Lecture 7

- Gave an intuitive definition of a closed and bounded region in the plane and stated the result that continuous functions on such domain attain their absolute extrema.
- Did the following example and looked at computer generated images (see "Materials") to help understand our calculations. Find the absolute extrema of f(x,y) = x^2 + 2 y^2 on the disk of radius one centered at (0,0).
- Discussed the method of Lagrange multipliers and justified it intuitively by looking at a sketch of level curves and using the fact that the gradient vectors are orthogonal to the corresponding level curves.
- Re-solved the previous example using the method of Lagrange multipliers.
- Did another example of Lagrange multipliers: On the curve x^2 + xy+y^2 which points are closest and furthest from the origin?
- Reviewed Taylor series and Taylor polynomials 1 variable.
- Gave the definition of the nth Taylor polynomial in two variables. Wrote out the 2nd order Taylor polynomial on 2 variables. Noted that (1) the 1st order Taylor polynomial gives the tangent plane equatin as expected., and (2) The 2nd order terms can be written [x-x_0 y-y_0] H(x_0,y_0) [x-x_0 y-y_0]^T (where T denoted transpose and H is the Hessian). The 2nd derivative test can be understood by learning the corresdpondence between properties of the Hessian H and the properties of the quadratic polynomial [x-x_0 y-y_0] H(x_ 0,y_0) [x-x_0 y-y_0]^T, but we will not discuss this in this course. But it is closely related to the bonus exercise in assignment 4.
- Showed an example using Maple of the the 2nd order Taylor polynomial approximating a surface. The code and output can be found in "materials".
- (Not covered in class but in the notes)

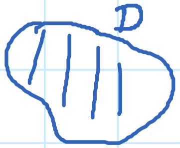
Where to find this material

- Adams_and_Essex 13.1, 13.2, 13.3
- Adams_and_Essex. 12.9. See "materials" for a copy of this sections.
- Corral, 2.5, 2.7
- Guichard, 14.7, 14.8
- Active Calculus. 10.7, 10.8

Absolute extrema (2)

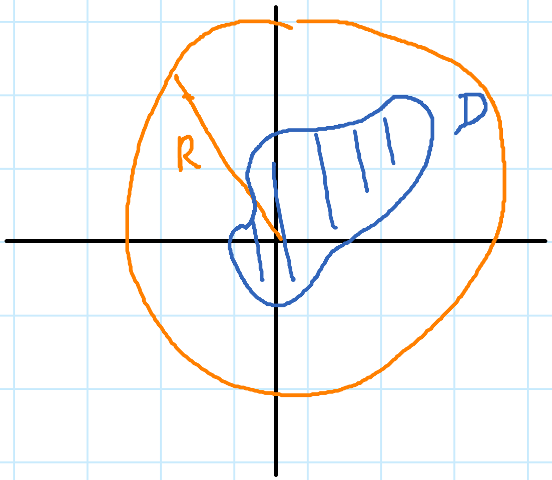
Theorem: A continuous function $f(x, y)$ on a closed and bounded domain $D \subset \mathbb{R}^2$ attains its absolute maximum and minimum on D.

closed.



D contains all its boundary points (see lecture #2)

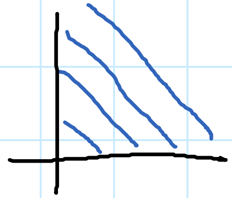
bounded



There exist $R > 0$ such that D lies inside the circle of radius R

example

① $D = \{ (x, y) \mid x \geq 0, y \geq 0 \}$



is closed and unbounded

How to find absolute extrema?

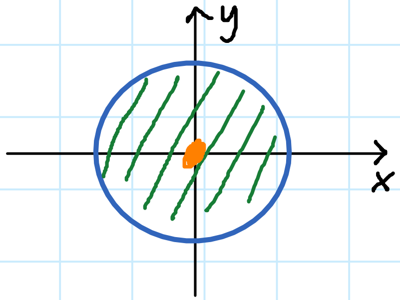
Need to look

1. **In the interior** - the extrema can only occur at critical points (relatively easy to find)

2. **On the boundary** - this is difficult as the boundary is in general a curve. We say that we a **finding extrema subject to the constraint** of being on the boundary. This is an example of **constrained optimization** which is a very general problem type that appears in a huge variety of applications.

Extrema example

Find the absolute extrema of $f(x,y) = x^2 + 2y^2$ on the closed disk of radius 1.



Interior : $x^2+y^2<1$

Boundary: $x^2+y^2=1$

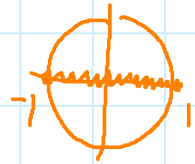
INTERIOR: Find the critical points

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= 2x = 0\\ \frac{\partial f}{\partial y} &= 4y = 0\end{aligned}\right\} \Rightarrow (x,y)=(0,0)$$

BOUNDARY We don't have a general method yet, but in this simple example we can manage with 1-variable methods by elimanting one of the variable.

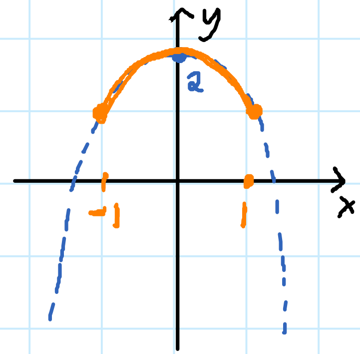
⊛ $x^2+y^2=1 \Rightarrow y^2=1-x^2$

So $f(x,y(x)) = x^2+2(1-x^2)$
$= 2-x^2$



on the domain $[-1,1]$

We now have a 1-variable problem



Critical point: $f'(x) = -2x = 0$
$\Rightarrow x=0$
⊛ $\Rightarrow y=\pm 1$

$(0,1)$, $(0,-1)$

Endpoints $x=-1$, $x=1$
$\Rightarrow y=0$

$(1,0)$, $(-1,0)$

Possible locations for the absolute extrema

LOCATION	VALUE
$(0,0)$	$f=0$
$(0,1)$	$f=2$
$(0,-1)$	$f=2$
$(1,0)$	$f=1$
$(-1,0)$	$f=1$

Abs max $f=2$ at $(0,1)$, $(0,-1)$

Abs min $f=0$ at $(0,0)$

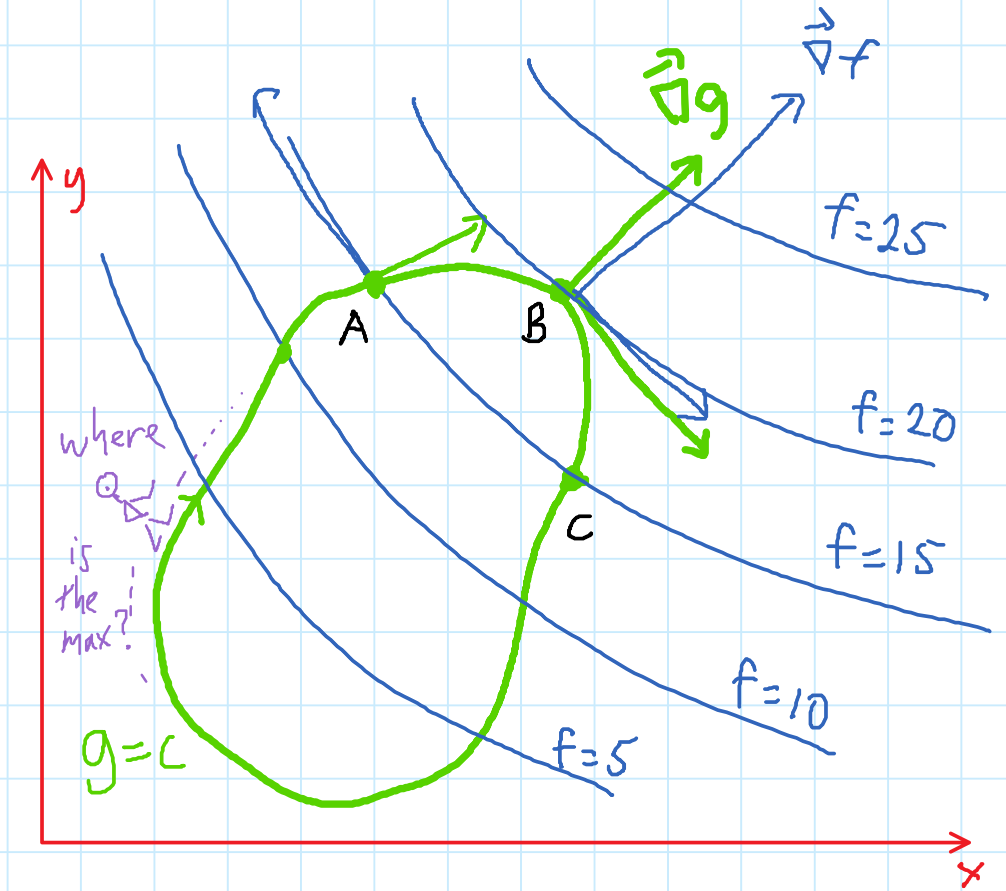
See software plots

Lagrange multipliers

The boundary part of the previous example (extrema values of $f(x,y) = x^2 + 2y^2$ on the curve $x^2 + y^2 = 1$) is a problem of the following form.

Find the absolute extrema of a function $f(x,y)$ subject to a constraint $g(x,y) = c$

Let's invent a method. Hmmm . . . . . ?



Questions

① Can the max be at A? No (cross the level curve)

② Can the max be at B? Yes (just touch the level curve)

③ Can the max be at C? No

The point B is special. Mathematically we can encode this by saying

- Tangents are parallel
- So, the normal are parallel
- So, $\vec{\nabla} g(B)$ is parallel to $\vec{\nabla} f(B)$
- So $\vec{\nabla} f(B) = (\text{constant})\, \vec{\nabla} g(B)$

$$\vec{\nabla} f(B) = \lambda \vec{\nabla} g(B)$$

Lagrange multiplier

Lagrange multipliers (2)

The method of Lagrange multipliers.
The absolute extrema of a function $f(x,y)$ subject to a constraint $g(x,y) = c$ can only occur at points where $\vec{\nabla} f = \lambda \vec{\nabla} g$ or $\vec{\nabla} f$ is undefined.

Let us look at the previous example using this method.

$$f(x,y) = x^2 + 2y^2.$$

Constraint. $g(x,y) = x^2 + y^2 = 1$

Solution: $\vec{\nabla} f = \langle 2x, 4y \rangle$, $\vec{\nabla} g = \langle 2x, 2y \rangle$

Solve $\left.\begin{array}{l} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 1 \end{array}\right\} \Rightarrow$

$$2x = \lambda 2x \quad (1)$$
$$2y = \lambda 4y \quad (2)$$
$$x^2 + y^2 = 1 \quad (3)$$

$(1) \Rightarrow x(1-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 1$

case $x = 0$: $(3) \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

points $(0,1)$, $(0,-1)$

case $\lambda = 1$ $(2) \Rightarrow 2y = 4y \Rightarrow y = 0$

Then $(3) \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

points $(1,0)$, $(-1,0)$

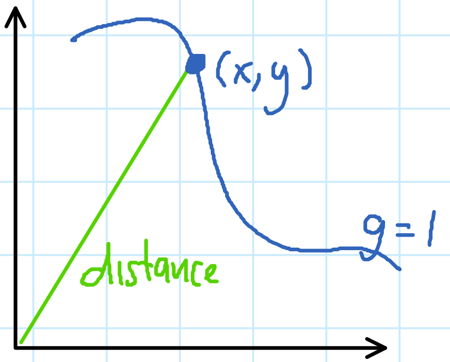
Conclusion The possible extrema occur at $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$

as before ✓

Example 2

Find the closest and furthest points on the curve $x^2 + xy + y^2 = 1$ from the origin.

Set-up



Let $g(x,y) = x^2 + xy + y^2$

Let $f(x,y) = \text{distance}^2$
$= x^2 + y^2$

Note: minimizing distance or distance ^2 gives the same locations.

Aim Minimize and maximize $f(x,y) = x^2 + y^2$ subject to the constraint $g(x,y) = x^2 + xy + y^2 = 1$

Solve $\vec{\nabla} f = \lambda \vec{\nabla} g$, $g = 1$, $\dfrac{\partial f}{\partial x} = 2x$, $\dfrac{\partial f}{\partial y} = 2y$

$\dfrac{\partial g}{\partial x} = 2x + y$, $\dfrac{\partial g}{\partial y} = 2y + x$

$\Rightarrow$
$$2x = \lambda(2x + y) \quad (1)$$
$$2y = \lambda(2y + x) \quad (2)$$
$$x^2 + xy + y^2 = 1 \quad (3)$$

Hint: Look at (1) − (2) (or (1) + (2))

$2(x - y) = \lambda(x - y) \Rightarrow (x-y)(2-\lambda) = 0$

$y = x$ or $\lambda = 2 \Rightarrow 2x = 4x + 2y$
$y = -x$

(3) $x^2 + x^2 + x^2 = 1$
$x = \pm 1/\sqrt{3}$

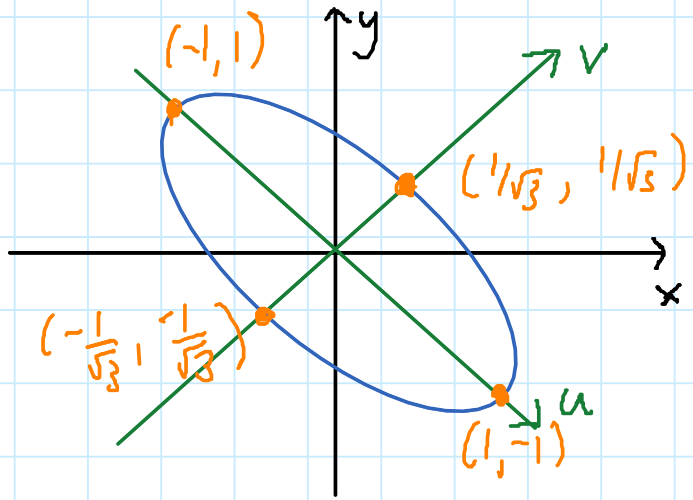
(3) $\Rightarrow x^2 = \pm 1$

what shape is the curve? (extra – not examinable)

change variables: $u = \dfrac{x-y}{2}$, $v = \dfrac{x+y}{2}$

Then $u + v = x$, $v - u = y$

$x^2 + xy + y^2 = (u^2 + v^2 + 2uv) + (v^2 - u^2) + (u^2 + v^2 - 2uv)$
$= u^2 + 3v^2 = 1$



Taylor series

1 variable review

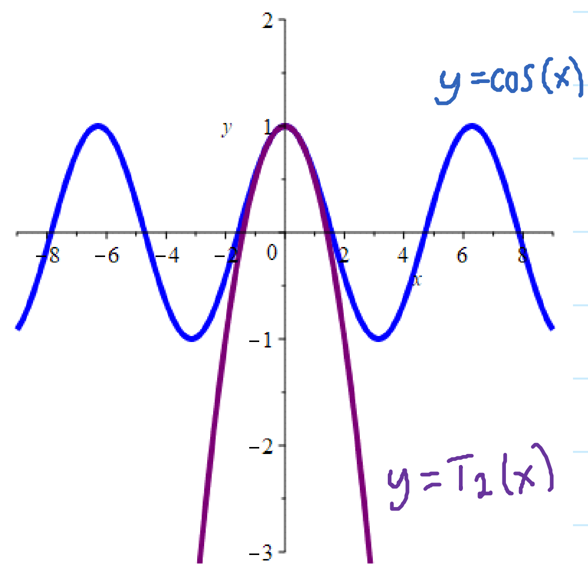
Function $f(x) = \cos(x)$

Taylor series (centered at $x=0$) $= T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

2nd degree Taylor polynomial $= T_2(x)$

$= 1 - \frac{x^2}{2!}$



Idea/use: Information about f and all it's derivates at $x = a$ is often enough to determine $f(x)$ for other values of x.

(1) In the previous example in fact $\cos(x) = T(x)$ for all x.

and $f(0)=0$

(2) If $f(x) = e^{-1/x^2}$ the Taylor series centered at $x = 0$ is $T(x) \equiv 0$. So $f(x) \neq T(x) \quad \forall x \neq 0$

Why are the coefficients $\frac{f^{(n)}(0)}{n!}$?

Idea: Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

How can we extract the coefficients a_0, a_1, a_2, a_3

$p(0) = a_0$ (constant term is easy)

Now, let's make b into the constant term.

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2$, $p'(0) = a_1$

$p''(x) = 2a_2 + 3\cdot 2 a_3 x$, $\frac{p''(0)}{2} = a_2$

$p'''(x) = 3\cdot 2\cdot a_3$, $\frac{p'''(0)}{3\cdot 2} = a_3$

$3\cdot 2 = 3!$

In general

$a_n = p^{(n)}(0)/n!$

Taylor polynomial in 2 variables

The same idea motivates the definition. Lets see how to extract the coefficinet of a term in a polynomial of x and y.

$3x^2y^4$

Say $p(x,y) = a_{ij}\, x^i y^j$. Then $a_{ij} = \dfrac{\partial^{i+j} p(0)}{\partial x^i \partial y^j} \Big/ i!\,j!$

The n^{th} degree Taylor polynomial of $f(x,y)$ is

$$\boxed{T_n f(x_0,y_0) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j} (x-x_0)^i (y-y_0)^j}$$

The 2^{nd} degree Taylor polynomial is (look at the case $(x_0,y_0)=(0,0)$ to shorten notation)

$$T_2 f(0,0) = \underbrace{f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y}_{\text{Linear approx}} + \frac{f_{xx}(0,0)\,x^2}{\underset{=2!}{2}} + \frac{f_{xy}(0,0)\,xy}{1!\,1!_{=1}} + \frac{f_{yy}(0,0)\,y^2}{\underset{=2!}{2}}$$

$$= f(0,0) + \begin{bmatrix} f_x(0,0) & f_y(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\vec{\nabla} f(0,0) \cdot \langle x, y \rangle$

Hessian (matrix) $= H$

Taylor series example

$f(x,y) = \sqrt{x^2 + y^3}$. Compute $T_2 f(x,y)$

centered at $(1,2)$

0^{th} degree term: $f(1,2) = \sqrt{1+8} = 3$

1^{st} degree terms: $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{}}$

$\frac{\partial f}{\partial x}(1,2) = \frac{1}{3}$

$\frac{\partial f}{\partial y} = \frac{\frac{3}{2}y^2}{\sqrt{}}$

$\frac{\partial f}{\partial y}(1,2) = \frac{6}{3} = 2$

2^{nd} degree terms: Arrrggbhh !!!!

Let's look at the polynomial and the graphs using computer software.

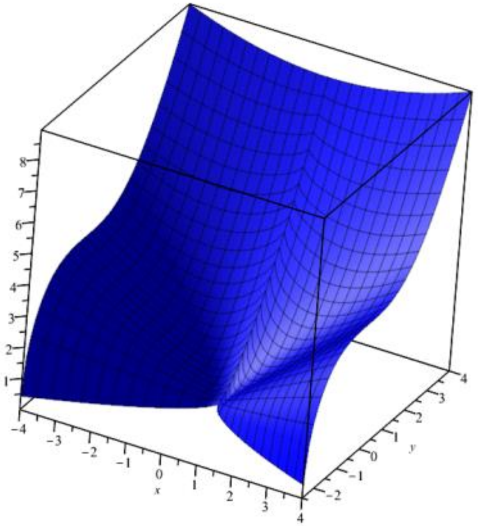
Note: in exams you will get functions that are quick to differentiate

$$T2f := -\frac{4}{3} + 2y + \frac{x}{3} + \frac{4(x-1)^2}{27} - \frac{2(y-2)(x-1)}{9} + \frac{(y-2)^2}{3}$$

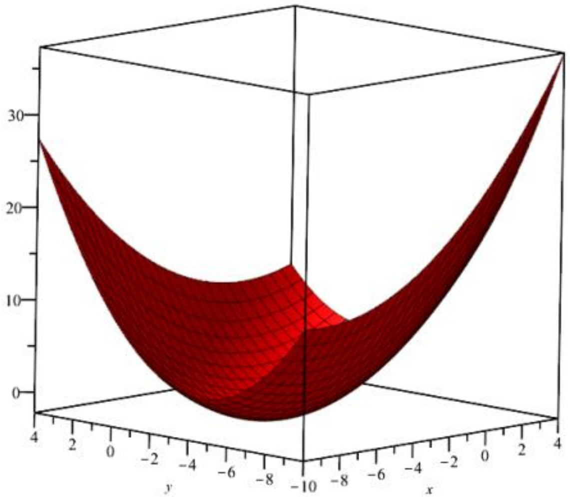
Plots are on the next page

(code and images are in MyCourses)

$$z = f(x,y) = \sqrt{x^2 + y^3}$$

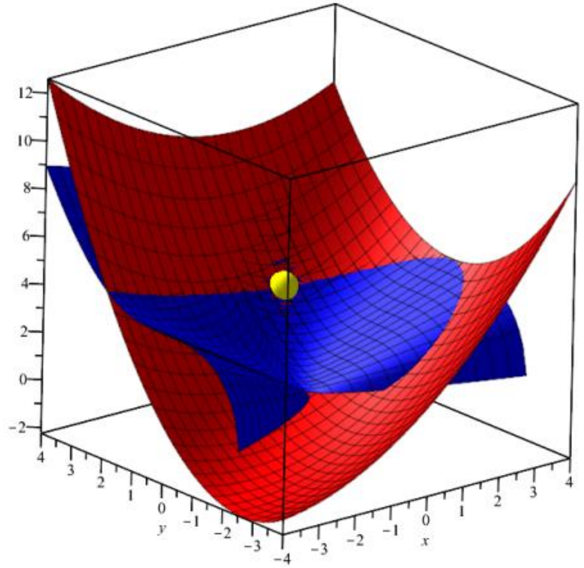


Note the cusp at (0,0,0)



$$z = T2f := -\frac{4}{3} + 2y + \frac{x}{3} + \frac{4(x-1)^2}{27} - \frac{2(y-2)(x-1)}{9} + \frac{(y-2)^2}{3}$$

The surface and it's quadratic approximation at (1,2)



Zooming in