Nonlinear dynamics & chaos

2D phase-plane analysis

Lecture V

Recap:
2D Linear systems

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

Matrix form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Classification of linear systems

Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda \mathbf{v}$$

Characteristic equation

$$\det(A - \lambda I) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

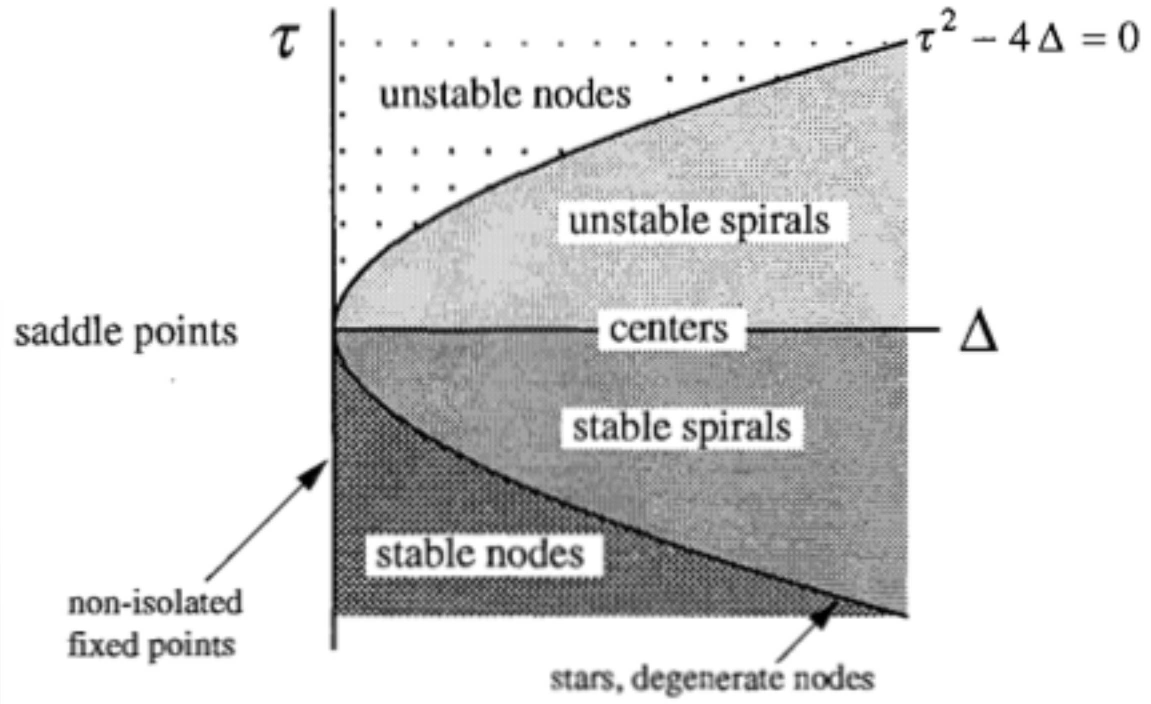
$$\begin{aligned} \tau &= \mathrm{trace}(A) = a + d \\ \Delta &= \det(A) = ad - bc \end{aligned}$$

Classification of fixed points

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2$$

Δ and τ are solved from

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - \tau\lambda + \Delta = 0$$



Phase portraits

The general form of a vector field on the phase plane:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

In vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

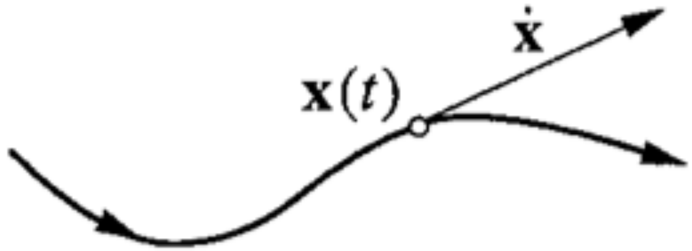
$$[\mathbf{x} = (x_1, x_2), \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))]$$

$$\mathbf{x} = \text{point in phase plane}$$

$$\dot{\mathbf{x}} = \text{velocity at that point}$$

Phase portraits

Solution **x**(t) describes a trajectory on the phase plane



The whole plane is filled with (non-intersecting) trajectories starting from different phase points.

For nonlinear systems there is no hope to find trajectories analytically + the analytical solutions would not provide much insight.

Our approach: determine the qualitative behavior of the solutions via phase portraits.

Example I

$$\begin{aligned} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{aligned}$$

Phase portrait: plot the nullclines.

The **nullclines** are the curves where

$$\dot{x} = 0 \qquad \text{or} \qquad \dot{y} = 0$$

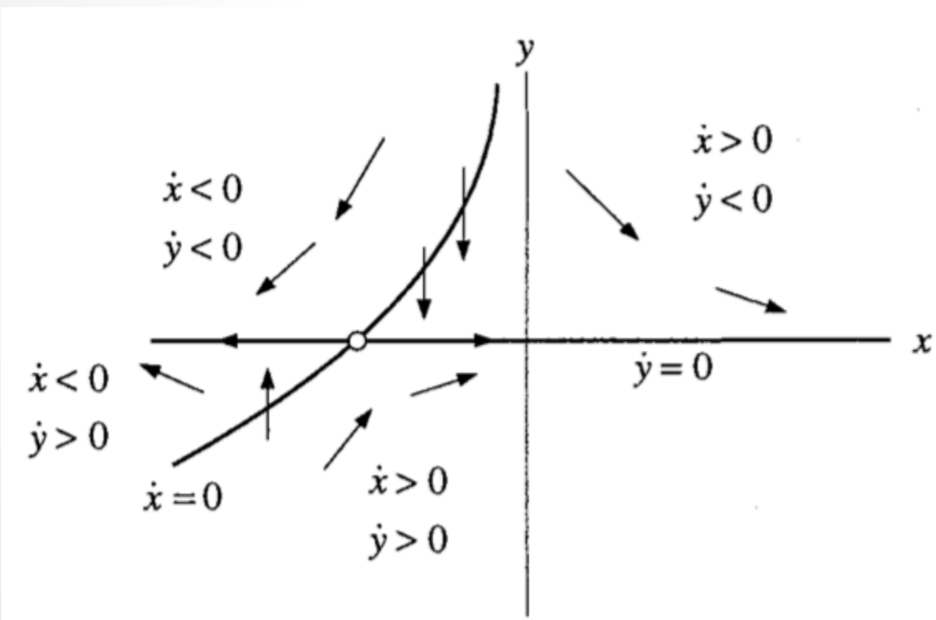
On the nullclines the flow is either purely horizontal or purely vertical

$$\begin{aligned} x + e^{-y} &= 0 \\ y &= 0 \end{aligned}$$

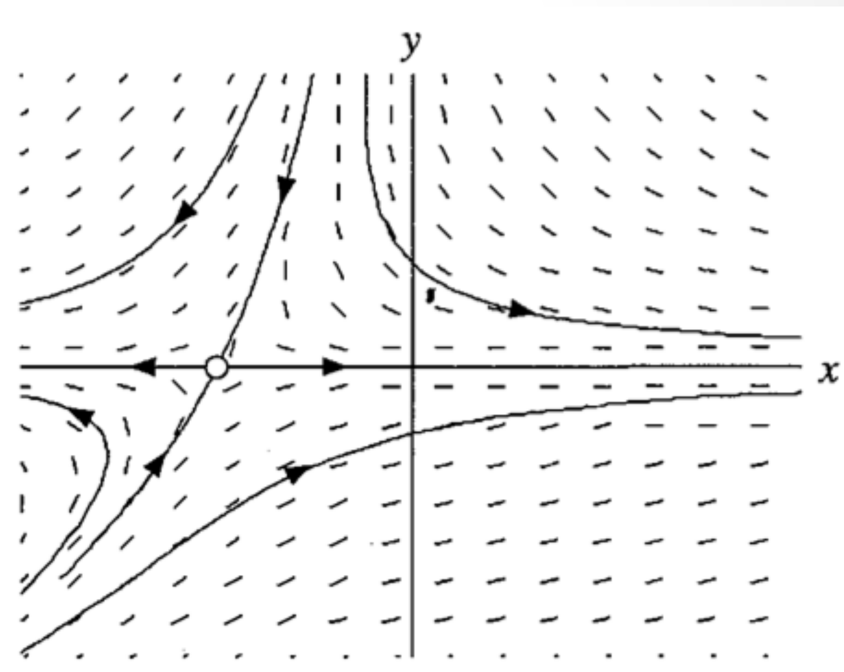
Example I

$$\begin{aligned}\dot{x} &= x + e^{-y} \\ \dot{y} &= -y\end{aligned}$$

Analysis:



Numerical solution:

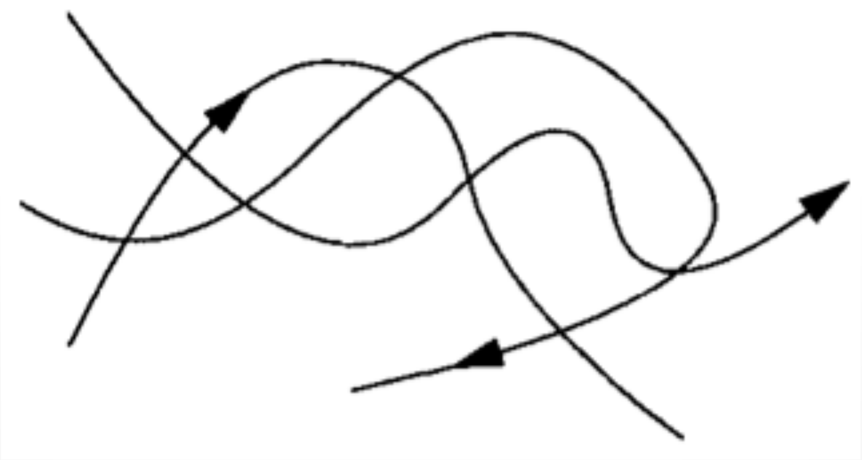


Existence, uniqueness and topological consequences

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x_0}$$

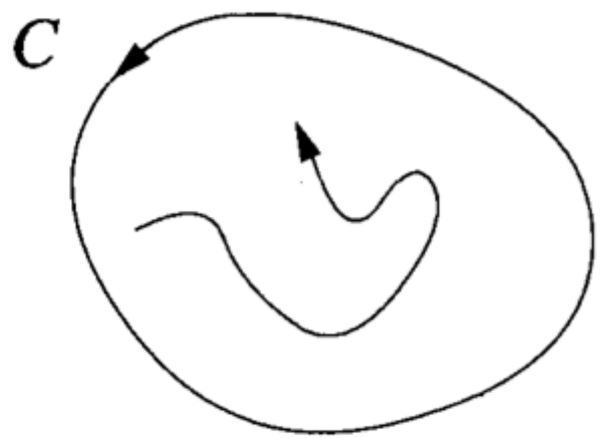
Corollary: different trajectories never intersect!

If two trajectories did intersect there would be two solutions starting from the same point (the crossing point).



Existence, uniqueness and topological consequences

Consequence in two dimensions: any trajectory starting from inside a closed orbit will be trapped inside it forever!



(End of recap.)

Fixed points and linearization

Aim: To approximate the phase portrait near a fixed point by that of a corresponding linear system.

The complete system

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

Fixed point (x^*, y^*)

$$f(x^*,y^*) = 0, \qquad g(x^*,y^*) = 0$$

Components of a small disturbance from the fixed point

$$u = x - x^*, \qquad v = y - y^*$$

Does the disturbance (perturbation) grow or decay?

$$\dot{u} = \dot{x}, \qquad \dot{v} = \dot{y}$$

Fixed points and linearization

$$\begin{aligned}\dot{u} &= \dot{x} \\ &= f(x^* + u, y^* + v) \\ &= f(x^*, y^*) + u\frac{\partial f}{\partial x}\bigg|_{(x^*,y^*)} + v\frac{\partial f}{\partial y}\bigg|_{(x^*,y^*)} + O(u^2, v^2, uv) \\ &= u\frac{\partial f}{\partial x}\bigg|_{(x^*,y^*)} + v\frac{\partial f}{\partial y}\bigg|_{(x^*,y^*)} + O(u^2, v^2, uv)\end{aligned}$$

Likewise:

$$\dot{v} = u\frac{\partial g}{\partial x}\bigg|_{(x^*,y^*)} + v\frac{\partial g}{\partial y}\bigg|_{(x^*,y^*)} + O(u^2, v^2, uv)$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \begin{pmatrix} u \\ v \end{pmatrix} + \text{ quadratic terms}$$

Fixed points and linearization

$$A = \left(\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right)_{(x^*,y^*)}$$

is the Jacobian matrix at the fixed point (x^*, y^*). It is the multivariate analog of the derivative $f'(x^*)$ for 1-dimensional systems.

Neglecting terms of the second and higher order we obtain the linearized system

$$\left(\begin{array}{c} \dot{u} \\ \dot{v} \end{array} \right) = \left(\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right)_{(x^*,y^*)} \left(\begin{array}{c} u \\ v \end{array} \right)$$

Fixed points and linearization

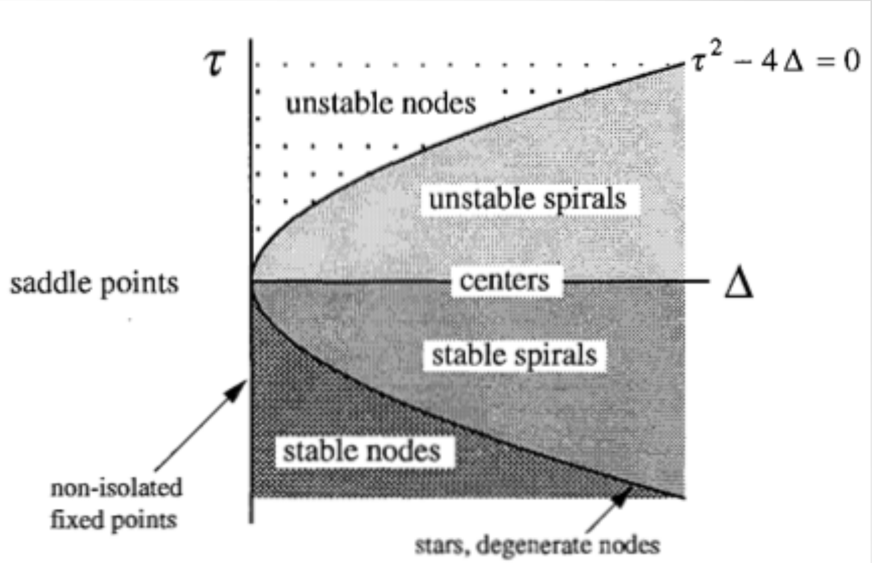
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} \begin{pmatrix} u \\ v \end{pmatrix}$$

Gain: The dynamics near the fixed points can be analyzed using the methods for linear systems.

The effect of small nonlinear terms:

If the fixed point is not one of the borderline cases (centers, degenerate nodes, stars, non-isolated fixed points) the predicted type of the linearized system is the correct one.

Example I



$$\begin{aligned}\dot{x} &= -x + x^3 \\ \dot{y} &= -2y\end{aligned}$$

Fixed points: (0,0), (1,0), (-1,0)

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} = \begin{pmatrix} -1 + 3{x^*}^2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$(0,0) \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad (\pm 1,0) \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\tau = -3,\ \Delta = 2;\ \tau^2 - 4\Delta = 1 \Rightarrow$

stable node

$\tau = 0,\ \Delta = -4 \Rightarrow$

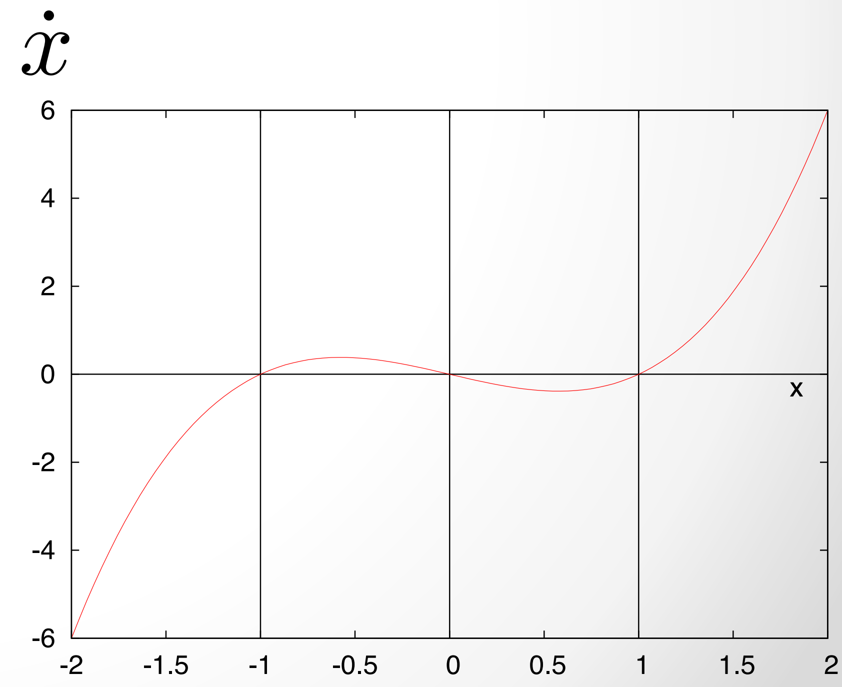
saddle points

Example I

$$\begin{aligned} \dot{x} &= -x + x^3 \\ \dot{y} &= -2y \end{aligned}$$

Let's check the result from linearization::

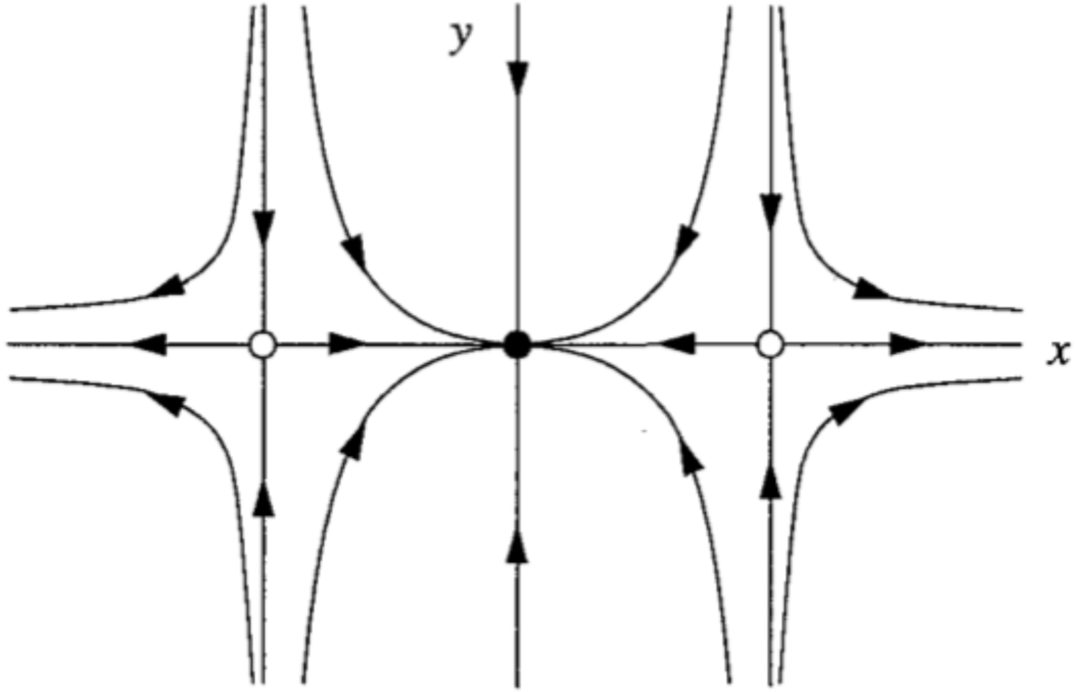
1) Equations for x and y are uncoupled.
2) y-direction: trajectories decay exponentially to $y = 0$.
3) x-direction: trajectories are attracted to $x = 0$ and repelled from $x = \pm 1$.
4) Vertical lines $x = 0$ and $x = \pm 1$ are invariant: a trajectory starting on these lines stays on them forever.
5) The horizontal line $y = 0$ is invariant.



Example I

$$\begin{aligned}\dot{x} &= -x + x^3 \\ \dot{y} &= -2y\end{aligned}$$

The phase portrait is symmetric with respect to the x- and the y -axes, since the equations are *invariant* under transformations $x \to -x$ and $y \to -y$.



Example II (a borderline case)

$$\begin{aligned}\dot{x} &= -y + ax(x^2 + y^2)\\ \dot{y} &= x + ay(x^2 + y^2)\end{aligned}$$

(0, 0) is a fixed point → linearisation. The Jacobian

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\tau = 0$, $\Delta = 1 > 0$ → the fixed point (0, 0) of the linearized system is a center.

To analyze the full system we switch to polar coordinates.

Example II

$$\begin{aligned}\dot{x} &= -y + ax(x^2 + y^2)\\ \dot{y} &= x + ay(x^2 + y^2)\end{aligned}$$

Polar coordinates: $\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$

Standard trick for deriving the differential equation for *r* in polar coordinates (remember this):

A: Use $x^2 + y^2 = r^2 \quad \rightarrow \quad x\dot{x} + y\dot{y} = r\dot{r}$

Substitute for $\dot{x}$ and $\dot{y}$ to get

$$r\dot{r} = x[-y + ax(x^2 + y^2)] + y[x + ay(x^2 + y^2)] = a(x^2 + y^2)^2 = ar^4$$

➔ $\dot{r} = ar^3$

Example II

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

B: Use $$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2}$$ (… and remember this)

Derivation: $\theta = \arctan(\frac{y}{x})$; $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$$\dot{\theta} = \frac{d}{dt}\arctan(\frac{y}{x}) = \frac{x\dot{y} - \dot{x}y}{x^2}\frac{x^2}{x^2+y^2} = \frac{x\dot{y} - \dot{x}y}{r^2}$$

Example II

$$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2} = \frac{x^2 + axy(x^2+y^2) + y^2 - axy(x^2+y^2)}{r^2} = 1$$

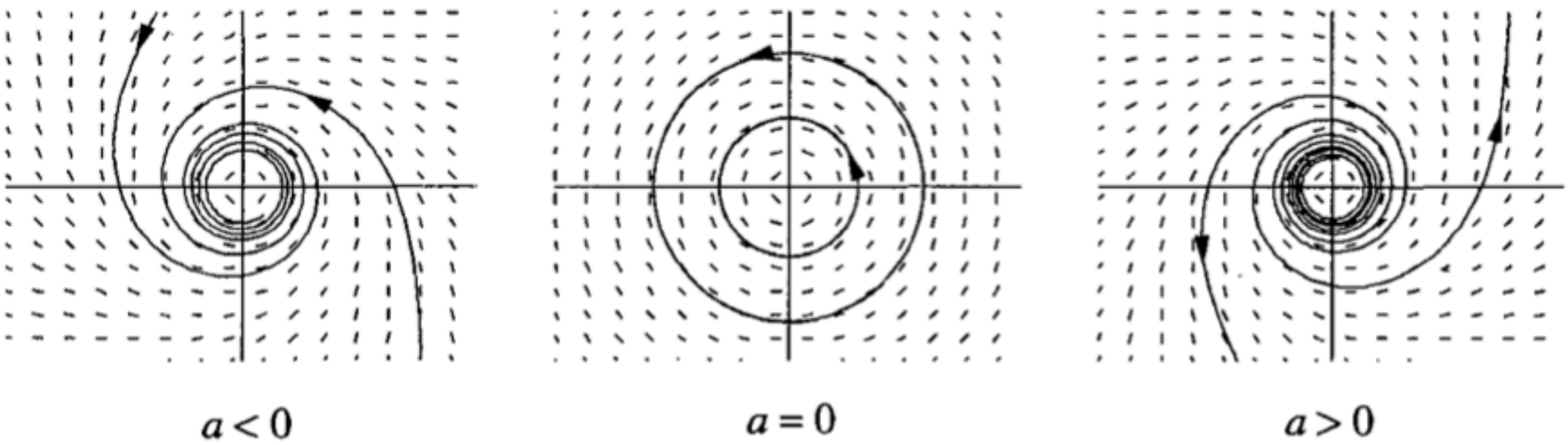
$$\dot{\theta} = 1$$

$$\Longrightarrow \qquad \begin{array}{rcl} \dot{r} & = & ar^3 \\ \dot{\theta} & = & 1 \end{array}$$

Example II

$$\begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned}$$

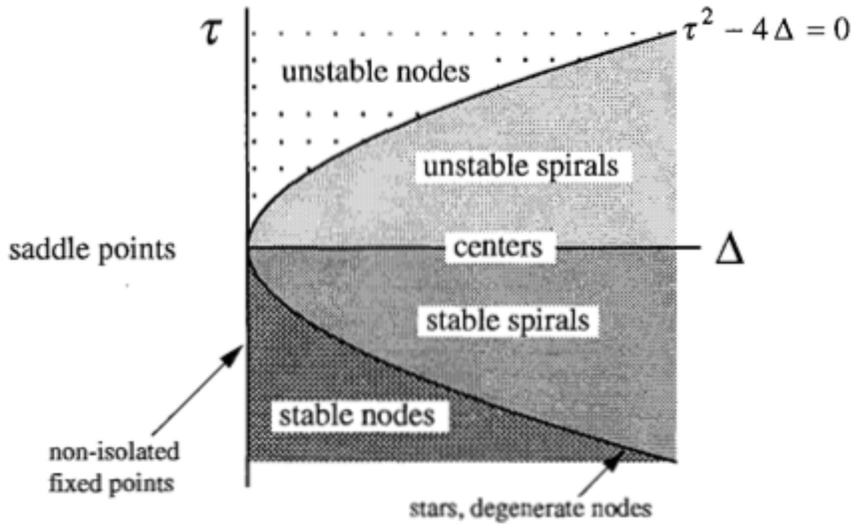
Radial and angular motions are independent



The fixed point is a spiral (stable for $a < 0$, unstable for $a > 0$). Centers ($a = 0$) are delicate: the orbit needs to close perfectly after one cycle, the slightest perturbation turns it into a spiral.

Fixed points and linearization

Stars and degenerate nodes can be altered by small nonlinearities; however, unlike in the case of centers their stability does not change! (Example: stable star ➔ stable spiral.)



In other words, stars and degenerate nodes stay well within regions of stability or instability: small perturbations will leave them in those areas.

Fixed points and linearization

Robust cases

1) *Repellers* (or *sources*): both eigenvalues have positive real part
2) *Attractors* (or *sinks*): both eigenvalues have negative real part
3) *Saddles*: one eigenvalue is positive, the other is negative

Marginal cases

1) *Centers*: both eigenvalues are purely imaginary
2) *Higher-order and non-isolated fixed points*: at least one eigenvalue is zero

Marginal cases are those where at least one eigenvalue satisfies $\mathrm{Re}(\lambda) = 0$.

Fixed points and linearization

If $\mathrm{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is called hyperbolic: in this case its type is predicted by the linearization. The condition $\mathrm{Re}(\lambda) \neq 0$ is the exact analog of $f'(x^*) \neq 0$ in one dimension for the stability of the FP to be accurately predictable by linearization.

$\mathrm{Re}(\lambda) \neq 0$, of course, applies also in higher-order systems.

Hartman-Grobman Theorem: The local phase portrait near a hyperbolic fixed point is *topologically equivalent* to the phase portrait of the linearization. (In other words, there is a *homeomorphism* that maps one to the other.)

Fixed points and linearization

Homeomorpism: Let X_1 and X_2 be topological spaces. A map $f : X_1 \rightarrow X_2$ is a homeomorphism if it is continuous and has an inverse $f^{-1} : X_2 \rightarrow X_1$, which is also continuous. If there exists a homeomorpism between X_1 and X_2, X_1 is said to be homeomorphic to X_2 and vice versa.

Examples: a) An open disc $D^2 = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2$.

b) A coffee cup is homeomorphic to a doughnut.

$\longrightarrow$

Fixed points and linearization

Intuitively, two phase portraits are topologically equivalent if one is a distorted (bending, warping, but not tearing) version of the other. Hence, closed orbits stay closed, trajectories connecting saddle points must not be broken, etc.

A phase portrait is structurally stable if its **topology cannot be changed** by an arbitrarily small perturbation of the vector field. Hence, the phase portrait of a saddle point is structurally stable, that of a center is not, since a small perturbation converts the center into a spiral

Rabbits versus Sheep

Rabbits versus Sheep

Lotka-Volterra model of competition between two species.

Rabbits and sheep are competing for the same limited resource (e.g. grass): no predators, seasonal effects, etc.

1) Each species would grow to its carrying capacity in the absence of the other → logistic growth.
2) When rabbits and sheep encounter each other, trouble starts: sheep push rabbits away → conflicts occur at a rate proportional to the size of each population, reducing the growth rate for each species.
3) Rabbits reproduce faster but they are more severely penalized by conflicts.

Rabbits versus Sheep

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - y - x)$$

$x(t) \geq 0 \rightarrow$ population of rabbits
$y(t) \geq 0 \rightarrow$ population of sheep

Fixed points

(0,0), (0,2), (3,0), (1,1)

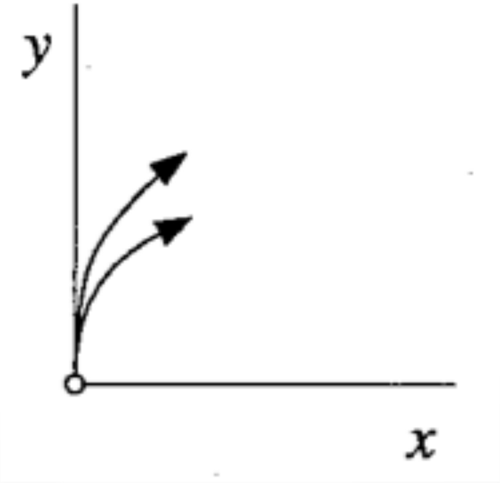
$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} = \begin{pmatrix} 3 - 2x^* - 2y^* & -2x^* \\ -y^* & 2 - x^* - 2y^* \end{pmatrix}$$

Rabbits versus Sheep

(0,0)

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues are $\lambda = 2, 3 \rightarrow$ the origin is an unstable node.
(Eigenvectors: (λ = 2) (0,1), (λ = 3) (1,0).)
Trajectories near a node are tangential to the slower eigendirection (here the y-axis, for which $\lambda = 2 < 3$).

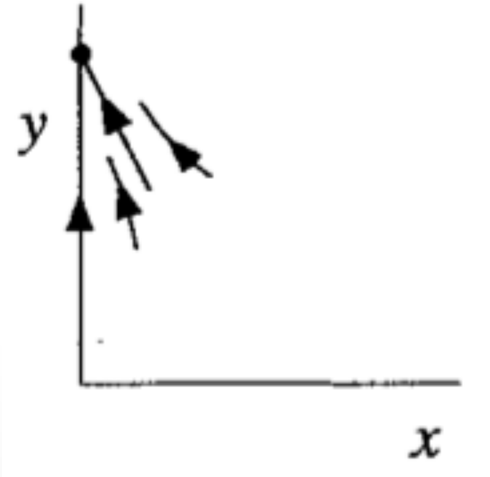


Rabbits versus Sheep

(0,2) $$A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

Eigenvalues are λ = -1, -2 → (0,2) is a stable node

Trajectories near a node are tangential to the slower eigendirection [here **v** = (1, -2), for which λ = -1 → |-1|<|-2|]



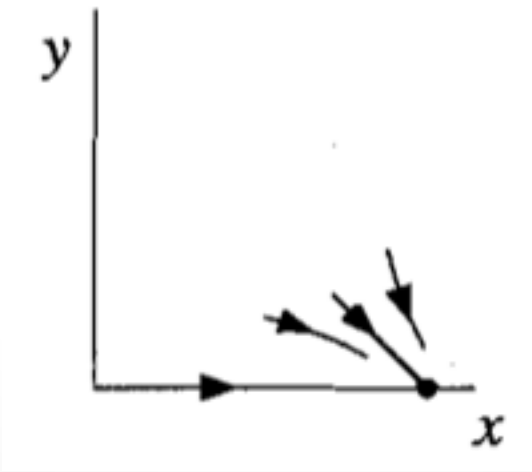
Rabbits versus Sheep

(3,0)

$$A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues are λ= -3, -1 → (3,0) is a stable node

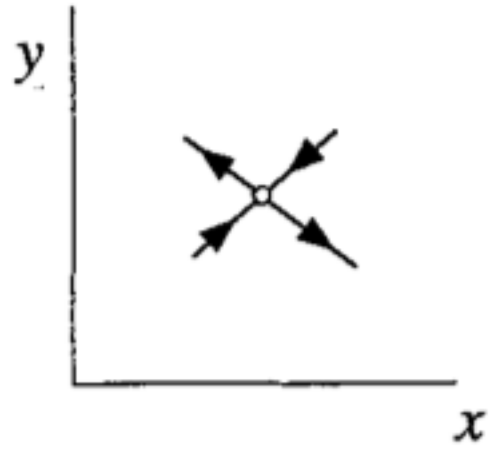
Trajectories near a node are tangential to the slower eigendirection [here **v** = (3, -1), for which λ = -1 → |-1|<|-3|)].



Rabbits versus Sheep

(1,1) $$A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

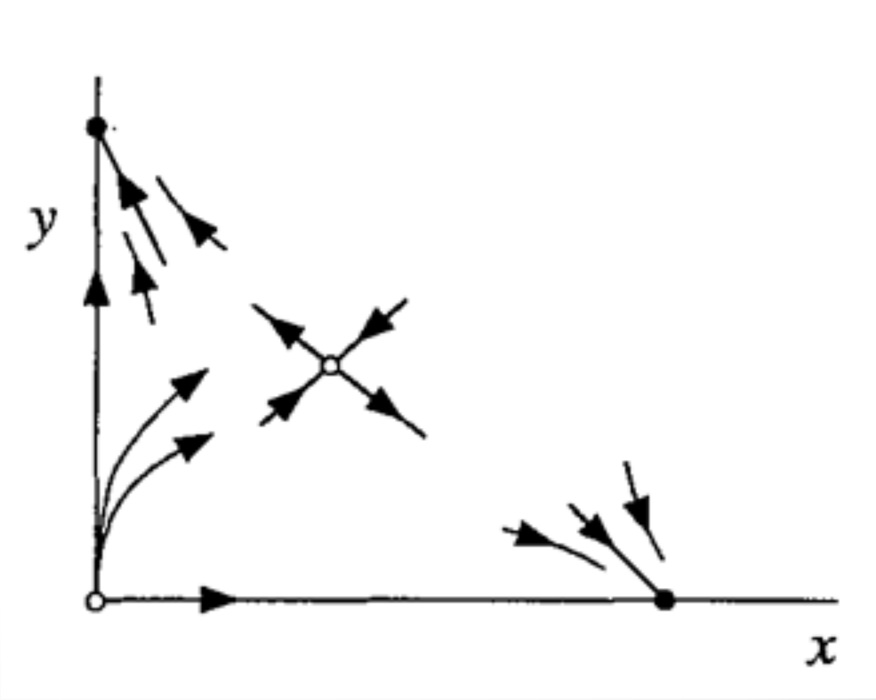
Eigenvalues are λ= $-1 \pm \sqrt{2}$ $\rightarrow$ (1,1) is a saddle point



Rabbits versus Sheep

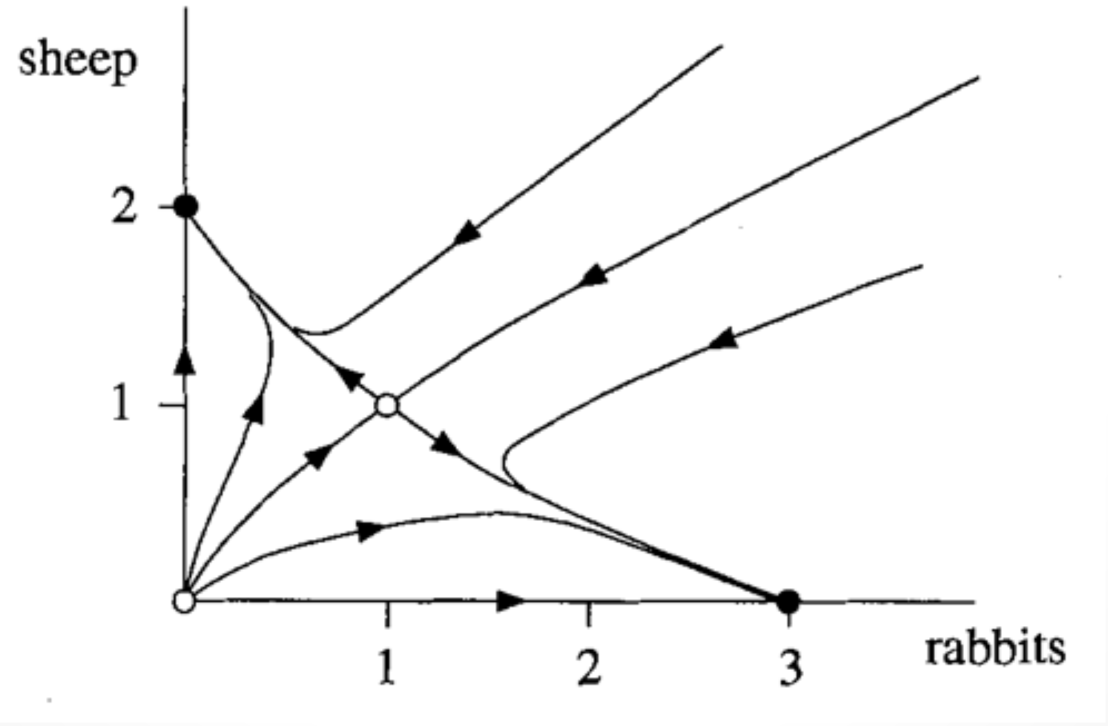
$$\begin{aligned}\dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y)\end{aligned}$$

Collecting the previous local portraits and adding the solutions dx/dt = 0 for x = 0 and dy/dt = 0 for y = 0 giving the horizontal and vertical trajectories:



Rabbits versus Sheep

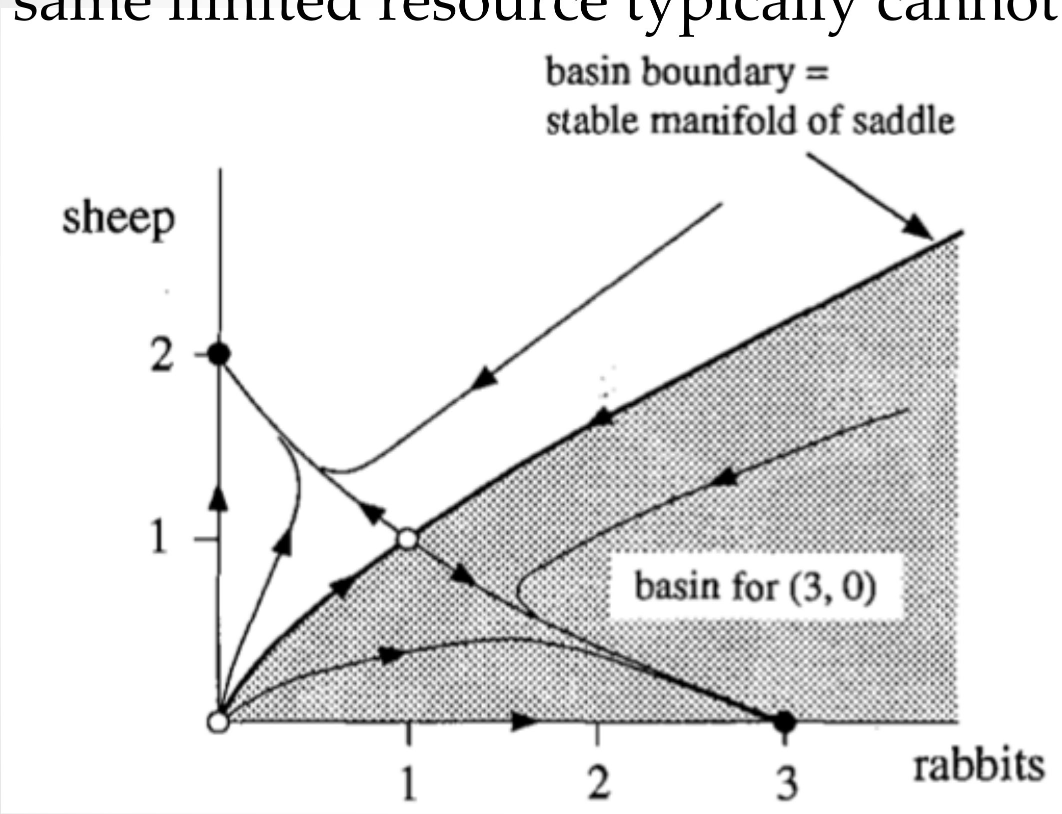
$$\begin{aligned}\dot{x} &= x(3 - x - 2y)\\ \dot{y} &= y(2 - x - y)\end{aligned}$$



Biological interpretation: one species drives the other to extinction.

Rabbits versus Sheep

Principle of competitive exclusion: two species competing for the same limited resource typically cannot coexist.



The basin of attraction of an attracting fixed point is the set of initial conditions $\mathbf{x}_0$ leading to that fixed point ($\mathbf{x}(t) \to \mathbf{x}^*$) as $t \to \infty$.

Because the stable manifold separates the basins for the two nodes it is called the basin boundary.

Conservative systems

Equation of motion of a mass m moving along the x-axis, subject to a nonlinear force $F(x)$:

$$m\ddot{x} = F(x)$$

$F(x)$ has no dependence on the velocity or time $\rightarrow$ no damping or friction, no time-dependent driving force.

The energy is conserved

$$F(x) = -\frac{dV}{dx} \quad \rightarrow \quad m\ddot{x} + \frac{dV}{dx} = 0$$

$V(x)$ is the potential energy.

Conservative systems

Standard trick (to be remembered), multiply by $\dot{x}$:

$$m\dot{x}\ddot{x} + \frac{dV(x(t))}{dx}\dot{x} = 0 \quad \rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right] = 0$$

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \text{ is a constant of motion}$$

Systems with a conserved quantity are called conservative.

General definition: given a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

a conserved quantity is a real-valued continuous function $E(x)$ that is constant on trajectories ($dE/dt = 0$), but nonconstant on every open set (to exclude e.g. $E(\mathbf{x}) \equiv 0$).

Example I

A conservative system cannot have any attracting fixed points.

If there were a fixed point $\mathbf{x}^*$, then all points in its basin of attraction would have to be at the same energy $E(\mathbf{x}^*)$ (since energy is constant on all trajectories leading to $\mathbf{x}^*$), so there would be an open set with constant energy.

No attracting fixed points. So, what kind of fixed points can occur in conservative systems?

Example II

Particle of mass $m = 1$ moving in a double-well potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

$$F(x) = -\frac{dV}{dx} = x - x^3 \quad \rightarrow \quad \ddot{x} = x - x^3$$

As a vector field:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 \end{aligned}$$

Fixed points: (0, 0), (±1, 0)

Example II

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 \end{aligned}$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} = \begin{pmatrix} 0 & 1 \\ 1 - 3{x^*}^2 & 0 \end{pmatrix}$$

$(0, 0) \rightarrow \Delta = -1 < 0 \rightarrow$ saddle point!

$(\pm 1, 0) \rightarrow \tau = 0, \Delta = 2 \rightarrow$ centers!

Question: Will the nonlinear terms destroy the center predicted by the linear approximation?

Answer: In the conserved system no!

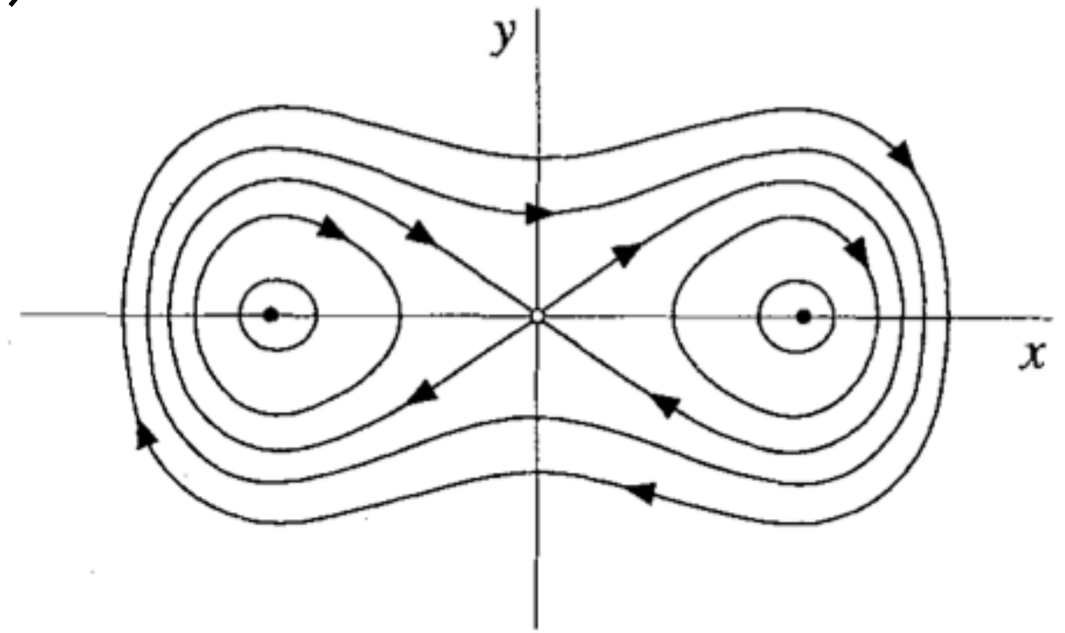
Example II

In conservative systems trajectories are (typically) closed curves defined by contours of constant energy. In this particular case:

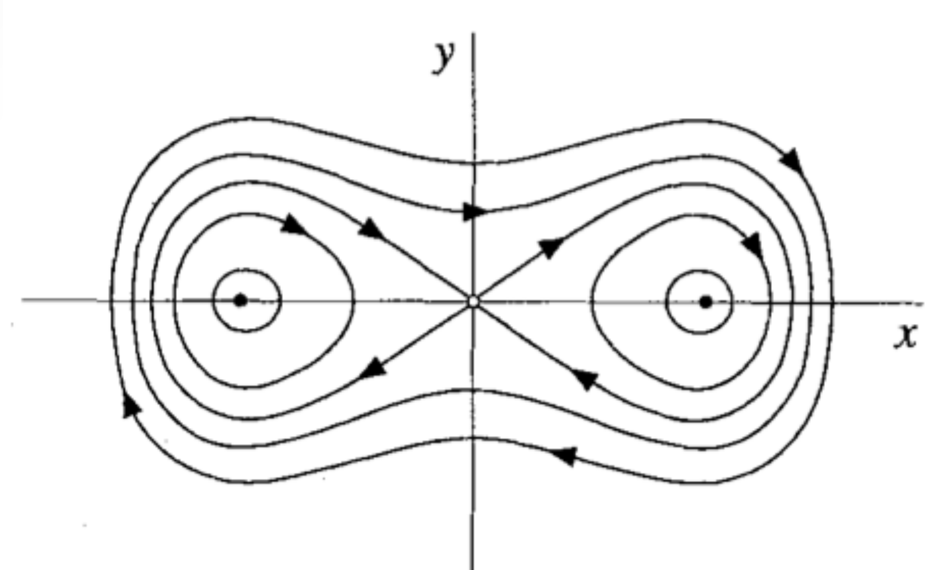
$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{ constant}$$

$$E_{kin} = \frac{1}{2}\dot{v}^2$$

$(m = 1)$



Example II



1) Near the centers there are small periodic orbits.
2) There are also large periodic orbits encircling all fixed points.
3) Solutions are periodic except for equilibria (fixed points) and the homoclinic orbits, which approach the origin when $t \to \pm\infty$. (Note: homoclinic orbits are ones starting and ending at the same point; not periodic, since it takes forever to reach a fixed point.)

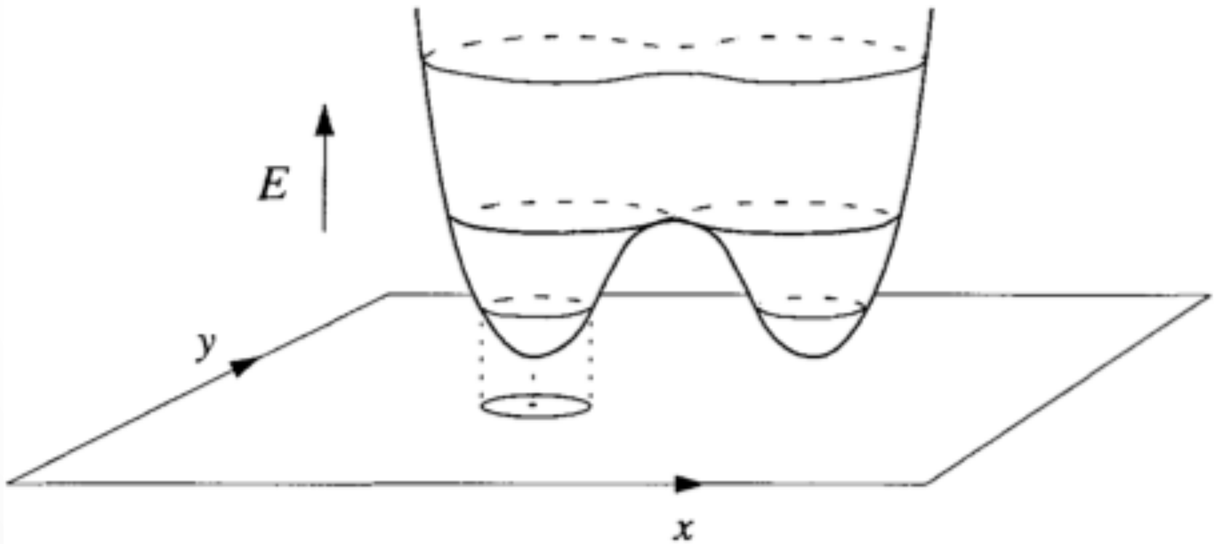
Example II

1) Neutrally stable equilibria correspond to the particle at rest at the bottom of either one of the wells.
2) Small closed orbits → small oscillations about equilibria.
3) Large closed orbits → oscillations taking the particle back and forth over the hump.
4) Saddle point? Homoclinic orbits?

Example II

Sketch the graph of the energy function

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$



1) Local minima of *E* project down to centers in the phase plane
2) Contours of slightly higher energy are small closed orbits
3) At *E*-value of local maximum (saddle point):homoclinic orbits
4) At higher *E*-values → large periodic orbits

Nonlinear centers

Theorem (nonlinear centers for conservative systems): Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbf{R}^2$ and $\mathbf{f}$ is continuously differentiable. Suppose that there exists a conserved quantity $E(\mathbf{x})$ and an isolated fixed point $\mathbf{x}^*$. If $\mathbf{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

Ideas behind the proof:

1) Since E is constant on trajectories, each trajectory is *contained in some contour of E*.
2) Near a local maximum (or minimum), *contours are closed*
3) The orbit is periodic, i.e. it does not stop at some point of the contour because $\mathbf{x}^*$ is isolated, so there are *no other fixed points in its close proximity*

Reversible systems

Many mechanical systems have time-reversal symmetry, i.e. their dynamics looks the same whether time runs forward or backward. (For example, think of a pendulum.)

Any mechanical system of the form

$$m\ddot{x} = F(x)$$

is symmetric under time reversal!

$$t \to -t \quad \longrightarrow \quad \ddot{x} \to \ddot{x}$$

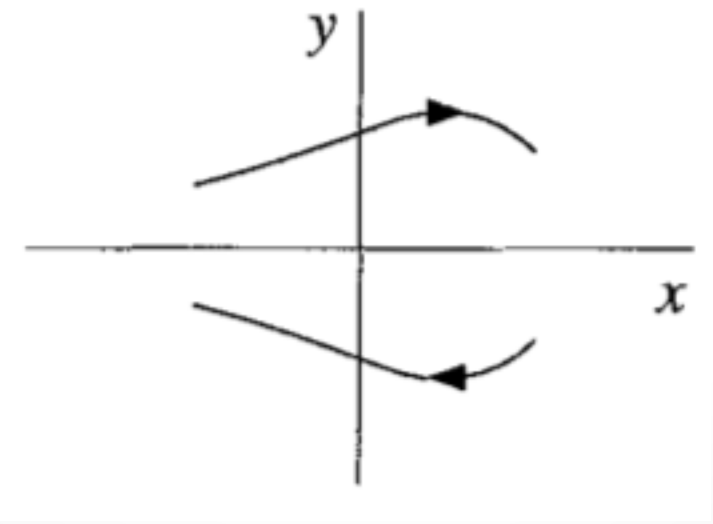
The acceleration does not change, the velocity changes sign!

Reversible systems

$$t \to -t, \; y \to -y \Rightarrow$$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{F(x)}{m} \end{aligned} \quad \to \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{F(x)}{m} \end{aligned}$$

Consequence: if $(x(t), y(t)$ is a solution, also $(x(-t), -y(-t))$ is a solution!



Reversible systems

More generally, any second-order system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

such that f is **odd** in y, $f(x,-y) = -f(x,y)$, and g is **even** in y, $g(x,-y) = g(x,y)$, is reversible!

Reversible systems are different from conservative systems, but they share some properties.

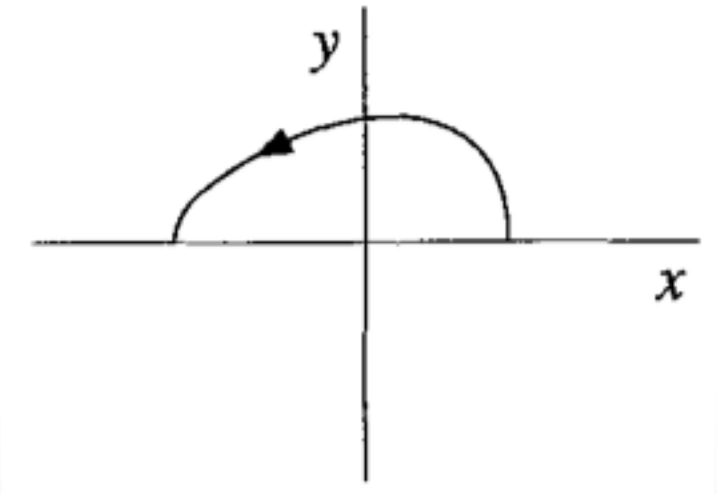
Theorem (nonlinear centers for reversible systems): Suppose the origin $\mathbf{x}^* = \mathbf{0}$ is a linear center of a reversible system. Then, sufficiently close to the origin, all orbits are closed.

In other words, for a reversible system a linear center is also a nonlinear center.

Reversible systems

Ideas behind the proof:

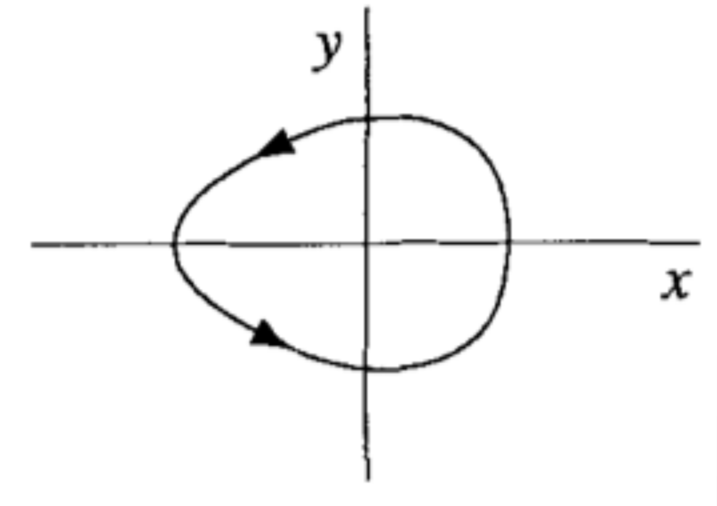
1) Let us take a trajectory starting on the positive x-axis near the origin.
2) Because of the influence of the linear center (if the system is close enough to it), the trajectory will bend and intersect the negative x-axis.



Reversible systems

Ideas behind the proof:

3) By using reversibility we can reflect the trajectory above the x-axis, obtaining a twin trajectory (we know that it is a solution of the equation of motion and it must be the only one).
4) The two trajectories form a closed orbit, as desired.



Example I

$$\begin{aligned} \dot{x} &= y - y^3 \\ \dot{y} &= -x - y^2 \end{aligned}$$

The system is reversible and the origin (0, 0) is a fixed point. What kind of a fixed point is it?

Jacobian at the origin:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

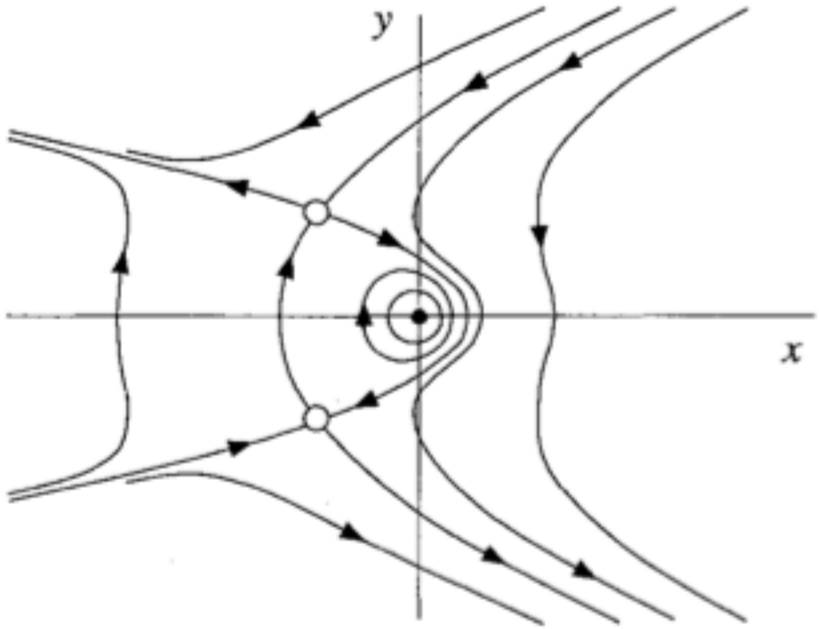
$\tau = 0$, $\Delta = 1 \rightarrow$ a linear center $\rightarrow$ also a nonlinear center (due to the theorem).

Other fixed points are (-1, 1) and (-1, -1)

$$A = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix}$$

$\Delta = -2 < 0 \rightarrow$ saddle points.

Example I



The twin saddle points are joined by a pair of trajectories, called heteroclinic orbits or saddle connections.

Homoclinic and heteroclinic orbits are common in conservative and reversible systems.

Example II

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^2 \end{aligned}$$

Show that there is a homoclinic orbit in the half-plane $x \geq 0$.

Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 2x & 0 \end{pmatrix}$$

Fixed points: (0, 0) $\tau = 0$, $\Delta = -1 \rightarrow$ **saddle point**.
(1, 0) $\tau = 0$, $\Delta = 1 \rightarrow$ **linear center** and due to reversibility also **nonlinear center**.

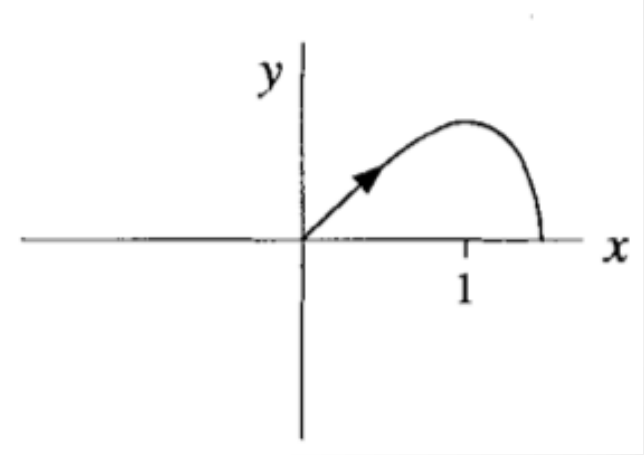
For FP (0,0) the eigenvectors corresponding to the eigenvalues 1 and -1 are $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$.

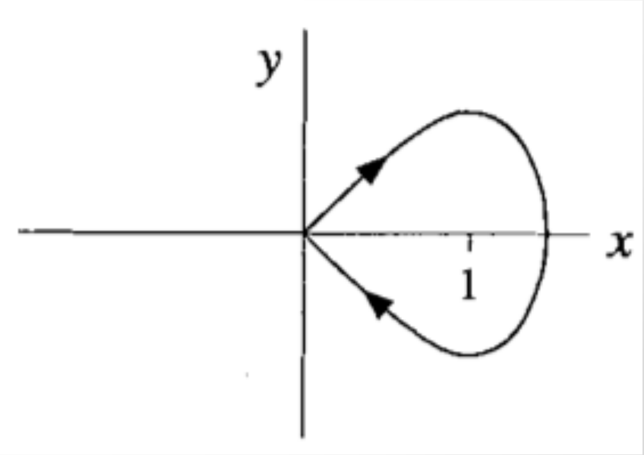
The unstable manifold leaves the origin along $\mathbf{v}_1 = (1, 1)$.

Example II

$f(x,y) = y = -f(x,-y)$; $g(x,y) = x - x^2 = g(x,-y)$, so the system is reversible: use this to plot the trajectories.

1) Initially we are in the first quadrant ($x > 0$ and $y > 0$).
2) Velocity in the x-direction is positive, in the y-direction it is positive until the system passes $x = 1$.
3) For $x > 1$ the velocity in the y-direction becomes negative and the particle ends up hitting the x-axis.
4) By reversibility there must be a twin trajectory with the same endpoints and arrows reversed.
5) The two trajectories together form a homoclinic orbit.





Reversibility

More general definition of reversibility: If there exists a mapping $R(\mathbf{x})$ of the phase space to itself that satisfies $R^2(\mathbf{x}) = \mathbf{x}$, then the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is invariant under the change of variables $t \to -t$, $\mathbf{x} \to \boldsymbol{R}(\mathbf{x})$. (Reflection about the x-axis has the property $R^2(\mathbf{x}) = \mathbf{x}$.)

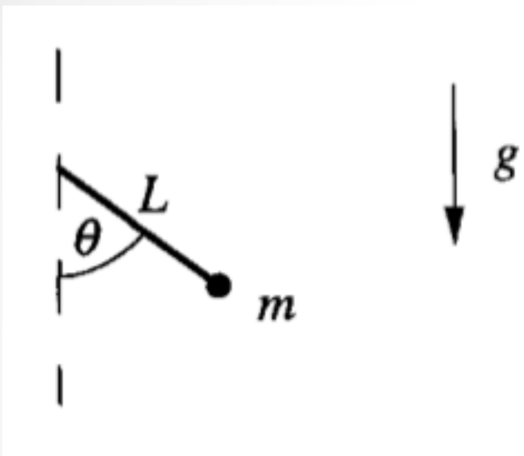
Example

$$\dot{x} = 2\cos x - \cos y$$

$$\dot{y} = 2\cos y - \cos x$$

This system is invariant under $t \to -t$, $x \to -x$, and $y \to -y$, so it is reversible, with $R(x, y) = (-x, -y)$. However, it is not conservative because it has an attractive fixed point at $(-\frac{\pi}{2}, -\frac{\pi}{2})$.

Pendulum



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

This was linearized at high school: sin $\theta \approx \theta$. Here we solve the system for all θ diagrammatically.

Nondimensionalization: $\omega = \sqrt{g/L}$, $\tau = \omega t$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \quad \rightarrow \quad \ddot{\theta} + \sin\theta = 0$$

$$\begin{aligned} \dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta \end{aligned}$$

Note: differentiation with respect to τ.

Pendulum

$$\begin{aligned}\dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta\end{aligned}$$

The system is reversible, since the equations are invariant under $\tau \rightarrow -\tau$ and $\nu \rightarrow -\nu$, that is, $f(\theta, -\nu) = -f(\theta, \nu)$ and $g(\theta, -\nu) = g(\theta, \nu)$.

Fixed points: $(\theta^*, \nu^*) = (k\pi, 0)$, where k is any integer.

Focus on the FPs (0, 0), (π, 0) (the other fixed points coincide with either of them, $\theta \rightarrow \theta + 2\pi$). The Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$$

(0, 0) $\rightarrow$ $\tau = 0$, $\Delta = 1 > 0$ $\rightarrow$ linear center $\rightarrow$ nonlinear center (reversible system).

(π, 0) $\rightarrow$ $\tau = 0$, $\Delta = -1 < 0$ $\rightarrow$ saddle point.

Pendulum

$$\begin{aligned}\dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta\end{aligned}$$

The system is reversible

$$\begin{aligned}\tau &\to -\tau \\ \nu &\to -\nu\end{aligned} \quad \to \quad \begin{aligned}\dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta\end{aligned}$$

The system is conservative (multiply the nondimensionalized equation by $d\theta/d\tau$):

$$\dot{\theta}(\ddot{\theta} + \sin\theta) = 0 \quad \to \quad \frac{1}{2}\dot{\theta}^2 - \cos\theta = \text{ constant}$$

The energy function

$$E(\theta, \nu) = \frac{1}{2}\nu^2 - \cos\theta$$

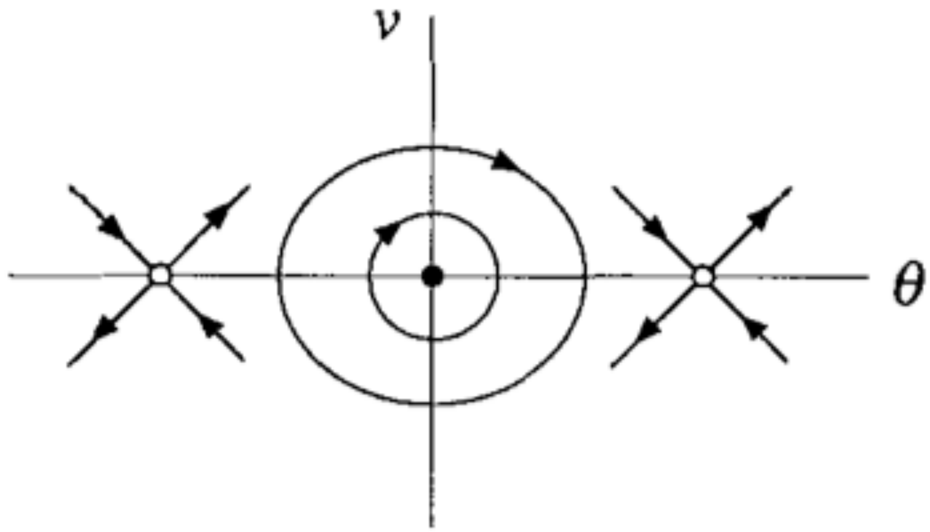
has a local minimum at (0, 0).

So, again: the origin is a nonlinear center.

Pendulum

$$\begin{aligned}\dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta\end{aligned}$$

The eigenvalues and -vectors at the saddle fixed point $(\pi,0)$ are $\lambda_1 = -1$, $\mathbf{v_1} = (1, -1)$; $\lambda_2 = 1$, $\mathbf{v_2} = (1, 1)$.

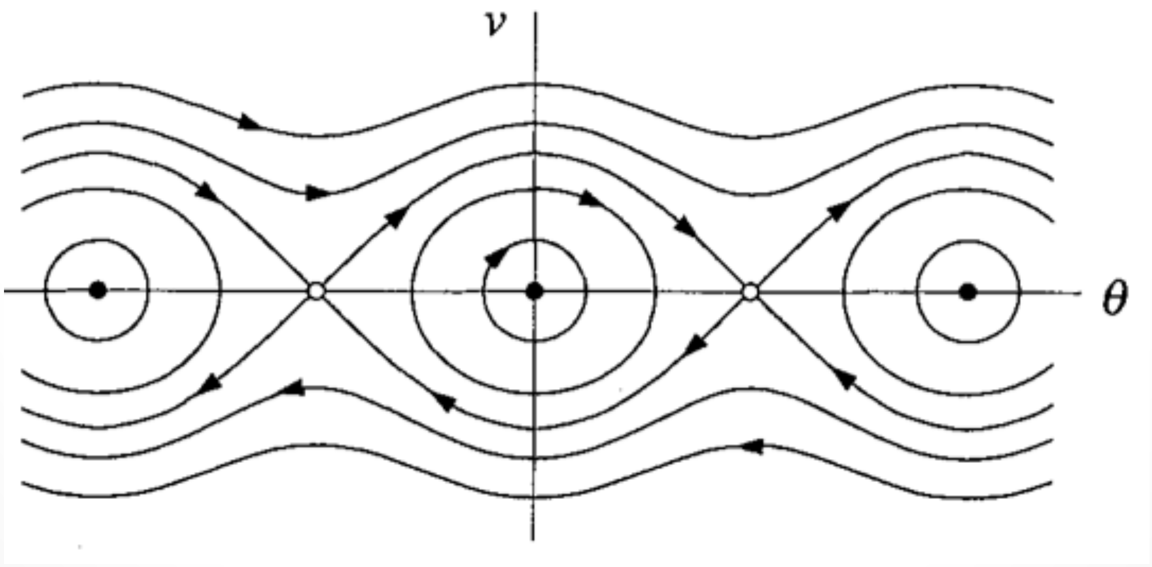


Pendulum

Now include the energy contours

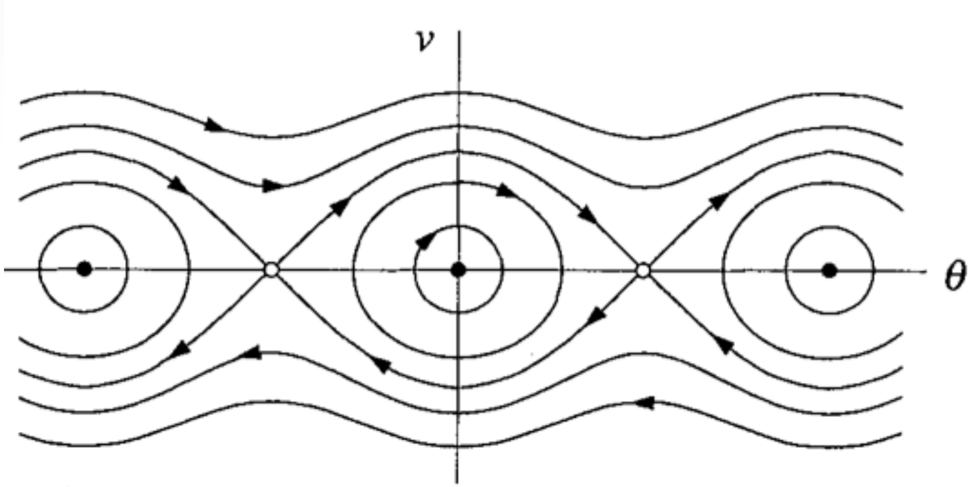
$$E(\theta, \nu) = \frac{1}{2}\nu^2 - \cos\theta$$

for different values of *E*:



The portrait is periodic in the θ-direction.

Pendulum



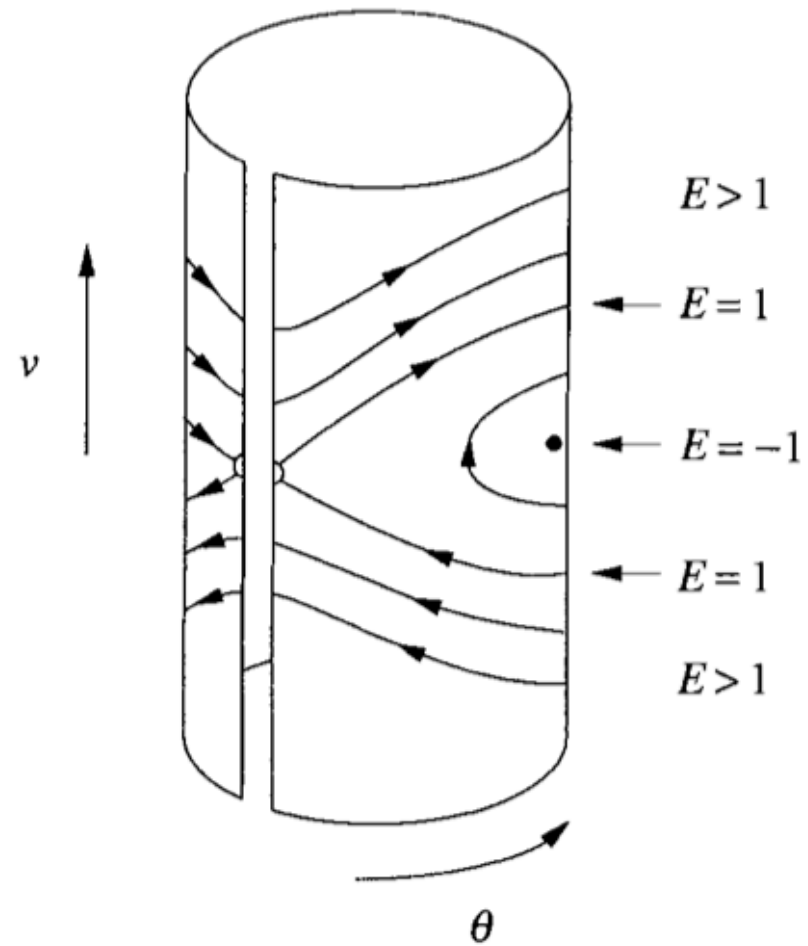
Physical interpretation:

1) The center is the neutrally stable equilibrium with the pendulum at rest straight down (minimum energy E = -1).
2) Small orbits about the center → small oscillations (librations).
3) If the energy increases, the amplitude of the oscillations increases. At the critical value $E = 1$ an unstable saddle (the pendulum straight up) is approached along the heteroclinic trajectory, and the pendulum slows down to a halt.
4) For $E > 1$ the pendulum whirls repeatedly over the top.

Pendulum

Cylindrical phase space

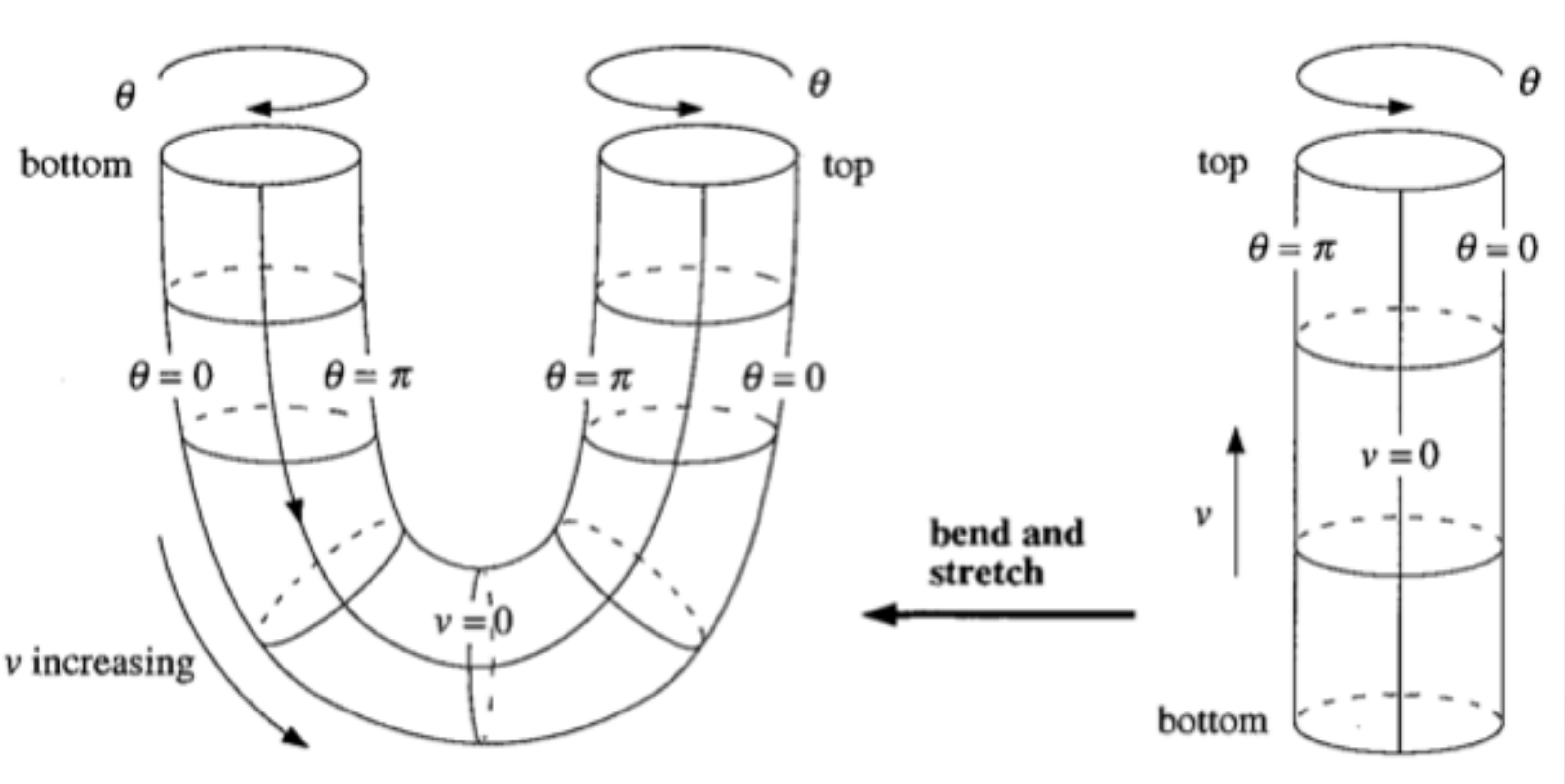
- Natural space for pendulum: one variable (θ) is periodic, the other (v) is not
- Periodic whirling motions (E > 1) look periodic
- Saddle points indicate the same physical state
- Heteroclinic trajectories become homoclinic orbits



Pendulum

Cylindrical phase space

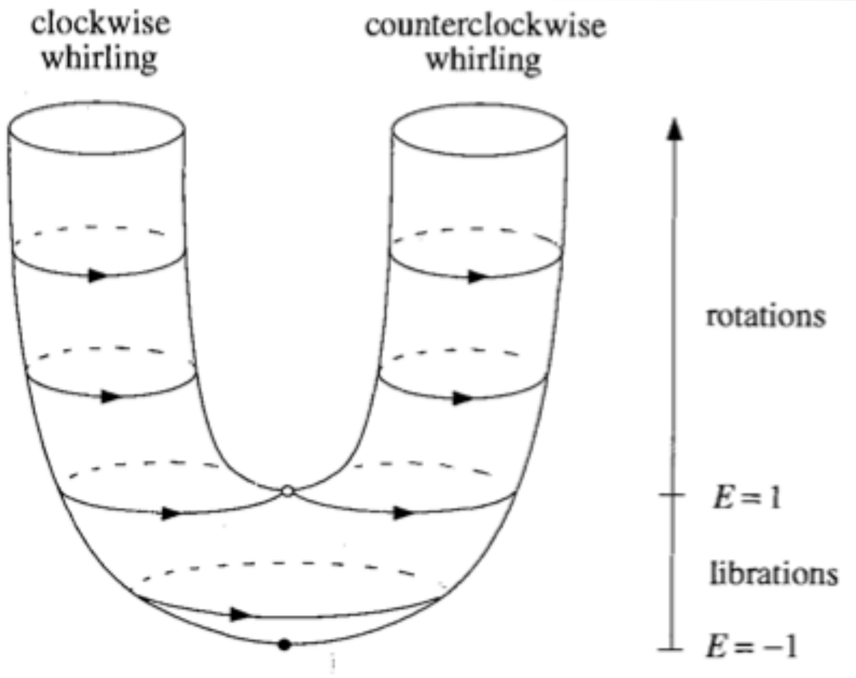
Plotting vertically the energy instead of the velocity: **U-tube**



Pendulum

Cylindrical phase space

- Orbits are sections at constant height/energy
- The two arms correspond to the two senses of rotations
- Homoclinic orbits lie at E = 1, borderline between librations and rotations

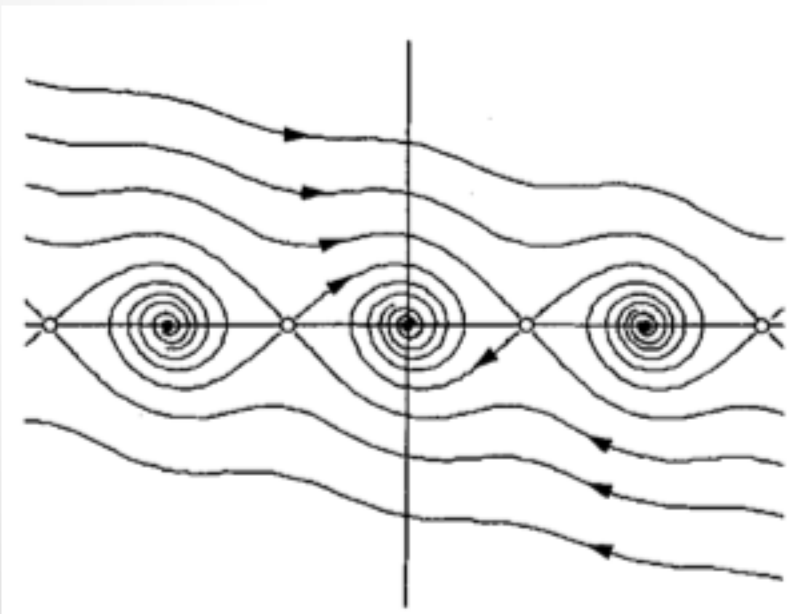


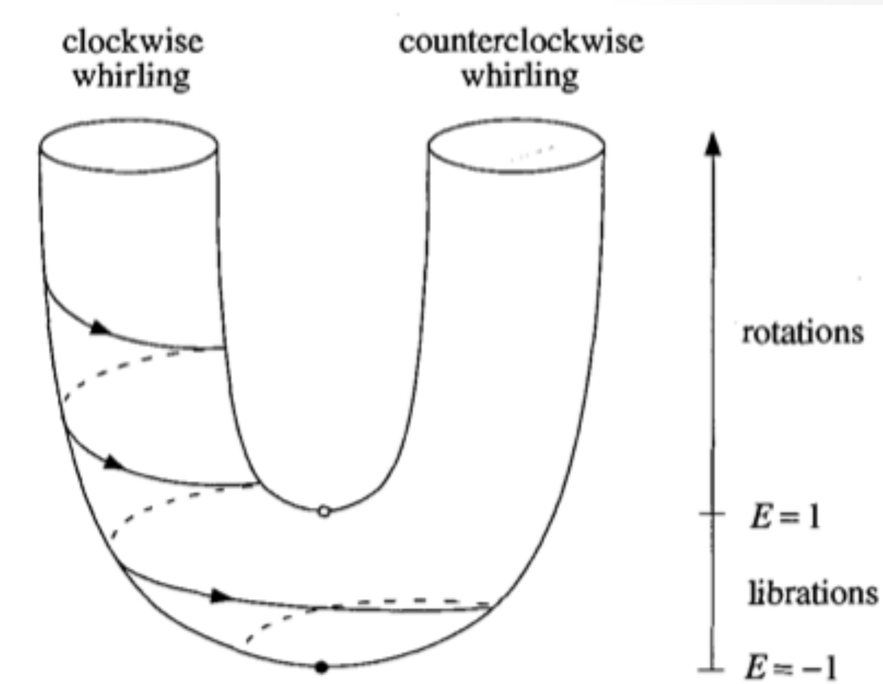
Pendulum

Damping

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0, \qquad \text{damping strength } b > 0$$

Centers $\rightarrow$ stable spirals
Saddle points $\rightarrow$ saddle points





All trajectories continuously lose altitude, except for the fixed points.

Pendulum

Damping

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0, \qquad \text{damping strength } b > 0$$

Change of energy along trajectory:

$$\frac{dE}{d\tau} = \frac{d}{d\tau}\left(\frac{1}{2}\dot{\theta}^2 - \cos\theta\right) = \dot{\theta}(\ddot{\theta} + \sin\theta) = -b\dot{\theta}^2$$

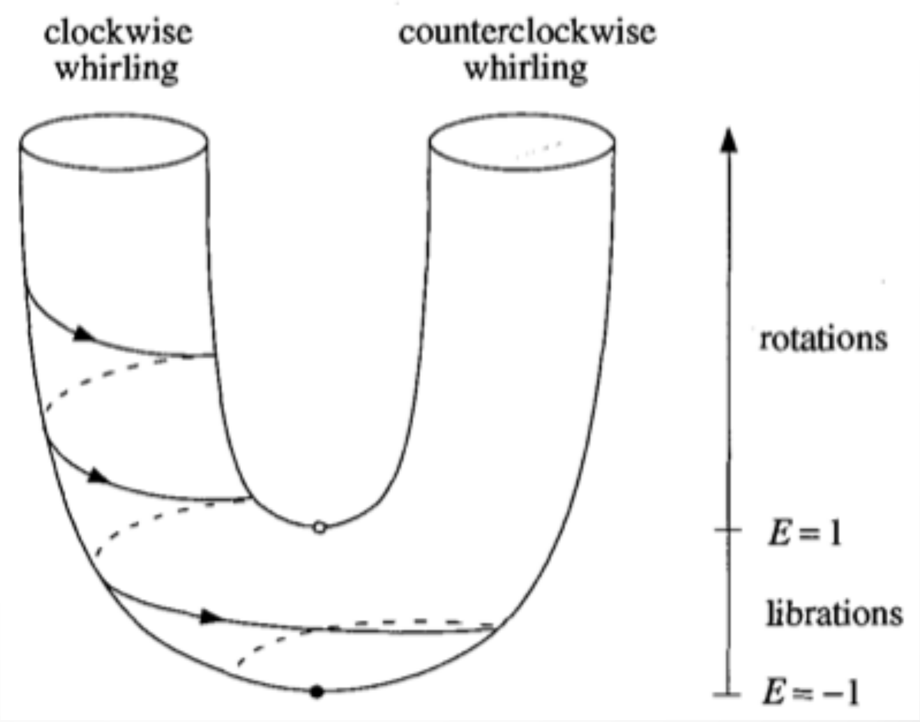
Consequence: E decreases monotonically along trajectories, except at fixed points (where $\dot{\theta} = 0$).

Pendulum

Damping

Physics: pendulum rotates over the top with decreasing energy, until it cannot complete the rotation and makes damped oscillations about equilibrium, where it eventually stops

$$\dot{\theta} = 0$$



Index theory

Global information about the phase portrait, as opposed to the local information provided by linearization

Questions:

1) Must a closed trajectory always encircle a fixed point?
2) If so, what types of fixed points are permitted?
3) What types of fixed points can coalesce in bifurcations?
4) Trajectories near higher-order fixed points?
5) Possibility of closed orbits?

Index of a closed curve C: integer that measures the winding of the vector field on C

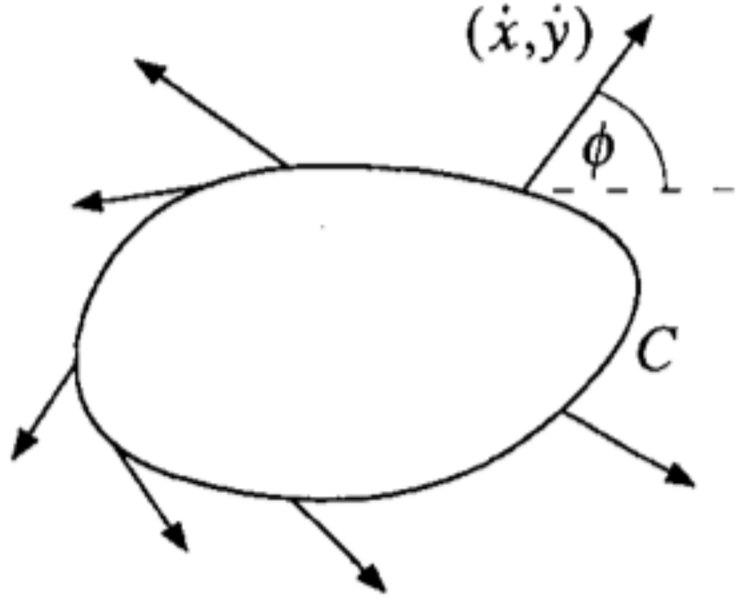
Similarity with electrostatics: from the behavior of electric field on a surface one may deduce the total amount of charges inside the surface; here one gets info on possible fixed points

Index theory

Suppose a smooth vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on the phase plane and consider a simple (= non-self-intersecting) closed curve *C*, which does not pass through fixed points of the system.

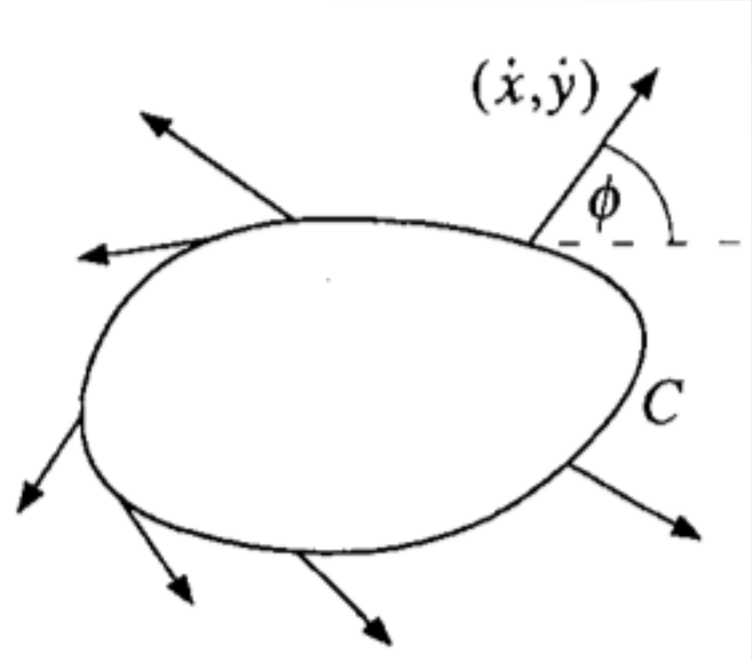
Then at each point of *C* the vector field makes a well-defined angle $\phi = \tan^{-1}(\dot{y}/\dot{x})$ $\quad (\arctan(x) \equiv \tan^{-1}(x))$

with the positive *x*-axis.



Index theory

As $\mathbf{x}$ moves counterclockwise around C, the angle φ changes continuously (the vector field is smooth) → when $\mathbf{x}$ comes back to the starting position φ has varied by a multiple of 2π.



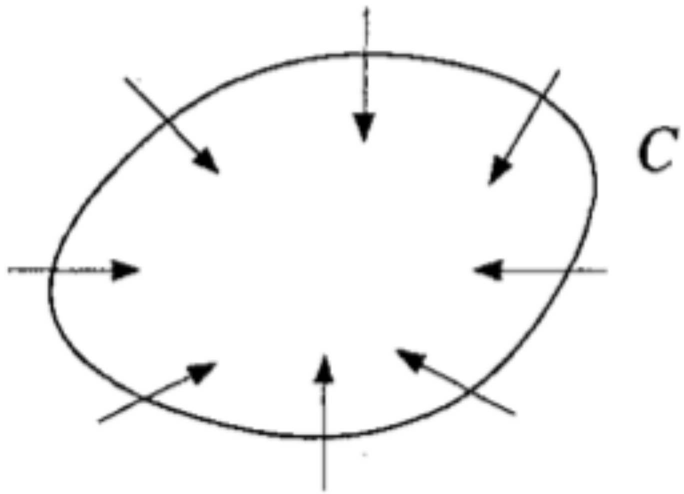
$[\varphi]_C$ = the net change in φ over one circuit

The index of the closed curve C:

$$I_C = \frac{1}{2\pi}[\phi]_C$$

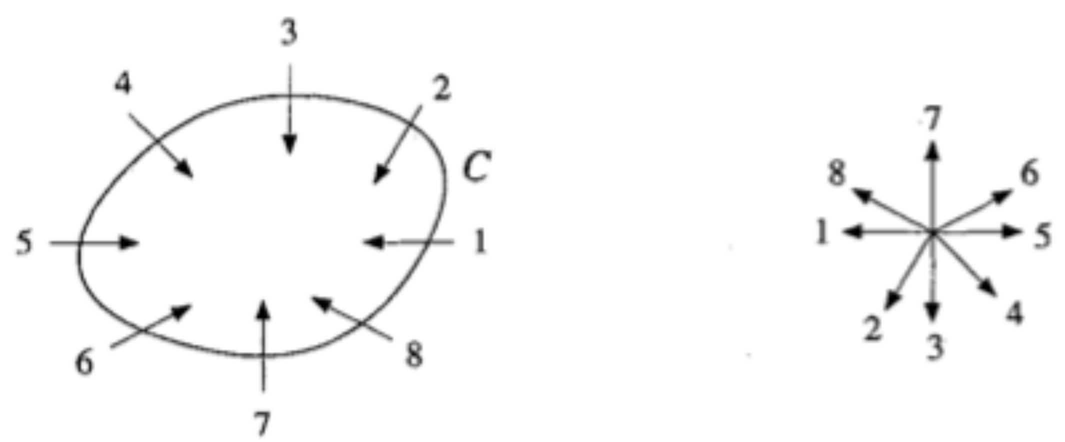
Example I

What's the index?



The vector field makes one complete rotation counterclockwise, so $I_C = +1$.

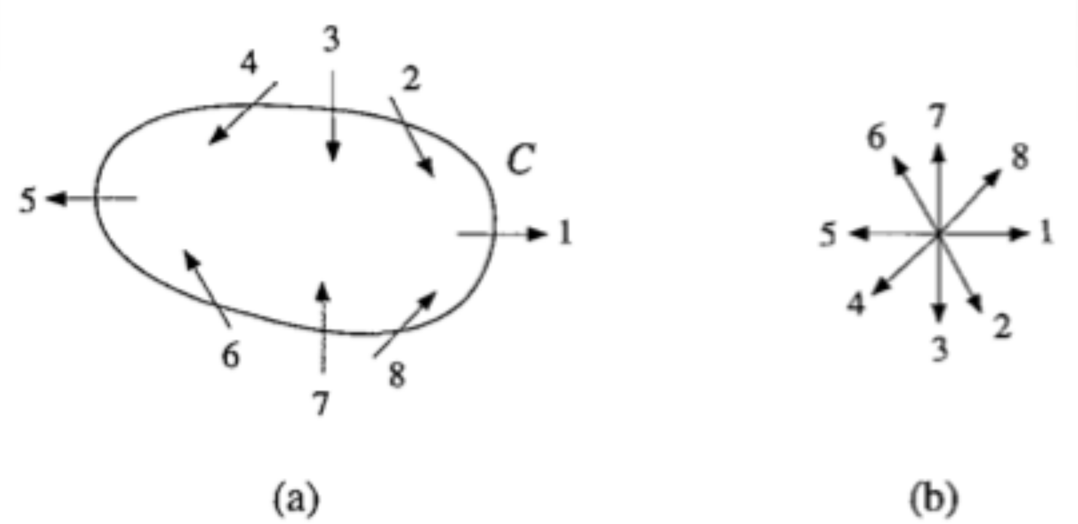
Trick



(a) (b)

The index is the net number of counterclockwise revolutions made by the numbered vectors in (b).

Example II



The vector field makes one complete rotation clockwise: $I_C = -1$.

Example III

The vector field

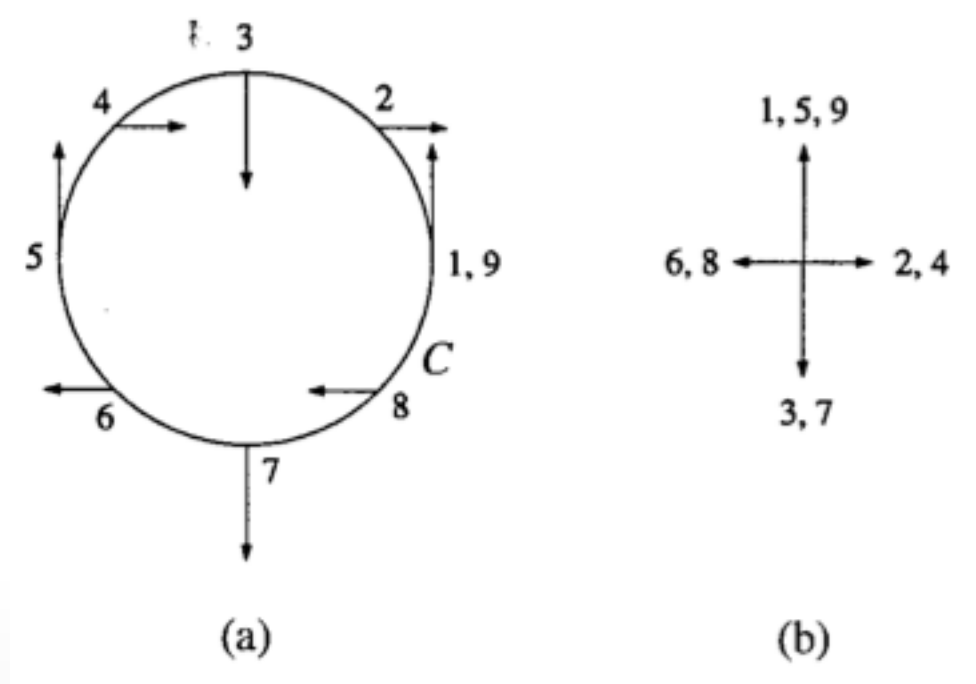
$$\begin{aligned}\dot{x} &= x^2 y \\ \dot{y} &= x^2 - y^2\end{aligned}$$

The curve C is the unit circle $x^2 + y^2 = 1$

What is I_C ?

$[\varphi]_C = -\pi + 2\pi - \pi = 0$

$I_C = 0$

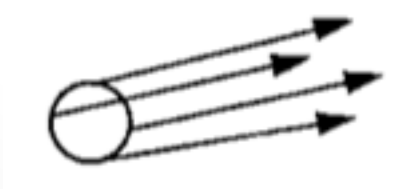


Properties of the index

1) If C can be continuously deformed into C' without passing through a fixed point, $I_C = I_{C'}$.

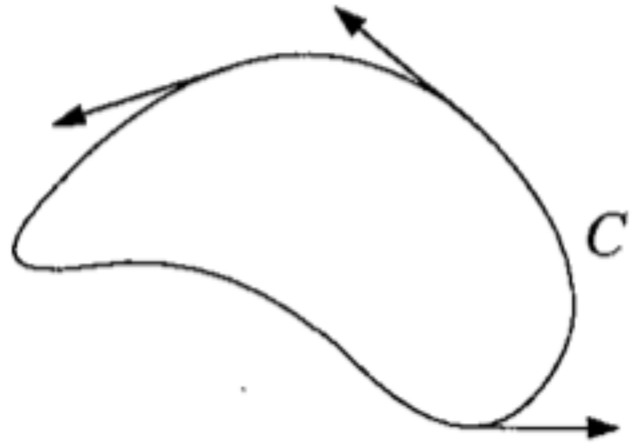
   Proof: The index cannot vary continuously, but only by integer values, so it cannot be altered by a continuous change of C.

2) If C does not enclose any fixed points, $I_C = 0$.
   Proof: By squeezing C until it becomes a very small circle the index does not change because of 1) and it equals zero because all vectors on the tiny circle point in the same direction.



Properties of the index

3) Under time reversal ($t \to -t$), the index is the same. Proof: The time reversal changes the signs of the velocity vectors, so the angles change from φ to $\varphi + \pi$, hence $[\varphi]_C$ stays the same
4) If C is a trajectory of the system, $I_C = +1$



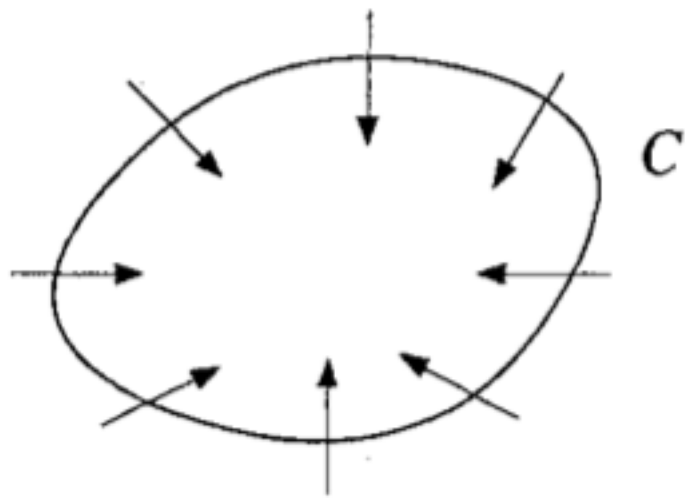
Index of a point

The index of an isolated fixed point $\mathbf{x}^*$ is the index of the vector field on any closed curve encircling $\mathbf{x}^*$ and no other fixed point.

By property 1), the value of the index is the same on any curve C, since it can be continuously deformed onto any other.

What is the index of a stable node?

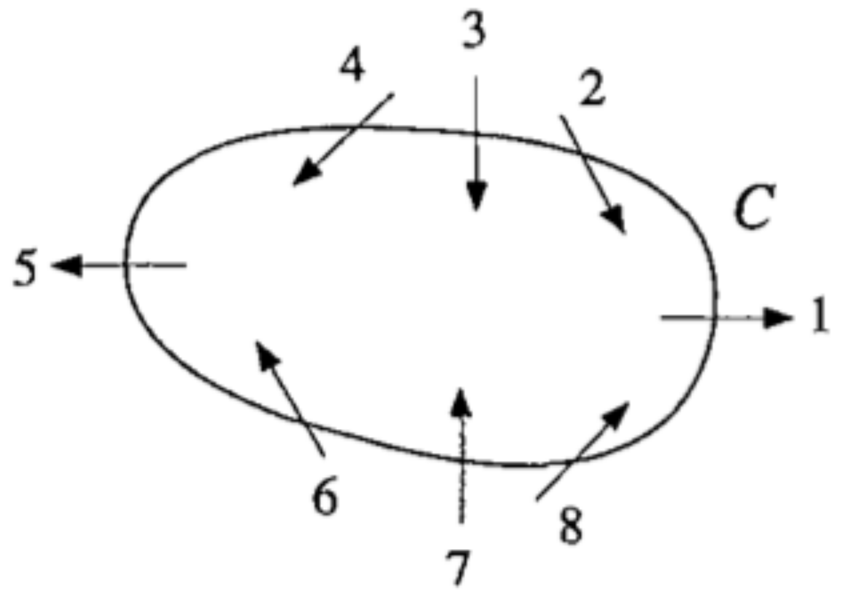
The vector field makes one complete rotation counterclockwise, so $I = +1$



The value is the same for unstable nodes as well, as the situation would be the same, only with reversed arrows (property 3).

Index of a point

What is the index of a saddle point?



The vector field makes one complete rotation clockwise: I = - 1.

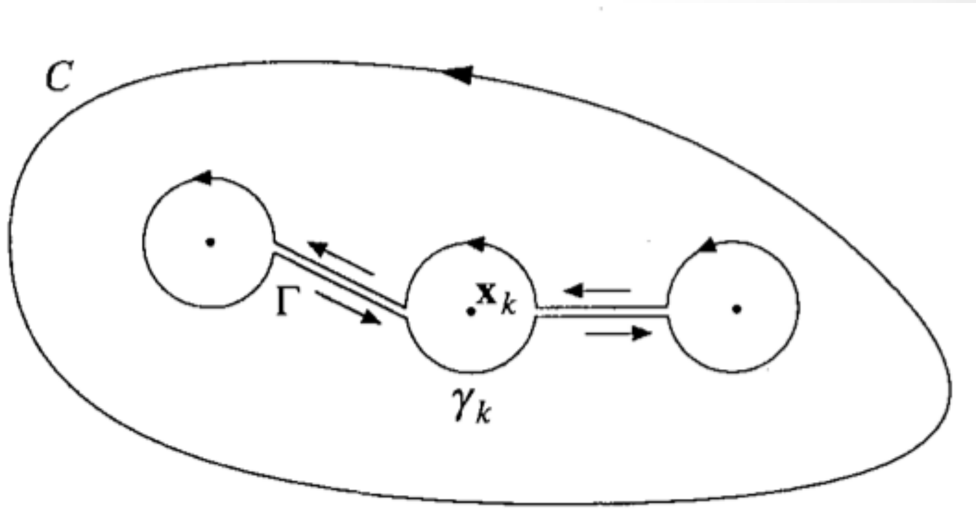
Spiral, centers, degenerate nodes and stars all have I = + 1, only saddle points have a different value.

Index of a point

Theorem: If a closed curve C surrounds n isolated fixed points, the index of the vector field on C equals the sum of the indices of the enclosed fixed points.

Proof:

C can be deformed to the contour Γ of the figure; the contributions of the bridges cancel as each bridge is crossed in both directions so the net changes in the angle are equal and opposite.



$$I_\Gamma = \frac{1}{2\pi}[\phi]_\Gamma = \frac{1}{2\pi}\sum_{k=1}^{n}[\phi]_{\gamma_k} = \frac{1}{2\pi}\sum_{k=1}^{n}2\pi I_k = \sum_{k=1}^{n} I_k$$

Index of a point

Theorem: Any closed orbit (trajectory) in the phase plane must enclose fixed points whose indices sum to +1.

Proof:
If C is a closed orbit, $I_C = +1$. From the previous theorem this is also the sum of the indices of the fixed points enclosed by C.

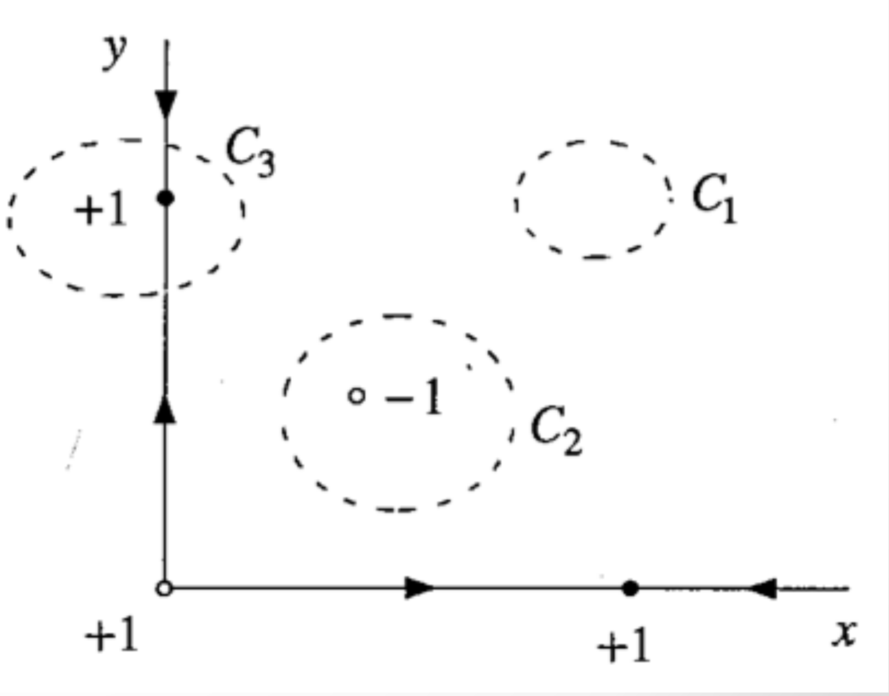
Consequence: Any closed orbit encloses at least one fixed point (if there were none, the index on the curve would be 0, instead of + 1). If there is a unique fixed point, it cannot be a saddle (as in this case the index would be – 1).

Example I

Show that closed orbits are impossible for the "rabbits versus sheep" system

$$\begin{aligned}\dot{x} &= x(3 - x - 2y)\\ \dot{y} &= y(2 - x - y)\end{aligned}$$

The only orbits enclosing good fixed points (i.e. with index + 1) would have to cross the x/y-axes, which contain trajectories of the system, and trajectories cannot cross (uniqueness)!



Example II

Show that the system

$$\begin{aligned} \dot{x} &= xe^{-x} \\ \dot{y} &= 1 + x + y^2 \end{aligned}$$

has no closed orbits.

Solution: The system has no fixed points, so it cannot have closed orbits, since the latter have to enclose at least one fixed point.