Lecture 9 - part 1

Topic: Newton's method

- Newton's method is a numerical method for solving (systems of) equations.
- Let's first look at one equation in one variable. We want to solve $f(x) = 0$. That is, we want to find where the graphs crosses the x-axis. The idea of the method is to guess an initial point x_0 (approximate solution). Approximate $f(x)$ at x_0 by its tangent line $y = f(x_0) + f'(x_0)(x-x_0)$. We can find where this tangent line crosses the x-axis. Putting $y =0$ and solving for x gives $x_1 = x = x_0 - f(x_0)/f'(x_0)$ which we hope is a better approximation to $f(x) = 0$. Iterate this procedure, with the general expression $x_{i+1} = x_i - f(x_i)/f'(x_i)$.
- For many example this algorithm works efficiently. In general there is a great deal to be said to investigate the subtitles of such numerical methods. Aalto offers many courses related to computational mathematics and numerical analysis.
- Now let us look at the case of n-equations in n unknowns. Let $n =2$ just for easy of writing, but there is no difference for larger numbers. We want to solve $f(x,y) = 0$ and $g(x,y) = 0$. Put these together as a column vector. Let $F = [f \ \ g]^T$ (where T denoted transpose). Let $z = [x \ y]^T$. So the system of equations is now written $F(z) = 0$. The analogue to the tangent line or plane is $L(z) = F(z_0) + J_F(z_0) (z - z_0)$. Letting $L = 0$ and solving for z gives $z_1 = z = z_0 - J_{F}^{(-1)}(z_0) F(z_0)$. Here the (-1) means matrix inverse.
- Showed an example of implementing this on Maple. The code and output can be found in "materials".

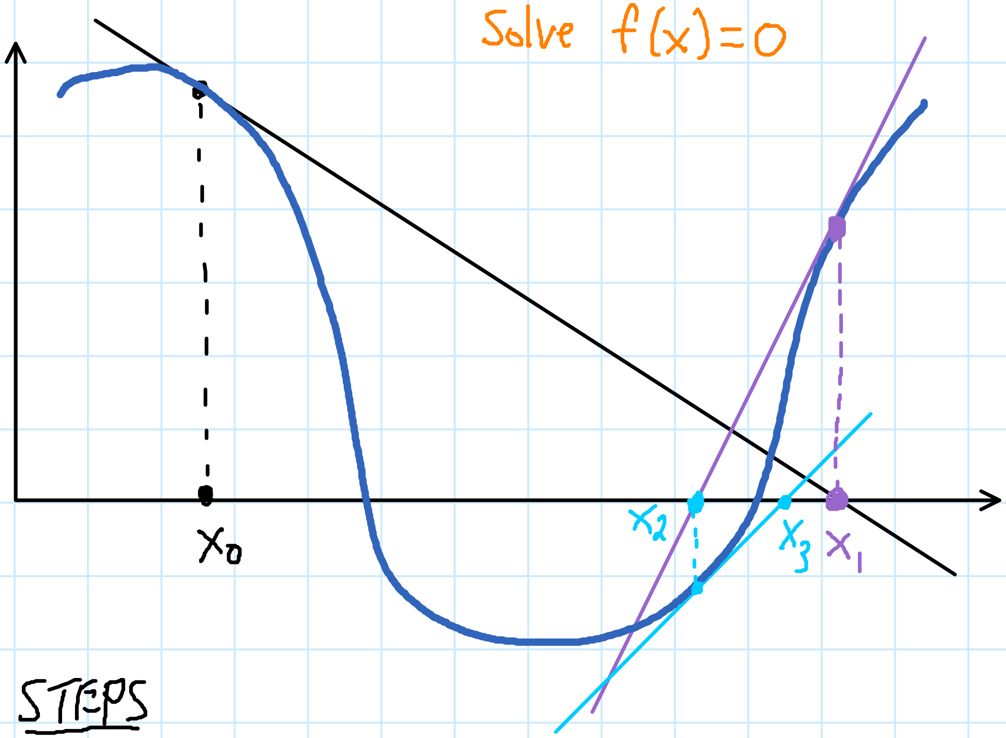
Where to find this material

- Adams and Essex. 13.7 (see "materials" for a copy of this section.)

Newton's method

This is a numerical method for finding zeros of a function.

Solve $f(x)=0$



STEPS

1. choose an initial guess x_0
2. draw the tangent line
3. x_1 is where this tangent intersects the x-axis (our new approximation)
4. Repeat: $x_2, x_3, \ldots$

Formula:

Tangent line at x_0 is $y = f(x_0) + f'(x_0)(x - x_0)$

Intercept is when $y = 0$, so

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

Solving for x gives $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

as long as $f'(x_0) \neq 0$

So the next approximation, x_1, is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeating this: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

⋮

The iterative formula is

NEWTON'S METHOD

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newton's methos - 2 variables

We deal the case of a function $F: \mathbb{R}^2 \to \mathbb{R}^2$

Let $F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

Solve $F(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \begin{cases} f = 0 \\ g = 0 \end{cases}$

Initial guess $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

To get the next iterate we solve

$$\vec{y} - F(\vec{x}_0) = J_F(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

for $\vec{x}$ when $\vec{y} = \vec{0}$

$$-F(\vec{x}_0) = J_F(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

In components

$$-\begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

As long as $J_F(x_0,y_0)$ is invertible,

$$(\vec{x} - \vec{x}_0) = -J_F^{-1}(x_0,y_0)\vec{F}(\vec{x}_0)$$

$$\Rightarrow \quad \vec{x}_1 = \vec{x}_0 - J_F^{-1}(x_0,y_0)\vec{F}(\vec{x}_0)$$

OR

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J_F^{-1}(x_0,y_0)\begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix}$$

This all works in any dimension: $F: \mathbb{R}^n \to \mathbb{R}^n$

$$\underbrace{x_{n+1}}_{\mathbb{R}^n} = \underbrace{x_n}_{\mathbb{R}^n} - \underbrace{J_F^{-1}}_{n \times n \text{ matrix}}(x_n,y_n)\,\underbrace{F(x_n)}_{\mathbb{R}^n}$$

Newton's method example

① $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - e^x + \sin(x)$

Find solutions to $f(x) = 0$

(See maple code and PDF on MyCourses)

② $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = \begin{bmatrix} x(1+y^2) - 1 \\ y(1+x^2) - 2 \end{bmatrix}$

Find solutions to $F(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

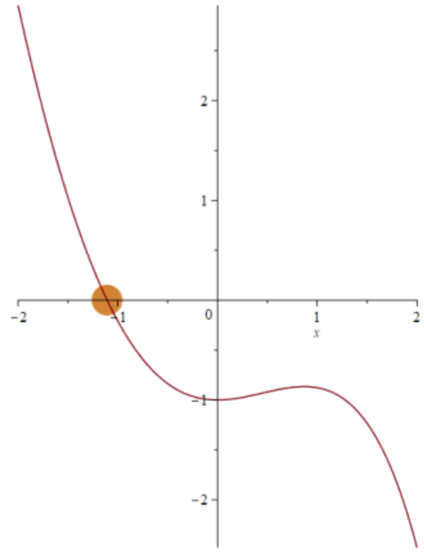
That is, find solutions to

$$\begin{cases} x(1+y^2) - 1 = 0 \\ y\left(1+x^2-2\right) = 0 \end{cases}$$

Think of this as the intersection of 3 surfaces

$$z = x(1+y^2) - 1 = y(1+x^2) - 2 = 0$$

①



②