$Q_1$ - Let $\underline{A} = \begin{bmatrix} 3 & -5 & 0 \\ 5 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Use the Gram-Schmidt process to find an orthogonal basis for column of $A$.

- Solution:

$$A = \begin{bmatrix} 3 & -5 & 0 \\ 5 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Longrightarrow S_1 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}, S_2 = \begin{bmatrix} -5 \\ -3 \\ 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

step 1) Let $\phi_1 = \frac{S_1}{\|S_1\|}$, $\|S_1\| = \sqrt{(3)^2 + (5)^2 + 0} = \sqrt{34}$

$$\Rightarrow \phi_1 = \frac{S_1}{\sqrt{34}} = \begin{bmatrix} \frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \\ 0 \end{bmatrix}$$

Step 2)

$$(\underline{S_2})_{\perp S_1} = \underline{S_2} - (\underline{S_2}^H \underline{\phi_1}) \cdot \underline{\phi_1}$$

$$\underline{S_2}^H \underline{\phi_1} = \begin{bmatrix} -5 & -3 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \\ 0 \end{bmatrix} = \frac{-15}{\sqrt{34}} - \frac{15}{\sqrt{34}} = \frac{-30}{\sqrt{34}}$$

$$\Rightarrow (\underline{S_2})_{\perp S_1} = \begin{bmatrix} -5 \\ -3 \\ 0 \end{bmatrix} - \left(\frac{-30}{\sqrt{34}}\right) \cdot \begin{bmatrix} \frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \\ 0 \end{bmatrix}$$

$$\Rightarrow (\underline{s_2})_{\perp S_1} = \begin{bmatrix} -5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{45}{17} \\ \frac{75}{17} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-85+45}{17} \\ \frac{-51+75}{17} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-40}{17} \\ \frac{24}{17} \\ 0 \end{bmatrix}$$

$$\Rightarrow \phi_2 = \frac{(\underline{s_2})_{\perp S_1}}{\|(\underline{s_2})_{\perp S_1}\|}$$

we need to calculate $\|(\underline{s_2})_{\perp S_1}\|$ :

$$\|(\underline{s_2})_{\perp S_1}\| = \sqrt{\left(\frac{40}{17}\right)^2 + \left(\frac{24}{17}\right)^2 + 0} = \frac{\sqrt{1600+576}}{17} = \frac{\sqrt{2176}}{17}$$

$$\simeq \frac{46.6}{17}$$

$$\Rightarrow \phi_2 = \frac{(\underline{s_2})_{\perp S_1}}{\|(\underline{s_2})_{\perp S_1}\|} = \begin{bmatrix} \left(\frac{-40}{17}\right) / \left(\frac{46.6}{17}\right) \\ \left(\frac{24}{17}\right) / \left(\frac{46.6}{17}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-40}{46.6} \\ \frac{24}{46.6} \\ 0 \end{bmatrix}$$

step 3)

$$(\underline{s_3})_{\perp S_2} = \underline{s_3} - (s_3^H \phi_1) \cdot \phi_1 - (s_3^H \phi_2) \cdot \phi_2$$

$$s_3^H \phi_1 = [0 \quad 0 \quad 2] \begin{bmatrix} \frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \\ 0 \end{bmatrix} = 0 \quad , \quad s_3^H \phi_2 = [0 \quad 0 \quad 2] \begin{bmatrix} \frac{-40}{46.6} \\ \frac{24}{46.6} \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow (\underline{s_3})_{\perp S_2} = s_3 \Rightarrow \phi_3 = \frac{(\underline{s_3})_{\perp S_2}}{\|(\underline{s_3})_{\perp S_2}\|} = \frac{s_3}{\|s_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$Q_2$ - Consider the set of three finite energy signals

$$s_1(t) = 1, \qquad 0 \leq t \leq 1$$

$$s_2(t) = \cos(2\pi t), \qquad 0 \leq t \leq 1$$

$$s_3(t) = \cos^2(\pi t), \qquad 0 \leq t \leq 1$$

Find an orthonormal basis for the signal space spanned by these three signals?

- solution :

Firstly, we can write :

$$s_3(t) = \cos^2(\pi t) = \frac{1}{2}(1 + \cos(2\pi t)) = \frac{1}{2}(1) + \frac{1}{2} \cdot \cos(2\pi t)$$

$$= \frac{1}{2} s_1(t) + \frac{1}{2}\cos(2\pi t) = \frac{1}{2} s_1(t) + \frac{1}{2} s_2(t)$$

Thus, $s_3(t)$ can be written as a linear combination of $s_1(t)$ and $s_2(t)$. As a result, the dimension of the signal subspace is two instead of three which means we only need to performe the Gram-Schmidt procedure for two of the signals.

step 1) $\quad v_1(t) = s_1(t) \quad , \quad \phi_1(t) = \dfrac{v_1(t)}{\|v_1(t)\|}$

$$\|v_1(t)\|^2 = \int_0^1 v_1(t)\, v_1^*(t)\, dt = \int_0^1 1\, dt = 1 \Rightarrow \underline{\|v_1(t)\| = 1}$$

$$\Rightarrow \phi_1(t) = \frac{v_1(t)}{\|v_1(t)\|} = s_1(t) = 1 \qquad 0 \leq t \leq 1$$

step 2)

$$v_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \cdot \phi_1(t)$$

$$\langle s_2(t), \phi_1(t) \rangle = \int_0^1 s_2(t) \cdot \phi_1^*(t)\, dt = \int_0^1 \cos(2\pi t) \cdot (1)\, dt$$

$$= \int_0^1 \cos(2\pi t)\, dt = \frac{1}{2\pi}\left[\sin(2\pi t)\right]_0^1 = \frac{1}{2\pi}\left[\underbrace{\sin(2\pi) - \sin(0)}_{0}\right]$$

$$= 0$$

$$\Rightarrow v_2(t) = s_2(t) = \cos(2\pi t), \quad \phi_2(t) = \frac{v_2(t)}{\|v_2(t)\|}$$

$$\|v_2(t)\|^2 = \int_0^1 \cos^2(2\pi t)\, dt = \int_0^1 \frac{1}{2}\left(1 + \cos(4\pi t)\right) dt$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^1 \cos(4\pi t)\, dt = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4\pi}\left[\sin(4\pi t)\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4\pi}\left[\underbrace{\sin(4\pi) - \sin(0)}_{0}\right] = \frac{1}{2}$$

$$\Rightarrow \|v_2(t)\| = \frac{1}{\sqrt{2}}$$

$$\phi_2(t) = \frac{\cos(2\pi t)}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \cdot \cos(2\pi t)$$

Q3 - Show the following statement is true for $x$ being a normal random variable:

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

- <u>solution</u>:

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \quad (1)$$

Let $y = \dfrac{(x-\mu)}{\sqrt{2}\,\sigma} \Longrightarrow dx = \sqrt{2}\,\sigma\,dy \quad (2)$

$$\Longrightarrow (1) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \overset{\text{substitute}}{\underset{(2)}{=}} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\,dy$$

Let $\int_{-\infty}^{+\infty} e^{-y^2}\,dy = I$

then $I^2 = \left[\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right]\left[\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Letting $x = r \cdot \cos(\theta)$ , $y = r \cdot \sin(\theta)$ (polar coordinates)

, we have:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \cdot dr\, d\theta = 2\pi \int_0^{\infty} e^{-r^2} \cdot r \cdot dr =$$

$$2\pi \cdot \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} = -\pi \left[ e^{-r^2} \right]_0^{\infty} = -\pi \left[ 0 - 1 \right] = \pi$$

$$\Rightarrow I = \sqrt{\pi} \quad \Rightarrow \int_{-\infty}^{+\infty} f_X(x)\, dx = \frac{1}{\sqrt{\pi}} \cdot I = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1 \quad \checkmark$$

Q4- show that the conditional pdf of $x$ given the event $B = (a < x \leq b)$ is as follows:

$$f_X(x \mid a < x \leq b) = \begin{cases} 0 & x \leq a \\ \dfrac{f_X(x)}{\int_a^b f_X(x')\, dx'} & a < x \leq b \\ 0 & x > b \end{cases}$$

- solution:

$$F_X(x \mid a < X \leq b) = P(X \leq x \mid a < X \leq b)$$

$$= \frac{P\{(X \leq x) \cap (a < X \leq b)\}}{P(a < X \leq b)}$$

Now $(X \leq x) \cap (a < X \leq b) = \begin{cases} \phi & x \leq a \\ a < X \leq x & a < x \leq b \\ a < X \leq b & x > b \end{cases}$

Thus,

$$F_X(x \mid a < X \leq b) = \frac{P(\phi)}{P(a < X \leq b)} = 0 \quad \text{for} \quad x \leq a$$

$$F_X(x \mid a < X \leq b) = \frac{P(a < X \leq x)}{P(a < X \leq b)} = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, \quad a < x \leq b$$

$$F_X(x \mid a < X \leq b) = \frac{P(a < X \leq b)}{P(a < X \leq b)} = 1, \quad \text{for} \quad x > b$$

Now, we use $f_X(x) = \dfrac{d\, F_X(x)}{dx}$

$$\Rightarrow f_X(x \mid a < X \leq b) = \begin{cases} 0 & x \leq a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} = \dfrac{f_X(x)}{\int_a^b f_X(x')\,dx'} & a < x \leq b \\ 0 & x > b \end{cases}$$

$Q_5$ - If $x \sim \mathcal{N}(\underline{\mu}, \underline{\Sigma})$ and $\underline{A}$ is a $q \times p$ matrix with $\text{rank}(A) = q \leq p$, show that $\underline{\mu}_z = \underline{A}\underline{\mu} + \underline{b}$ and $\underline{\Sigma}_z = \underline{A}\,\underline{\Sigma}\,\underline{A}^T$ in the case that $\underline{z} = \underline{A}\,\underline{x} + \underline{b}$.

- solution:

Simply using the definition of mean and covariance matrix, we can write:

$$\underline{\mu}_z = E\{\underline{z}\} = E\{\underline{A}\,\underline{x} + \underline{b}\} = E\{\underline{A}\underline{x}\} + E\{\underline{b}\}$$

$$= A\,E\{\underline{x}\} + \underline{b} = \underline{A}\underline{\mu} + \underline{b} \quad \checkmark$$

$$\underline{\Sigma}_z = E\left\{(\underline{z} - \underline{\mu}_z)(\underline{z} - \underline{\mu}_z)^T\right\}$$

$$= E\left\{\left(\underline{A}\underline{x} + \underline{b} - (\underline{A}\underline{\mu} + \underline{b})\right)\left(\underline{A}\,\underline{x} + \underline{b} - (\underline{A}\underline{\mu} + \underline{b})\right)^T\right\}$$

$$= E\left\{(A(\underline{x} - \underline{\mu}))\left(A(\underline{x} - \underline{\mu})\right)^T\right\} = E\left\{A(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T A^T\right\}$$

$$= A \cdot \underbrace{E\left\{(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T\right\}}_{\underline{\Sigma}} A^T = \underline{A} \cdot \underline{\Sigma}\,\underline{A}^T \quad \checkmark$$