Lecture 8

Because of the midterm exam this is a short 30min lecture on Jacobians.

- We looked at the derivative of functions f: R -> R, f : R->R^2 And the gradient vector of functions f : R^2 -> R. These can be though of as linear transformations (by matrix multiplication). We are talking here about these derivatives once evaluated at a given point. Instead of thinking of f'(a) as being a number we can think of it as a linear transformation (function) from R to R given by L(x) = f'(a)x. Similarly for f : R^2 -> R we have L(x,y) = [f_x(a) f_y(b)] [x, y]^T. Here T denotes transpose. So the previous expression is just a row times a column. The formulas give the standard equations for the tangent plane and tangent line if the origin is shifted to the point in question. eg. L(x) - f(a) = f'(a)(x-a).
- For a function F : R^n -> R^m, the derivative is a linear transformation from R^n to R^m, and is thus a m x n matrix. This is called the Jacobian matrix and is denoted by J_F or D(F). There are other common notations. See for example https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

Where to find the material:

- Adams_and_Essex. 12.6. See "materials" for a copy of these sections.
- See also the change of variables/Jacobian. Guichard 15.7. (see future lectures on changes of variabes in double and triple integrals).

What is a derivative?

Goal: Come up with the definition of the derivative of a function $f: \mathbb{R}^m \to \mathbb{R}^n$. In vector calculus this if usually refered to as the Jacobian

Firect, let's look at the cases we know about with some new notation

(1) $f: \mathbb{R} \to \mathbb{R}$

$$x \mapsto f(x)$$

(2) $f: \mathbb{R} \to \mathbb{R}^2$ (before $\vec{r}(t) = \langle x(t), y(t) \rangle$)

$$x \mapsto f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$$

(3) $f: \mathbb{R}^2 \to \mathbb{R}$ (before $f(x,y)$)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

"Recall"

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 5 \end{bmatrix} \quad 2 \times 3 \text{ matrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ 4x_1 + 0 + 5x_3 \end{bmatrix}$$

Think of A as a linear map

$$A: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(multiplication by A)

write $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Familiar derivatives in terms of matrices.

FUNCTION	LINEARIZATION at x_0	TANGENT OBJECT
$f: \mathbb{R} \to \mathbb{R}$	$J: \mathbb{R} \to \mathbb{R}$ $J[x] = f'(x_0)[x]$ 1x1 matrix $J = [f'(x_0)]$	$[y - y_0] = \underbrace{f'(x_0)}_{= J}[x - x_0]$
$f: \mathbb{R} \to \mathbb{R}^2$	$J: \mathbb{R} \to \mathbb{R}^2$ $J[x] = \begin{bmatrix} f_1'(x_0) \\ f_2'(x_0) \end{bmatrix} \boxed{x}$ $= \begin{bmatrix} x f_1'(x_0) \\ x f_2'(x_0) \end{bmatrix}$ 2x1 matrix $J = \begin{bmatrix} f_1'(x_0) \\ f_2'(x_0) \end{bmatrix}$ $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$	$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} f_1(x_0) \\ f_2(x_0) \end{bmatrix} = \underbrace{\begin{bmatrix} f_1'(x_0) \\ f_2'(x_0) \end{bmatrix}}_{= J}[x - x_0]$ compare to $\vec{r}(t) = \vec{r}(t_0) + (t - t_0)\,\vec{r}\,'(t_0)$
$f: \mathbb{R}^2 \to \mathbb{R}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$	$J: \mathbb{R}^2 \to \mathbb{R}$ 1x2 matrix $J = \left[\frac{\partial f}{\partial x_1}(x_0) \quad \frac{\partial f}{\partial x_2}(x_0) \right]$ $= \nabla f(x_0)$ $J \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$ $= f_{x_1}(x_0)\, x_1 + f_{x_2}(x_0)\, x_2$	$z - f(x_0)$ $\approx$ regular tangent plane equation $= \underbrace{[f_{x_1}(x_0) \quad f_{x_2}(x_0)]}_{= J} [x - x_0]$ $= f_{x_1}(x_0)(x_1 - x_{10})$ $+ f_{x_2}(x_0)(x_2 - x_{20})$

The Jacobian

∇f_i

Answer: The linearization of the function.

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Then $J_f = \mathbb{R}^n \to \mathbb{R}^m$ (linear map - matrix)

Df //

Notation $x \in \mathbb{R}^n$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

write $f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$

Think: $f_i(x) : \mathbb{R}^m \to \mathbb{R}$

$$\nabla f_1(x) = \left[\frac{\partial f_1}{\partial x_1} \; \cdots \; \frac{\partial f_1}{\partial x_n} \right]$$

Definition: The Jacobian of f at x_0 is

$$J_f(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

Note, if $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ Then $Jf(x_0) a \in \mathbb{R}^m$

So $J : \mathbb{R}^n \to \mathbb{R}^m$

Note The chain rule still works the same way, and in fact is just matrix multiplication (we do not discuss)

Example

Let $f(x,y) = x(1+y^2) - 1$

$g(x,y) = y(1+x^2) - 2$

(use later also)

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $F(\vec{x}) = \begin{bmatrix} f(\vec{x}) \\ g(\vec{x}) \end{bmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$= \begin{bmatrix} x(1+y^2)-1 \\ y(1+x^2)-2 \end{bmatrix}$

$$JF = \begin{bmatrix} \overbrace{\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}}^{=\nabla f} \\ \underbrace{\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}}_{=\nabla g} \end{bmatrix}$$

$$= \begin{bmatrix} 1+y^2 & 2xy \\ 2xy & 1+x^2 \end{bmatrix}$$

Let $P_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$J_F(P_0) = \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix}$$

$J_F(P_0) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} 5x+4y \\ 4x+2y \end{bmatrix} \in \mathbb{R}^2$$

What's next

We will see two applications

① Newton's method (self-study
- notes coming
soon)

② Change of variable in
integration