Lecture 9 - part 2

Topics: Double integrals and ~~polar coordinates~~

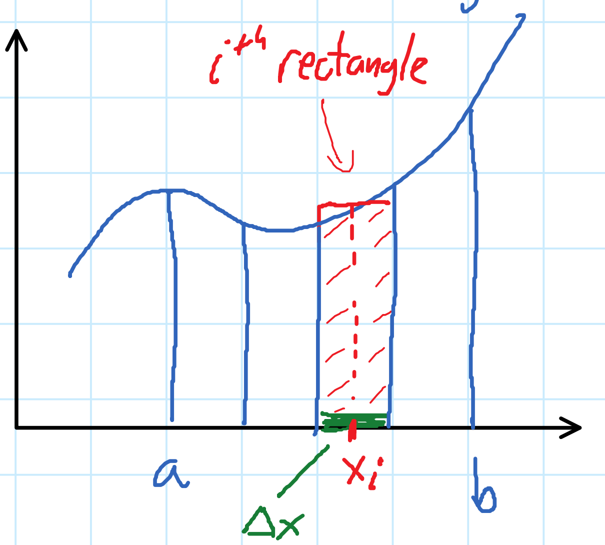
- Defined the double integral using Riemann sums.
- Using intuition from slicing volumes in different ways we understood how to compute the double integral in terms of iterated integrals. This was formalized in Fubini's Theorem.
- Learned how to integrate over general (non-rectangular) regions.
- Did examples of switching the order of integration. In one example we saw that one order of integration was impossible to calculate while the other order was easy.
- ~~Introduced polar coordinates. Showed geometrically that dA = r dr d theta.~~ (will cover in the next lecture)

Where to find this material

- Adams and Essex 14.1, 14.2, 14.4
- Corral, 3.1, 3.2, 3.5
- Guichard, 15.1, 15.2, 15.7 (check out the beautiful picture in exercise 15.1.30)
- Active Calculus. 11.1 - 11.3, 11.5

Double integrals

Single integral



$f(x)$ defined on $[a,b]$

N interval

$$\Delta x = \frac{b-a}{N}$$

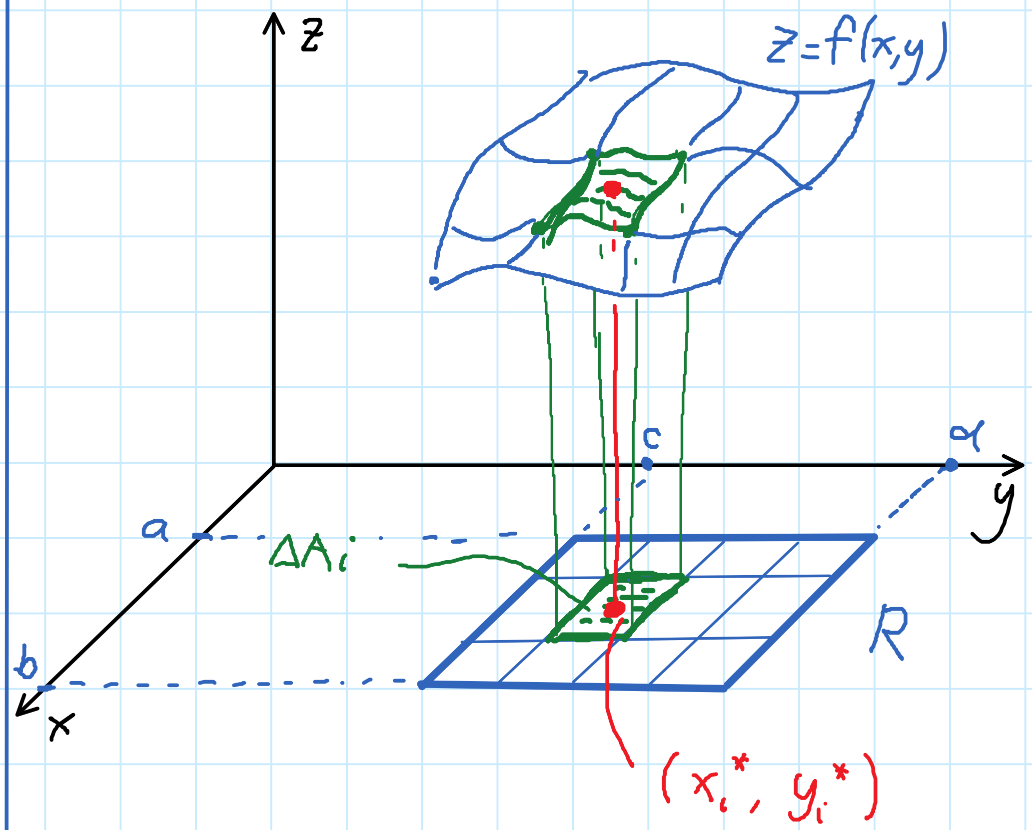
$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x$$

Double integral

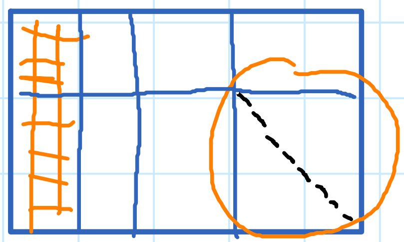
more general regions later

$f(x,y)$ defined on rectangle R

As above, we will think of $f(x,y)$ as the height above (x,y)



$R = [a,b]\times[c,d] = \{(x,y) \mid a\le x\le b,\ c\le y\le d\}$



$P =$ a partition

= a set of subrectangles that make up R

Let $\|P\|$ = maximum length diagonal why ?

Saying $\|P\| \to 0$ is a precise way of saying that the number of subrectangles goes to inifinity and the size goes to zero.

height above (x,y)

Saying $||P|| \to 0$ is a precise way of saying that the number of subrectangles goes to inifinity and the size goes to zero.

Double integral (definiton)

R = rectangle , P = partition of R into N subrectangles of sizes ΔA_i , $i = 1, \dots, N$

$f : R \to \mathbb{R}$

Definition of the double integral

$$\iint\limits_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{i=1}^{N} \overbrace{f(x_i, y_i)\Delta A_i}^{\text{volume of the column}}$$

Theorem: If $f(x,y)$ is continuous on R then the limit exists and so the double integral is defined

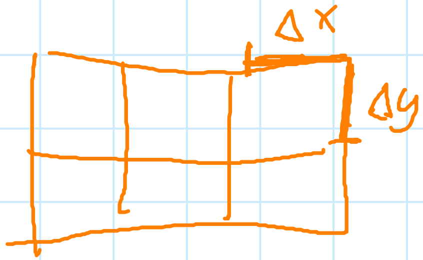
In fact the integral can be defined for much more general types of functions (we do not discuss these topics).

How to compute ?

Note that in the textbooks you may see

$\Delta x = \dfrac{b-a}{M}$, $\Delta y = \dfrac{d-c}{N}$

$\Delta A = \Delta x \Delta y$



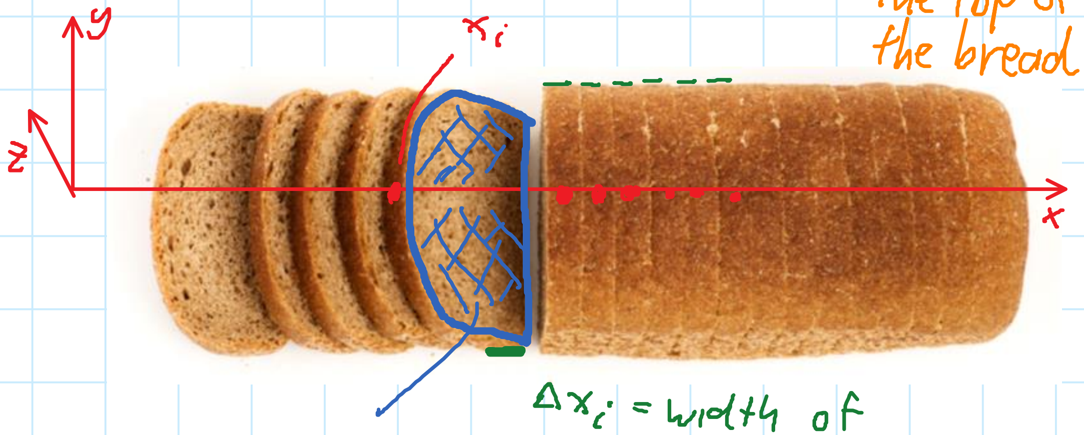
$$\iint\limits_R f(x,y)\,dA = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j)\,\Delta A$$

Both these definitions are only given here at the intuitive level.

Since we are not proving things, it does not matter which definition we use. A rigorous treatment of these definitons and theorems can be found in books on real analysis . See also the course on metric spaces for foundations needed for such topics.

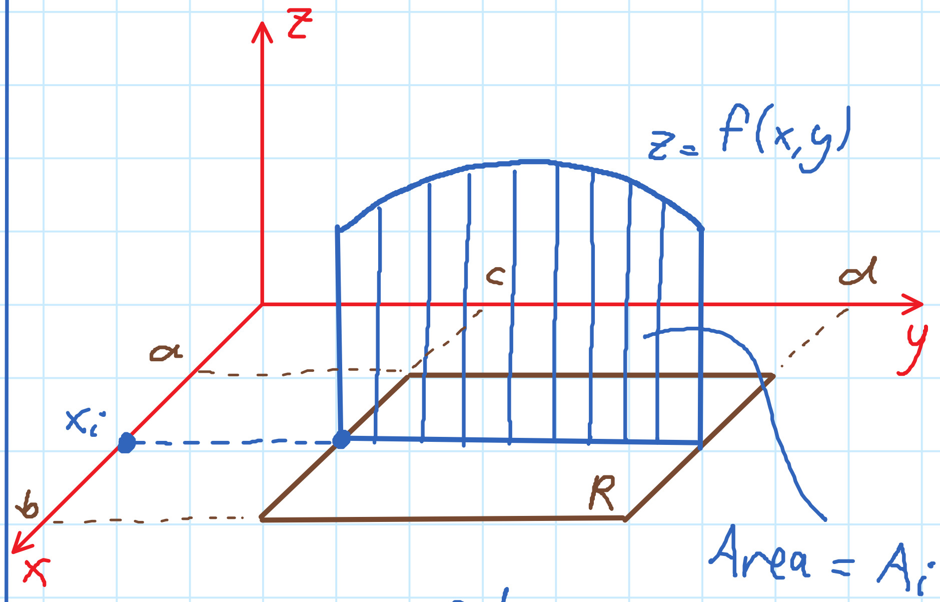
Fubini's theorem

$z = f(x,y)$ = height of the top of the bread



Δx_i = width of the i^{th} slice

Area of the i^{th} slice $= A_i$



Let $A(x)$ be the cross-sectional area at position x.

Then $A_i = A(x_i)$

Volume of the i^{th} slice $\approx A(x_i)\,\Delta x_i$

$$\text{Volume} = \lim_{\#\text{slices} \to \infty} \sum_{i=1}^{\#\text{slices}} A(x_i)\,\Delta x_i$$

$$= \int_a^b A(x)\,dx$$

$$A_i = A(x_i) = \int_c^d f(x_i, y)\,dy$$

$$\text{Volume} = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

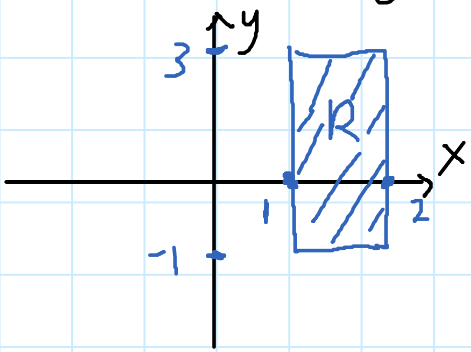
Fubini's Theorem: If $f(x,y)$ is continuous on R then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(y,x)\,dx\,dy$$

Examples

① Compute the double integral of $f(x,y) = x^2y$ over $[1,2] \times [-1,3]$



$$\iint_R x^2y\,dA = \int_1^2 \left(\int_{-1}^3 x^2y\,dy \right) dx$$

$$= \int_1^2 \left(\frac{x^2y^2}{2} \Big|_{y=-1}^{y=3} \right) dx$$

$$= \int_1^2 x^2 \left(\frac{9}{2} - \frac{1}{2} \right) dx$$

$$= \int_1^2 \underbrace{4x^2}_{= A(x)} dx = \frac{4}{3}x^3 \Big|_1^2$$

$$= \frac{28}{3}$$

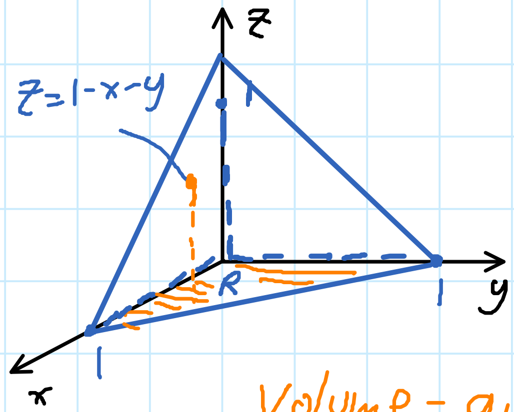
HW.

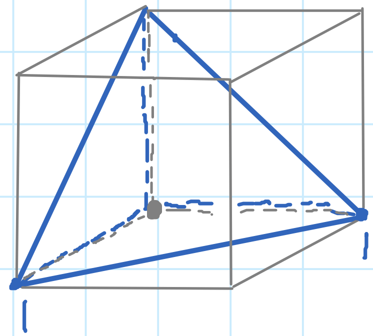
Same answer

$\int_{\cdot}^{\cdot} \int_{\cdot}^{\cdot} - \,dx\,dy$

② Find the volume of the prism bounded by the 3 coodinate planes and the plane $z = 1 - x - y$.

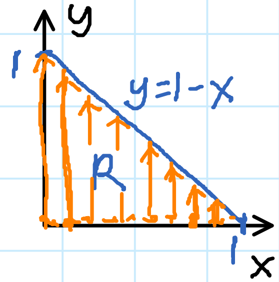
✓ xy-plane, xz-plane, yz-plane





Volume = guess = $\frac{1}{3}$? $\frac{1}{4}$? $\frac{1}{6}$? ✓

Calculate



R : $0 \le x \le 1$

$0 \le y \le 1-x$ line

Volume $= \iint_R (1-x-y)\,dA$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

R is a Region that is not a rectangle

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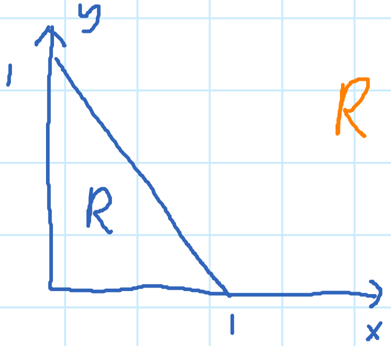
Calculation from previous example



$$R: \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \end{cases}$$

$$\int_0^1 \int_0^{1-x} 1-x-y \, dy \, dx = \int_0^1 (1-x)y - \frac{1}{2}y^2 \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \underbrace{\left[(1-x)(1-x) - \frac{1}{2}(1-x)^2 \right] - [0]}_{=A(x)} dx$$

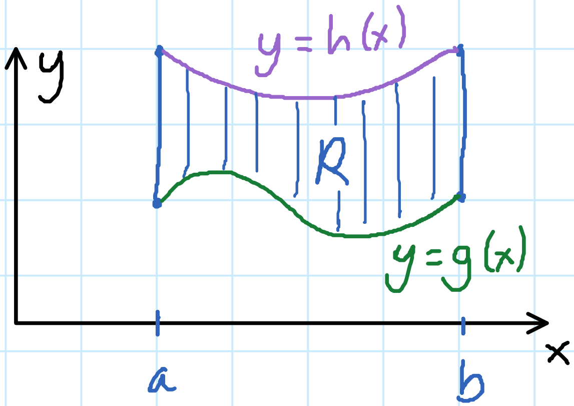
$$= \frac{1}{2} \int_0^1 (1-x)^2 \, dx$$

$$= \frac{1}{2} \left(\frac{-1}{3} \right) (1-x)^3 \Big|_0^1$$

$$= \frac{-1}{6} (0 - 1)$$

$$= \frac{1}{6}$$

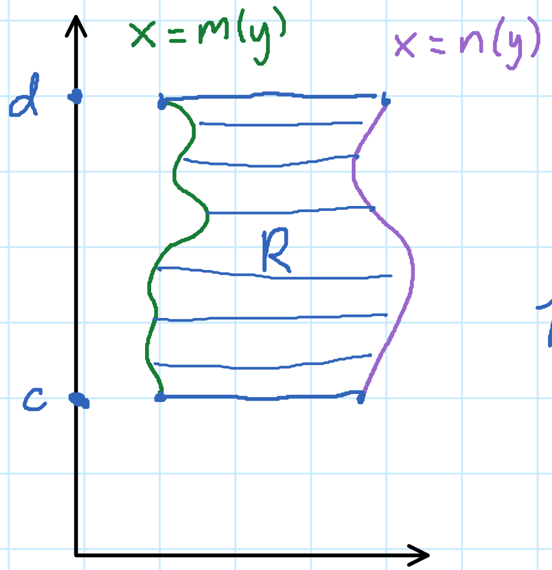
Non-rectanglular regions in general



$R:\quad a \leq x \leq b$

$\qquad g(x) \leq y \leq h(x)$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$



$R:\quad c \leq y \leq d$

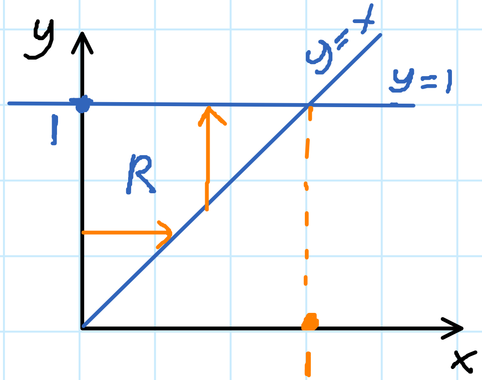
$\qquad m(y) \leq x \leq n(y)$

$$\iint_R f(x,y)\,dA$$

$$= \int_c^d \int_{m(y)}^{n(y)} f(x,y)\,dx\,dy$$

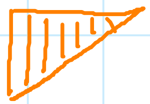
Example of switching the order of integration:

Let R be the triangle formed by the y-axis, the line y=1 and the line y=x. Evaluate $\iint_R \frac{\sin(y)}{y}\,\mathrm{d}A$



Note: $\int \frac{\sin(y)}{y}\,dy$ can not be expressed in terms of elementary functions

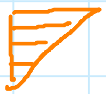
(A) slice vertically: $\iint_R f\,dA = \int_0^1 \int_x^1 \frac{\sin(y)}{y}\,dy\,dx$



$\eqqcolon \operatorname{sinc}(y)$

(B) slice horizontally: $R:\quad 0 \leq y \leq 1$

$\qquad 0 \leq x \leq y$



line $y=x$

$\Leftrightarrow x=y$

$$\int_0^1 \int_0^y \frac{\sin(y)}{y}\,dx\,dy$$

$$= \int_0^1 \frac{\sin y}{y}\, x \Big|_{x=0}^{x=y} dy = \int_0^1 \frac{\sin(y)}{\cancel{y}} \cdot y\,dy$$

$$= \int_0^1 \sin(y)\,dy$$

easy

$$= -\cos(1) + 1$$

$= -\cos(1) + 1 \longleftarrow = \int_0^1 \sin(y)\,dy$ easy