnetic saturation and spatial harmonics can be taken into account
- Core losses can be modeled outside the lossless field system

Lossless Magnetic Field[3,4]

Stored magnetic field energy W_m

- is a state function, depending only on its independent state variables
- is independent of the path used to reach the state
- can be determined completely if the electrical port relations are known
- can be evaluated by means of numerical techniques (e.g. FEM) or measurements



[3] Woodson and Melcher, *Electromechanical Dynamics*. John Wiley & Sons, 1968.

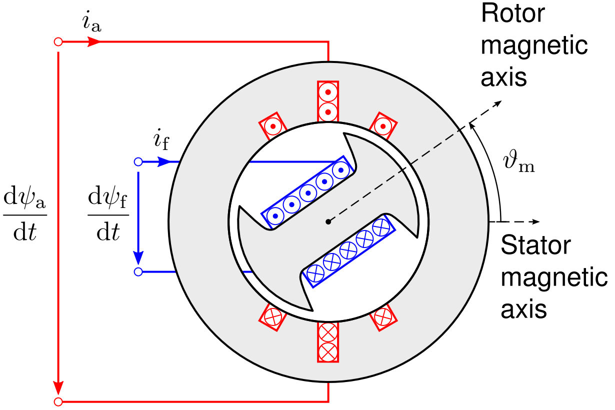
[4] Fitzgerald, Kingsley, and Umans, *Electric Machinery*. McGraw-Hill, 2003.

Example System: Single-Phase Machine With Field Winding

- Two electrical ports and one mechanical port
- Stored field energy

$$W_{\mathrm{m}} = W_{\mathrm{m}}(\psi_{\mathrm{a}}, \psi_{\mathrm{f}}, \vartheta_{\mathrm{m}})$$

  where ψ_{a}, ψ_{f}, and ϑ_{m} are independent state variables
- This example system is considered in the following



- Power balance

$$\frac{\mathrm{d}W_\mathrm{m}}{\mathrm{d}t} = i_\mathrm{a}\frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t} + i_\mathrm{f}\frac{\mathrm{d}\psi_\mathrm{f}}{\mathrm{d}t} - T_\mathrm{m}\frac{\mathrm{d}\vartheta_\mathrm{m}}{\mathrm{d}t}$$

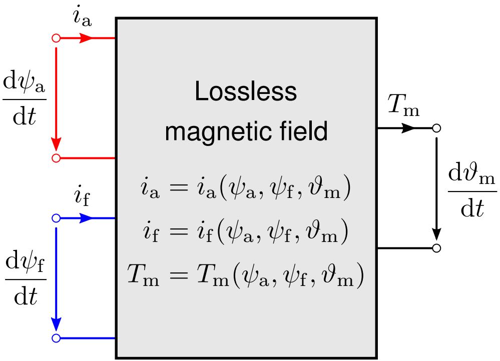
- Currents

$$i_\mathrm{a} = \frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \psi_\mathrm{a}}$$

$$i_\mathrm{f} = \frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \psi_\mathrm{f}}$$

- Torque

$$T_\mathrm{m} = -\frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \vartheta_\mathrm{m}}$$



Lossless Field System Should Satisfy Reciprocity Conditions

- Incremental mutual inductances should be equal in any operating point

$$\frac{\partial i_\mathrm{a}}{\partial \psi_\mathrm{f}} = \frac{\partial i_\mathrm{f}}{\partial \psi_\mathrm{a}}$$

- If multiple mechanical ports, analogous conditions hold for them as well
- Conditions between electrical and mechanical ports

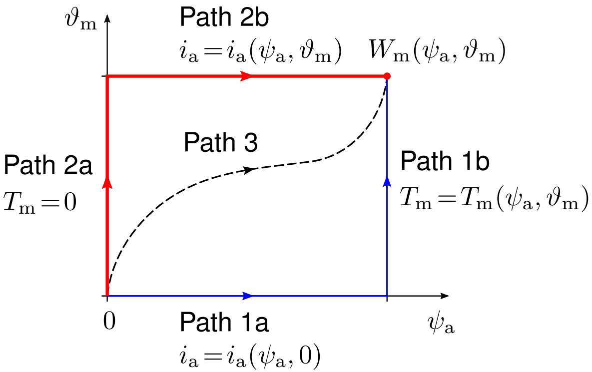
$$\frac{\partial i_\mathrm{a}}{\partial \vartheta_\mathrm{m}} = -\frac{\partial T_\mathrm{m}}{\partial \psi_\mathrm{a}} \qquad \frac{\partial i_\mathrm{f}}{\partial \vartheta_\mathrm{m}} = -\frac{\partial T_\mathrm{m}}{\partial \psi_\mathrm{f}}$$

Integration Path for the Field Energy Can Be Chosen Freely

- For illustration purposes
  $\psi_\mathrm{f} = 0$ assumed
- Integration along Path 1

$$W_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m}) = \int_0^{\psi_\mathrm{a}} i_\mathrm{a}(\psi_\mathrm{a}, 0)\mathrm{d}\psi_\mathrm{a} - \int_0^{\vartheta_\mathrm{m}} T_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m})\mathrm{d}\vartheta_\mathrm{m}$$

- We should know $T_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m})$



- Integration along Path 2

$$W_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m}) = \int_0^{\psi_\mathrm{a}} i_\mathrm{a}(\psi_\mathrm{a}, \vartheta_\mathrm{m}) \mathrm{d}\psi_\mathrm{a}$$

since $T_\mathrm{m}(0, \vartheta_\mathrm{m}) = 0$

- Torque is not needed in Path 2

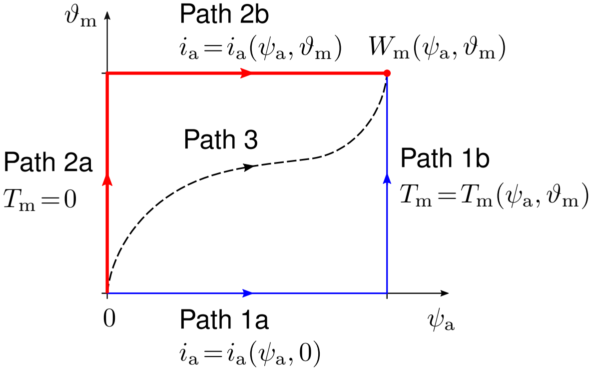
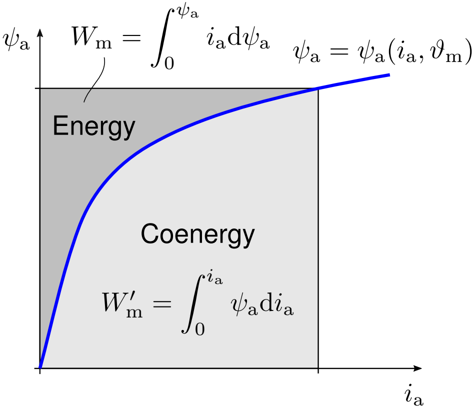


Illustration of Field Energy and Coenergy

- For illustration purposes $\psi_\mathrm{f} = 0$ assumed
- Area of the rectangle $\psi_\mathrm{a} i_\mathrm{a}$
- Relation of coenergy to field energy

$$W_\mathrm{m} + W'_\mathrm{m} = \psi_\mathrm{a} i_\mathrm{a}$$

- Magnetically linear case: $W_\mathrm{m} = W'_\mathrm{m}$



Field Energy and Coenergy

- Field energy

$$W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m}) = \int_0^{\psi_\mathrm{a}} i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})\mathrm{d}\psi_\mathrm{a} + \int_0^{\psi_\mathrm{f}} i_\mathrm{f}(0, \psi_\mathrm{f}, \vartheta_\mathrm{m})\mathrm{d}\psi_\mathrm{f}$$

- Coenergy

$$W'_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m}) = \int_0^{i_\mathrm{a}} \psi_\mathrm{a}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})\mathrm{d}i_\mathrm{a} + \int_0^{i_\mathrm{f}} \psi_\mathrm{f}(0, i_\mathrm{f}, \vartheta_\mathrm{m})\mathrm{d}i_\mathrm{f}$$

- Relation of coenergy to field energy

$$W_\mathrm{m} + W'_\mathrm{m} = \psi_\mathrm{a} i_\mathrm{a} + \psi_\mathrm{f} i_\mathrm{f}$$

- Torque is typically easier to calculate from coenergy

Torque from Coenergy

- Power balance

$$\frac{\mathrm{d}W'_\mathrm{m}}{\mathrm{d}t} = \psi_\mathrm{a}\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} + \psi_\mathrm{f}\frac{\mathrm{d}i_\mathrm{f}}{\mathrm{d}t} + T_\mathrm{m}\frac{\mathrm{d}\vartheta_\mathrm{m}}{\mathrm{d}t}$$

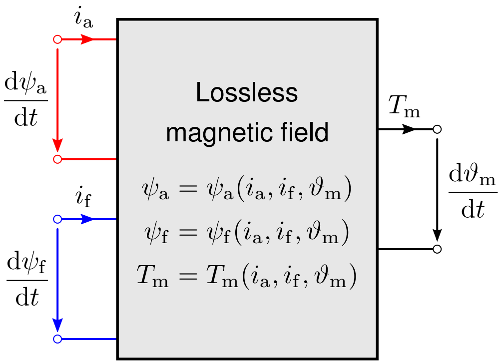
- Flux linkages

$$\psi_\mathrm{a} = \frac{\partial W'_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})}{\partial i_\mathrm{a}}$$

$$\psi_\mathrm{f} = \frac{\partial W'_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})}{\partial i_\mathrm{f}}$$

- Torque

$$T_\mathrm{m} = \frac{\partial W'_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \vartheta_\mathrm{m}}$$



Analytical Example

- Assume a magnetically linear machine with the flux linkages

$$\psi_\mathrm{a} = L_\mathrm{a}(\vartheta_\mathrm{m})i_\mathrm{a} + L_\mathrm{af}(\vartheta_\mathrm{m})i_\mathrm{f}$$
$$\psi_\mathrm{f} = L_\mathrm{af}(\vartheta_\mathrm{m})i_\mathrm{a} + L_\mathrm{f}i_\mathrm{f}$$

  where the inductances are

$$L_\mathrm{a}(\vartheta_\mathrm{m}) = L_0 + L_2\cos(2\vartheta_\mathrm{m}) \qquad L_\mathrm{af}(\vartheta_\mathrm{m}) = M\cos(\vartheta_\mathrm{m})$$

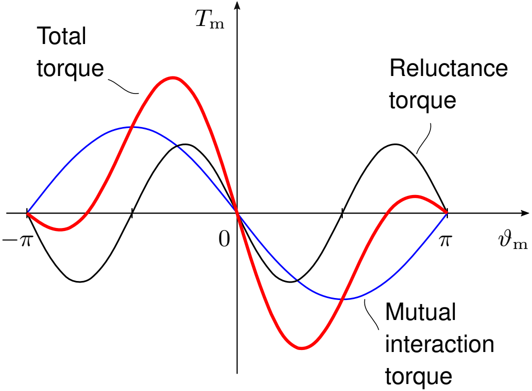
- Coenergy

$$W'_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m}) = \frac{1}{2}\left[L_0 + L_2\cos(2\vartheta_\mathrm{m})\right]i_\mathrm{a}^2 + M\cos(\vartheta_\mathrm{m})i_\mathrm{a}i_\mathrm{f} + \frac{1}{2}L_\mathrm{f}i_\mathrm{f}^2$$

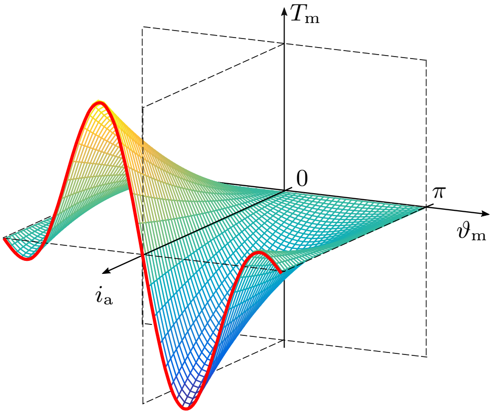
- Torque

$$T_\mathrm{m} = -M\sin(\vartheta_\mathrm{m})i_\mathrm{a}i_\mathrm{f} - L_2\sin(2\vartheta_\mathrm{m})i_\mathrm{a}^2$$

Torque

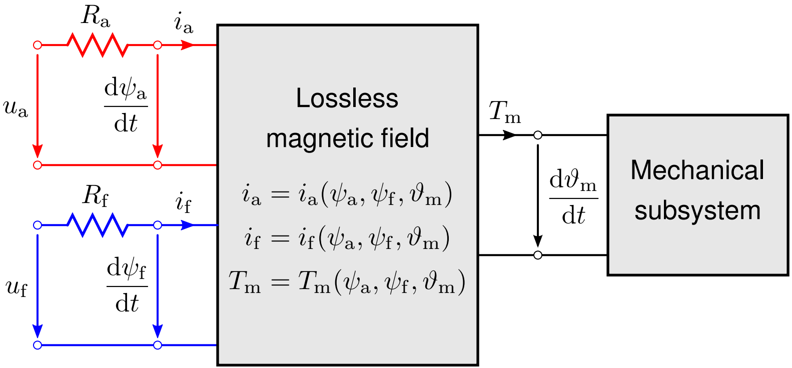


Currents i_a and i_f are constant



Field current i_f is constant

Inclusion of Voltage Equations



Voltage Equations: Flux Linkages as State Variables

- Voltage equations

$$\frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t} = u_\mathrm{a} - R_\mathrm{a} i_\mathrm{a} \qquad \frac{\mathrm{d}\psi_\mathrm{f}}{\mathrm{d}t} = u_\mathrm{f} - R_\mathrm{f} i_\mathrm{f}$$

  where the currents are known static functions of the state variables

$$i_\mathrm{a} = i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m}) \qquad i_\mathrm{f} = i_\mathrm{f}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$$

- Electromagnetic torque is the input for the mechanical subsystem

$$T_\mathrm{m} = T_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$$

  and the state variable ϑ_m is the output of the mechanical subsystem
- This set of equations is very simple to implement

The expression $T_\mathrm{m} = T_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})$ could be used as well since the currents $i_\mathrm{a} = i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$ and $i_\mathrm{f} = i_\mathrm{f}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$ are known.

Voltage Equations: Currents as State Variables

- If the currents are used the state variables, the representation of the voltage equations becomes complex, for example

$$\frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t} = \frac{\partial\psi_\mathrm{a}}{\partial i_\mathrm{a}}\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} + \frac{\partial\psi_\mathrm{a}}{\partial i_\mathrm{f}}\frac{\mathrm{d}i_\mathrm{f}}{\mathrm{d}t} + \frac{\partial\psi_\mathrm{a}}{\partial\vartheta_\mathrm{m}}\frac{\mathrm{d}\vartheta_\mathrm{m}}{\mathrm{d}t} = u_\mathrm{a} - R_\mathrm{a}i_\mathrm{a}$$

- In general case, all the partial derivatives are functions of i_a, i_f, and ϑ_m
- In the magnetically linear example case

$$L_\mathrm{a}(\vartheta_\mathrm{m})\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} + L_\mathrm{af}(\vartheta_\mathrm{m})\frac{\mathrm{d}i_\mathrm{f}}{\mathrm{d}t} + \left[\frac{\partial L_\mathrm{a}(\vartheta_\mathrm{m})}{\partial\vartheta_\mathrm{m}}i_\mathrm{a} + \frac{\partial L_\mathrm{af}(\vartheta_\mathrm{m})}{\partial\vartheta_\mathrm{m}}i_\mathrm{f}\right]\frac{\mathrm{d}\vartheta_\mathrm{m}}{\mathrm{d}t} = u_\mathrm{a} - R_\mathrm{a}i_\mathrm{a}$$

and similarly for the rotor voltage equation