Computational Algebraic Geometry

Applications of algebraic geometry in robotics

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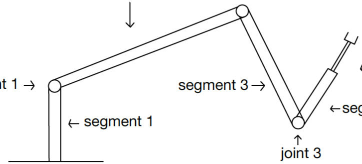
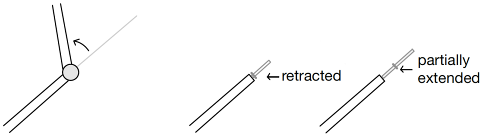
Geometric description of robotics

Setup



- Robots are constructed from rigid links or segments that are connected by various types of joints
- Segments are connected in series (as in human limbs)
- One end of the robot arm will be usually in a fixed position
- At the other end will be the hand that is sometimes considered the final segment of the robot

Joints



a revolute joint

a prismatic joint

- Since the segments of our robots are rigid, the possible motions are determined by the motions of joints:
  - planar revolute joints - permits a rotation of one segment related to another
  - prismatic joints - sliding or translation by axis

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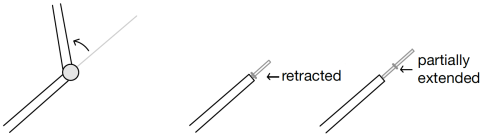

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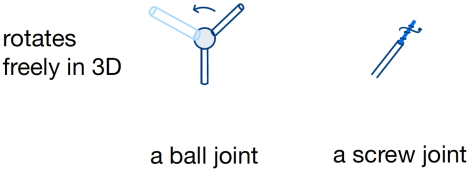
    - planar revolute joints - permits a rotation of one segment related to another
    - prismatic joints - sliding or translation by axis
- We will assume that
    - the joints all lie in the same plane,
    - the axes of rotation of all revolute joints are perpendicular to that plane, and
    - the translation axes for the prismatic joints all lie in the plane of joints

Real robots



- Real robots: ball joints, screw joints combining rotation and translation along the axis of rotation, several planar revolute joints with nonparallel axes of rotation
- These other kinds of joints can be considered using similar algebraic methods, but we will not do it today
- Our goal today is to give a general idea how to use affine varieties in the study of robotics

Mathematical setup

- The setting of a revolute joint between the segments i and $i+1$ is determined by the angle between these segments.
- Angles can be identified with the circle S^1 or the segment $[0, 2\pi]$ with the endpoints identified.
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- The setting of a prismatic joint is determined by the length that it is extended.
- All possible settings are given by a finite interval.
- The possible settings of all of the joints in a planar robot with r revolute joints and p prismatic joints:

$$\mathcal{J} = S^1 \times \cdots \times S^1 \times I_1 \times \cdots \times I_p.$$

  The set $\mathcal{J}$ is called the joint space of the robot.

Mathematical setup

- The possible configurations of the hand are described by
  - its position $(a, b) \in U \subseteq \mathbb{R}^2$ and
  - its orientation $u \in V \subseteq S^1$ given by a unit vector aligned with some feature of the hand.
- The configuration space or operation space of the robot's hand:

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- The configuration space or operation space of the robot's hand:
$$\mathcal{C} = U \times V \subseteq \mathbb{R}^2 \times S_1.$$
- Each setting of joints will place the hand at a uniquely determined position.
- Hence we have a mapping $f : \mathcal{J} \to \mathcal{C}$ which describes how the different joint settings yield different hand configurations.

Two basic problems

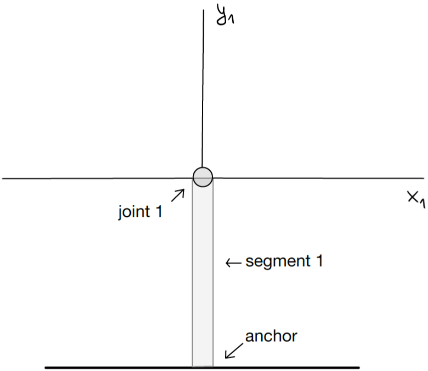
Forward kinematic problem: Can we give an explicit description or formula for f in terms of the joint settings and the dimensions of the segments of the robot arm?

Inverse kinematic problem: Given $c \in \mathcal{C}$, can we determine one or all of $j \in \mathcal{F}$ such that $f(j) = c$?

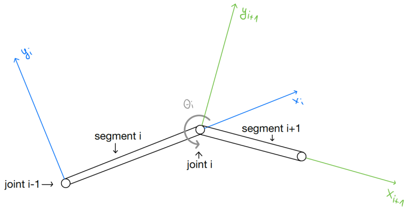
The forward kinematic problem

Coordinate systems

- The first segment is fixed (or anchored).
- The origin of the global coordinate system is placed at joint 1.



Coordinate systems



- There is a local coordinate system at each of the revolute joints.
- At joint i, we introduce (x_{i+1}, y_{i+1}) coordinate system in the following way:
  1. The origin is placed at joint i.
  2. The positive x_{i+1}-axis lies along the direction of the segment $i+1$.
  3. The positive y_{i+1}-axis forms a normal right handed coordinate system.
  4. For each $i \geq 2$, the (x_i, y_i) coordinates of joint i are $(l_i, 0)$, where l_i is the length of the segment i.

Coordinate systems

Relate (x_{i+1}, y_{i+1})-coordinates of a point with the (x_i, y_i)-coordinates of the point.

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- Let θ_i be the counterclockwise angle from the x_i-axis to the x_{i+1}-axis.
- First rotate by θ_i and then translate by $(l_i, 0)$.
- The rotation is obtained by multiplying by the rotation matrix

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}.$$

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- The translation is obtained by adding the vector $(l_i, 0)$.
- Thus

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix}.$$

Coordinate systems

- The last identity can be also written as

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_i \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}.$$

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- We have

$$A_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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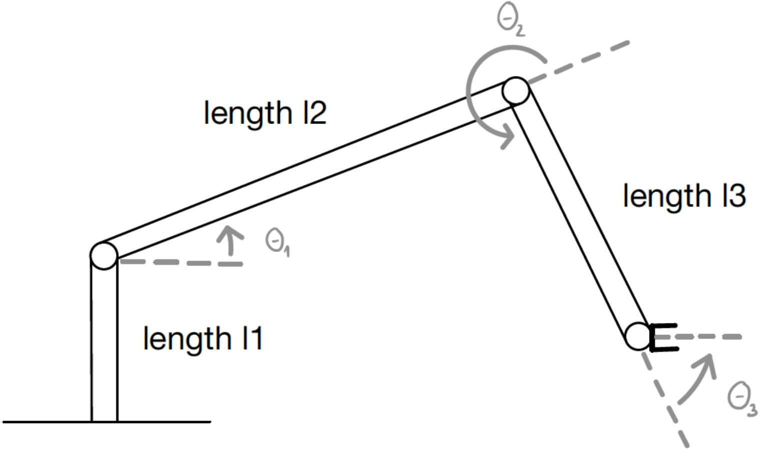
- Global coordinates can be obtained by starting in the last coordinate system and working our way back to the global (x_1, y_1) coordinate system one joint at the time.


θ_2
length l2
length l3
θ_1
length l1
θ_3

Example

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2)+l_2\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & l_3\sin(\theta_1+\theta_2)+l_2\sin\theta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}$$

Since the (x_4, y_4) coordinates of the hand are $(0, 0)$, we set $x_4 = y_4 = 0$:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} l_3\cos(\theta_1+\theta_2)+l_2\cos\theta_1 \\ l_3\sin(\theta_1+\theta_2)+l_2\sin\theta_1 \\ 1 \end{pmatrix}$$

Example

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Combining these two things gives the map $f : \mathcal{J} \to \mathcal{C}$:

$$f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 \\ l_3 \sin(\theta_1 + \theta_2) + l_2 \sin\theta_1 \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}.$$

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How to convert such representations to polynomial or rational mappings?

$$c_i = \cos\theta_i$$
$$s_i = \sin\theta_i$$

subject to $c_i^2 + s_i^2 - 1 = 0$ for $i = 1, 2, 3$. The variety defined by these three equations in $\mathbb{R}^6$ is a realization of the joint space $\mathcal{J}$.

Example

The (x_1, y_1) coordinates of the hand are:

$$\begin{pmatrix} l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}.$$

We have defined a polynomial map from

$$\mathcal{J} = \mathbb{V}(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1) \text{ to } \mathbb{R}^2.$$

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It is not possible to express the hand orientation as a polynomial in c_i and s_i, but it can be handled similarly.

The inverse kinematic problem

Inverse kinematic problem

Given a point $(x_1, y_1) = (a, b) \in \mathbb{R}^2$ and an orientation, we wish to determine whether it is possible to place the hand of the robot at that point with that orientation. If it is possible, we wish to find all combinations of joint settings that will accomplish this.

1. Determine the image of $f : \mathcal{J} \to \mathcal{C}$
2. Determine the inverse image $f^{-1}(c)$

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1. Determine the image of $f : \mathcal{J} \to \mathcal{C}$
2. Determine the inverse image $f^{-1}(c)$

Ignoring the hand orientation, we need to solve

$$
\begin{aligned}
a &= l_3(c_1 c_2 - s_1 s_2) + l_2 c_1, \\
b &= l_3(c_1 s_2 + c_2 s_1) + l_2 s_1, \\
0 &= c_1^2 + s_1^2 - 1, \\
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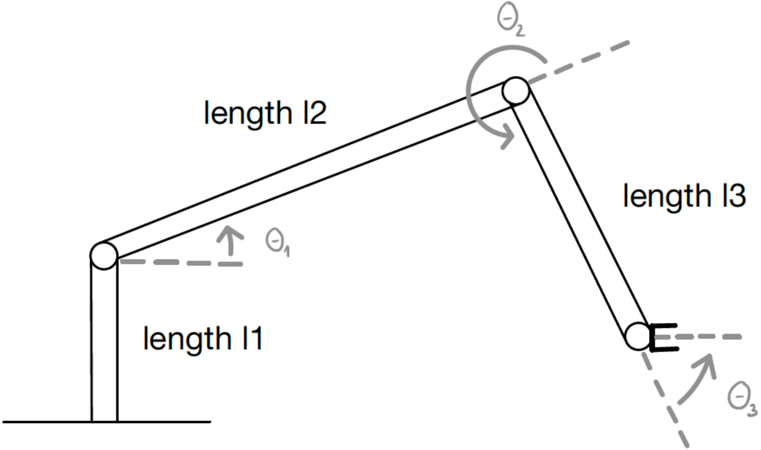
\end{aligned}
$$



Θ_2
length l2
length l3
Θ_1
length l1
Θ_3

Groebner basis

To solve these equations, we compute a Groebner basis using lex order with the variables ordered $c_2 > s_2 > c_1 > s_1$. The Groebner basis will depend on a, b, l_2, l_3 that appear as symbolic parameters.

$$
\begin{aligned}
& c_2 - \frac{a^2 + b^2 - l_2^2 - l_3^2}{2 l_2 l_3}, \\
& s_2 + \frac{a^2 + b^2}{a l_3} s_1 - \frac{a^2 b + b^3 + b(l_2^2 - l_3^2)}{2 a l_2 l_3}, \\
& c_1 + \frac{b}{a} s_1 - \frac{a^2 + b^2 + l_2^2 - l_3^2}{2 a l_2}, \\
& s_1^2 - \frac{a^2 b + b^3 + b(l_2^2 - l_3^2)}{l_2(a^2 + b^2)} s_1 \\
& + \frac{(a^2 + b^2)^2 + (l_2^2 - l_3^2)^2 - 2a^2(l_2^2 + l_3^2) + 2b^2(l_2^2 - l_3^2)}{4 l_2^2 (a^2 + b^2)}
\end{aligned}
$$

Specialization

- In practice, we want to compute the Groebner basis for specific values of parameters a, b, l_2, l_3.
- Substituting symbolic parameters by specific values is called specialization.

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- Substituting symbolic parameters by specific values is called specialization.
- There is a proper subvariety $W \subset \mathbb{R}^4$ such that the Groebner basis above specializes to a Groebner basis when a, b, l_2, l_3 take special values in $\mathbb{R}^4 - W$.
  - Vanishing of the denominators is one problem.
  - There can be other problems.
  - In this particular example, W is the variety that is defined by the vanishing of the denominators. I.e., W is the variety defined by a, l_2, l_3 and $a^2 + b^2$.

Solutions

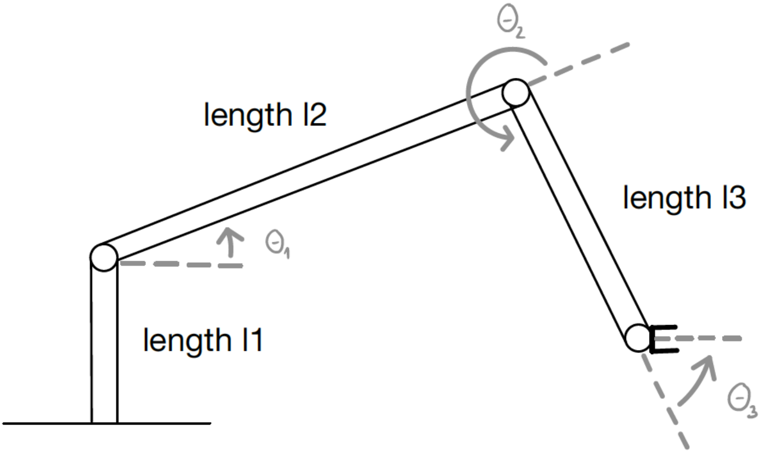
$$c_2 - \frac{a^2 + b^2 - l_2^2 - l_3^2}{2 l_2 l_3},$$

$$s_2 + \frac{a^2 + b^2}{a l_3} s_1 - \frac{a^2 b + b^3 + b(l_2^2 - l_3^2)}{2 a l_2 l_3},$$

$$c_1 + \frac{b}{a} s_1 - \frac{a^2 + b^2 + l_2^2 - l_3^2}{2 a l_2},$$

$$s_1^2 - \frac{a^2 b + b^3 + b(l_2^2 - l_3^2)}{l_2(a^2 + b^2)} s_1 + \frac{(a^2 + b^2)^2 + (l_2^2 - l_3^2)^2 - 2a^2(l_2^2 + l_3^2) + 2b^2(l_2^2 - l_3^2)}{4 l_2^2 (a^2 + b^2)}$$

- Any zero of s_1 can be extended uniquely to a full solution of the system.
- Since the last polynomial is quadratic in s_1, then s_1 can have at most two solutions.
- One has to study which solutions are real.



Quiz: Assume $l_2 = l_3 = 1$. What are the positions that the hand can reach?

Example $l_2 = l_3 = 1$

We will study the specialization $l_2 = l_3 = 1$. The Groebner basis in $\mathbb{R}(a, b)[s_1, c_1, s_2, c_2]$ is

$$
\begin{aligned}
& c_2 - \frac{a^2 + b^2 - 2}{2}, \\
& s_2 + \frac{a^2 + b^2}{a} s_1 - \frac{a^2 b + b^3}{2a}, \\
& c_1 + \frac{b}{a} s_1 - \frac{a^2 + b^2}{2a}, \\
& s_1^2 - b s_1 + \frac{(a^2 + b^2)^2 - 4a^2}{4(a^2 + b^2)}.
\end{aligned}
$$

This Groebner basis gives a Groebner basis for specializations satisfying $a \neq 0$ and $a^2 + b^2 \neq 0$.

Example $a \neq 0$

- If $a \neq 0$, then this implies $a^2 + b^2 \neq 0$, since $a, b \in \mathbb{R}$.
- We can find the solutions for s_1 by using the quadratic formula for the last equation:

$$s_1 = \frac{b}{2} \pm \frac{|a|\sqrt{4 - (a^2 + b^2)}}{2\sqrt{a^2 + b^2}}.$$

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- These solutions are real if and only if $0 < a^2 + b^2 \leq 4$. When $a^2 + b^2 = 4$, then we have a double root.
- The distance from joint 1 to 3 is at most $l_2 + l_3 = 2$ and position of distance 2 can be reached only in one way, by setting $\theta_2 = 0$.

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- The distance from joint 1 to 3 is at most $l_2 + l_3 = 2$ and position of distance 2 can be reached only in one way, by setting $\theta_2 = 0$.
- Given s_1, we can solve for c_1, s_2, c_2.
- The values for s_1, c_1, s_2, c_2 uniquely determine the angles θ_1 and θ_2.

Example $a = b = 0$

- Let $a = b = 0$.
- Then most polynomials in the Groebner basis are not defined.
- Geometrically this means that the joint 3 is placed at the origin of (x_1, y_1).
- There are infinitely many ways to do it: First choose θ_1 arbitrarily and then take $\theta_2 = \pi$.
- These are in fact the only possibilities for setting $(a, b) = (0, 0)$.

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$$a = l_3(c_1c_2 - s_1s_2) + l_2c_1, b = l_3(c_1s_2 + c_2s_1) + l_2s_1, 0 = c_1^2 + s_1^2 - 1, 0 = c_2^2 + s_2^2 - 1.$$

  to compute the Groebner basis.
- The Groebner basis is

$$c_2 - \frac{b^2 - 2}{2}, s_2 - bc_1, c_1^2 + \frac{b^2 - 4}{4}, s_1 - \frac{b}{2}.$$

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- The form of the Groebner basis is different under this specialization: the equation for c_1 instead of the equation for s_1 has degree 2.
- The system has two solutions when $|b| < 2$, one solution when $|b| = 2$ and no solutions when $|b| > 2$.

Example conclusion

The system has:

- infinitely many solutions when $a^2 + b^2 = 0$;
- two solutions when $0 < a^2 + b^2 < 4$;
- one solution when $a^2 + b^2 = 4$;
- no solutions when $a^2 + b^2 > 4$.

The cases $a^2 + b^2 = 0, 4$ are known as kinematic singularities of the robot.

Kinematic singularities

Let J_f denote the Jacobian matrix of the map $f : \mathcal{J} \to \mathcal{C}$.

$$f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 \\ l_3 \sin(\theta_1 + \theta_2) + l_2 \sin\theta_1 \\ 1 \end{pmatrix}$$

$$J_f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} -l_3 \sin(\theta_1 + \theta_2) - l_2 \sin\theta_1 & -l_3 \sin(\theta_1 + \theta_2) & 0 \\ l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 & l_3 \cos(\theta_1 + \theta_2) & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$J_f(\theta_1, \theta_2, \theta_3)$ defines a linear map that is the best linear approximation of f at $(\theta_1, \theta_2, \theta_2) \in \mathcal{J}$.

Kinematic singularities

- The dimensions $\dim(\mathcal{J})$ and $\dim(\mathcal{C})$ are the independent degrees of freedom of setting joints and the configuration.
- $\dim(\mathcal{J}) = 3$, since each planar joint contributes one degree of freedom
- $\dim(\mathcal{C}) = 3$

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- $\dim(\mathcal{J}) = 3$, since each planar joint contributes one degree of freedom
- $\dim(\mathcal{C}) = 3$
- In general $\dim(\mathcal{J}) = m$, $\dim(\mathcal{C}) = n$ and $\text{rank}(J_f(j)) \leq \min(m, n)$.
- $J_f(j)$ has maximal rank if it is equal to $\min(m, n)$ and otherwise it has deficient rank.
- In the latter case, the image is smaller than one would expect.

Kinematic singularities

- The dimensions $\dim(\mathcal{J})$ and $\dim(\mathcal{C})$ are the independent degrees of freedom of setting joints and the configuration.
- $\dim(\mathcal{J}) = 3$, since each planar joint contributes one degree of freedom
- $\dim(\mathcal{C}) = 3$
- In general $\dim(\mathcal{J}) = m$, $\dim(\mathcal{C}) = n$ and $\operatorname{rank}(J_f(j)) \leq \min(m, n)$.
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- In the latter case, the image is smaller than one would expect.

Definition

A kinematic singularity for a robot is a point $j \in \mathcal{J}$ such that $J_f(j)$ has rank strictly less than $\min(m, n)$.

Kinematic singularities

In our example, we have a kinematic singularity if and only if when the determinant of

$$J_f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} -l_3 \sin(\theta_1 + \theta_2) - l_2 \sin\theta_1 & -l_3 \sin(\theta_1 + \theta_2) & 0 \\ l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 & l_3 \cos(\theta_1 + \theta_2) & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is zero. This gives $\sin\theta_2 = 0$ or equivalently $\theta_2 = 0$ or $\theta_2 = \pi$.

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- The second case means that segment 3 folds back.
- These are the two cases from earlier when we have one or infinitely many ways to get a solution (a, b).

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Proposition

Let $f : \mathcal{J} \to \mathcal{C}$ for a robot with at least three revolute joints. Then there exist kinematic singularities $j \in \mathcal{J}$.

Planning the motions of a robot

The methods that we have described can be used for planning the motions of robots:

1. The first problem is to find a parametrized path $c(t) \in \mathcal{C}$ starting at the initial hand configuration and ending at the desired hand configuration.
2. The second problem is to find a corresponding path $j(t) \in \mathcal{J}$ such that $f(j(t)) = c(t)$ for all t.

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There might be further restrictions:

- If $c(t)$ starts and ends at the same point, then $j(t)$ should start and end at the same point. This is important for repetitive tasks, so that the same motion can be repeated.
- One would like to limit the joint speeds. Fast and rough movements can damage the mechanisms.
- One would like to do as little joint movement as possible.

Planning the motions of a robot

Kinematic singularities are important for motion planning. Assume we have a path $c(t)$ in the configuration space and the corresponding path $j(t)$ in the joint space, i.e. $f(j(t)) = c(t)$. The multivariable chain rule gives

$$c'(t) = J_f(j(t)) \cdot j'(t).$$

Then $c'(t)$ can be interpreted as the velocity of the configuration space path and $j'(t)$ as the corresponding joint space velocity.

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1. At a kinematic singularity, $c'(t) = J_f(j(t)) \cdot j'(t)$ might not have a smooth solution.
2. Near a kinematic singularity, very large joint space velocity might be needed.

Conclusion

Today:

- Applications of algebraic geometry in robotics
- Forward kinematic problem
- Inverse kinematic problem
- Kinematic singularities
- Motion planning

Next time: Numerical algebraic geometry and homotopy continuation