Lecture 11

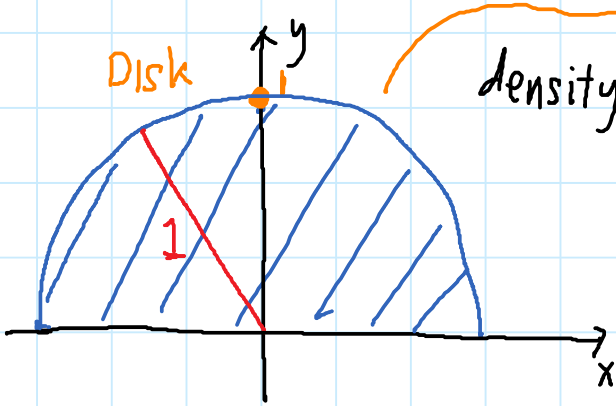
Topics: Center of mass, surface area, triple integrals. Cylindrical coordinates.

- Did the example of finding the center of mass of the top half of a disk of constant density. To reduce calculations we argued by symmetry that the x-coordinate of the center of mass is zero.
- Derived the formula for the surface area of parametric surfaces and then the special case when the surface is the graph of a function $f(x, y)$. We only do examples in this special case. Almost all of the work need was already done earlier in the course when we derived the formula for the tangent plane (see also the change of variable in double integrals). We cut the surfaces into small tiles by slicing in the x and y directions. Each tile was then approximated by a parallelogram shaped piece of the tangent plane. This is completely analogous to how we found arc length by using small pieces of the tangent line. The vectors that make the edges of the parallelogram were found by parametrizing the curves and differentiating. The area of the parallelogram is the magnitude of the cross product of these vectors. From earlier in the course we compute the cross product to be $\langle -f_x, -f_y, 1\rangle$. So the magnitude is $\sqrt{f_x^2 + f_y^2 + 1}$. The surface area is the double integral of this quantity.
- Computed the surface area of a cone. Note that this can be done without calculus.
- Gave the definition of the triple integral. Stated Fubini's theorem which enables us to compute a triple integral by computing an iterated integral. For double integrals there are two possible orders of integratino, dx dy and dy dx. There are 6 possible orders for a triple integral. dz dy dx, dz dx dy, dy dx dz and so on.
- Did an example of finding all 6 orders of integration for a non-rectangular region (although we only wrote down 3 in class). The region (in 3D space) of integration was given by$0 \le x \le 2, -1 \le y \le 1$ and $0 \le z \le y^2$. Drawing this and finding the projections onto each of the coordinate planes xy, yz and xz is where most of the work is.
- Introduced cylindrical coordinates (r, θ, z). This is realtively easy as it is just polar coords with z added. In polar coords we have $\mathrm{d}A = r dr\, \mathrm{d}\theta$. Now we have $\mathrm{dV} = r dz dr\, \mathrm{d}\theta$. Sketched the simplest surfaces in this coordinate system. For example z= constant, r = constant, θ = const
- Did a volume computation example in cylindrical coords.

Where to find this material

- Adams and Essex 14.5, 14.6, 14.7
- Corral, 3.3, 3.6, 4.4 (pages 156, 157)
- Guichard, 15.3 - 15.6
- Active Calculus.11.4, 11.6, 11.7, 11.8

Example of center of mass



density = constant
$= \rho(x,y)$
$= 1$

We need to compute M, M_x, M_y

M = mass = Area × density (as density is constant)

$= \frac{1}{2}\pi(1)^2 \cdot 1 = \frac{\pi}{2}$

$\bar{x} = 0$

Because the density and region are both symmetrical about the y-axis

$\bar{y} = < \frac{1}{2}$ Guess ✓

$$M_x = \iint_R y\,\rho(x,y)\,dA = \iint_R y\,dA$$

Cartesian

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} y^2 \Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{-1}^{1} 1 - x^2\,dx = \ldots$$

$= 2/3$

polar

$$= \int_0^{\pi}\int_0^1 \overbrace{r\sin\theta}^{=y}\ \overbrace{r\,dr\,d\theta}^{dA}$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta\ r^3\Big|_0^1 d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta\,d\theta$$

$= 2/3$

So $\bar{y} = \frac{M_x}{M} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}$ ≈ 0.42

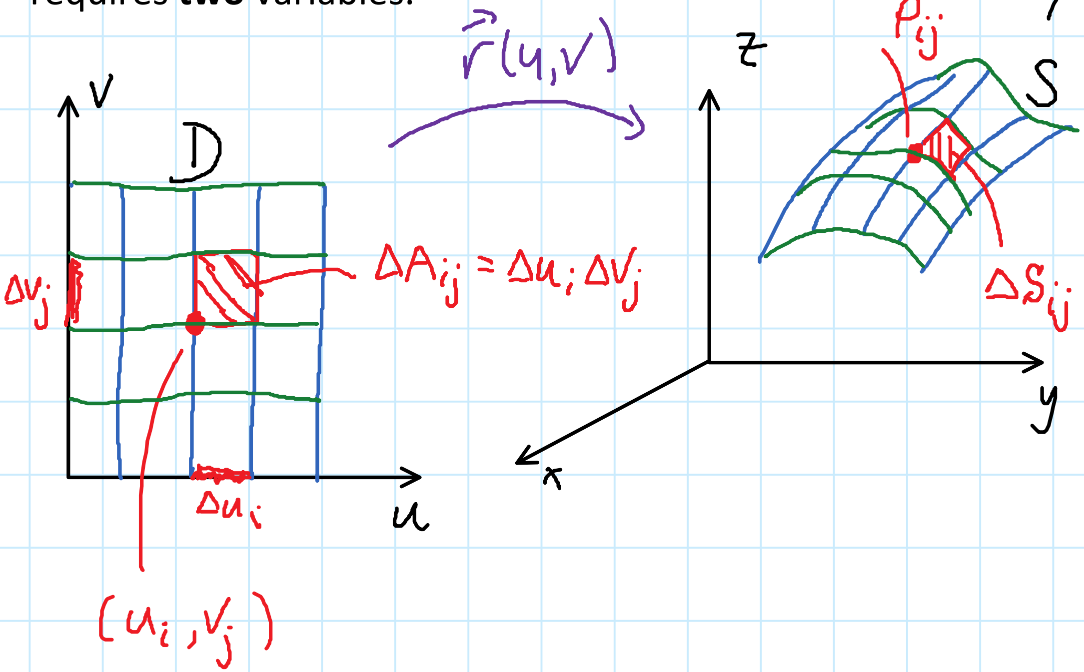
Center of mass $\left(0, \frac{4}{3\pi}\right)$

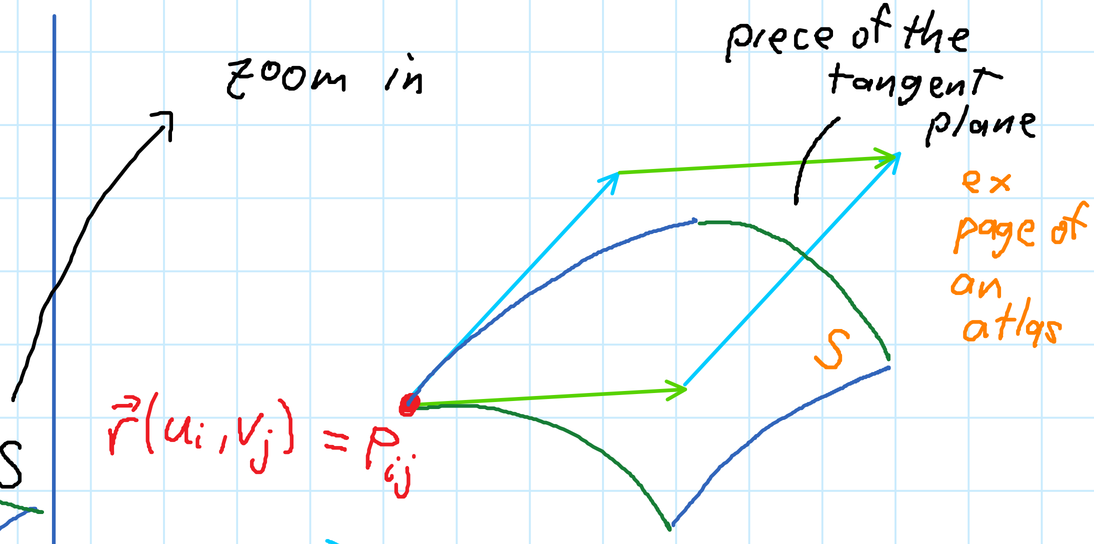
Surface area

A **curve** is **one**-dimensional object. It's parametrization requires **one** variable.

ex: Earth, use latitude and longitude

A **surface** is **two**-dimensional object. It's parametrization requires **two** variables.





$$= \frac{\partial \vec{r}}{\partial v}(u_i,v_j)\,\Delta v_j$$

$$= \frac{\partial \vec{r}}{\partial u}(u_i,v_j)\,\Delta u_i$$

See lecture 2 and lecture 10

Area of the small piece of the tangent plane $\approx$ Area of the small piece of the surface $= \Delta S_{ij}$

$=$ Area of the parallelogram

$= \| \nearrow \times \rightarrow \|$

$$= \left\| \frac{\partial \vec{r}}{\partial u}(u_i,v_j) \times \frac{\partial \vec{r}}{\partial v}(u_i,v_j) \right\| \Delta u_i \Delta v_j$$

Aim: Find a formula for the surface area.

Surface area (2)

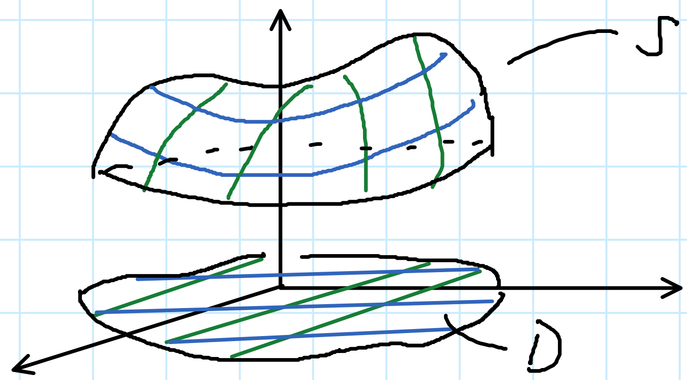
$$\Delta S_{ij} = \left\| \frac{\partial \vec{r}}{\partial u}(u_i, v_j) \times \frac{\partial \vec{r}}{\partial v}(u_i, v_j) \right\| \Delta u_i \Delta v_j$$

Summing these up, taking the limit as $\Delta u \to 0$ and $\Delta v \to 0$, and recalling the definition of the double integral, we get

Surface area of a surface S given parametrically by $\vec{r}(u,v)$ on a domain D

$$= \iint_D \left\| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right\| \underset{\substack{\| \\ du\,dv}}{dA}$$

Special case: The surface is the graph of a function $f(x,y)$



We can use (x,y) as the parameters.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

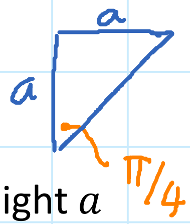
$$\frac{\partial \vec{r}}{\partial x} = \left\langle 1, 0, \frac{\partial f}{\partial x} \right\rangle, \quad \frac{\partial \vec{r}}{\partial y} = \left\langle 0, 1, \frac{\partial f}{\partial y} \right\rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle \quad (= \text{a normal vector})$$

$$\| \vec{r}_x \times \vec{r}_y \| = \sqrt{f_x^2 + f_y^2 + 1}$$

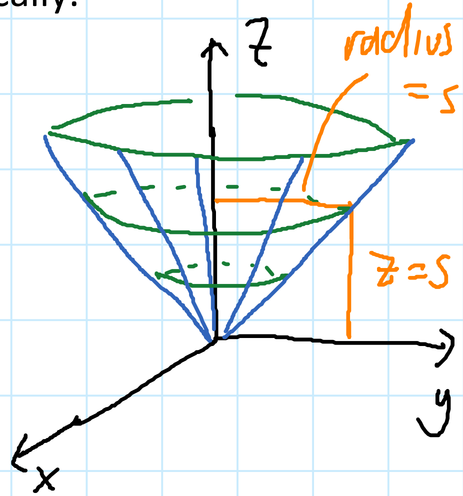
$$\text{Surface Area} = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \underset{\substack{\| \\ dx\,dy}}{dA}$$

Surface area example



Find the surface area of a right-angled cone of height a

Method A: Cone given parametrically.



$$\vec{r}(s,\theta) = \langle \underset{x}{\underset{\parallel}{s\cos\theta}}, \underset{y}{\underset{\parallel}{s\sin\theta}}, \underset{z}{\underset{\parallel}{s}} \rangle$$

$$x^2 + y^2 = z^2$$

$$\frac{\partial \vec{r}}{\partial s} = \langle \cos\theta, \sin\theta, 1 \rangle$$

$$\frac{\partial r}{\partial \theta} = \langle -s\sin\theta, s\cos\theta, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -s\sin\theta & s\cos\theta & 0 \end{vmatrix}$$

$$= \langle -s\cos\theta, -s\sin\theta, s(\cos^2\theta + \sin^2\theta) \rangle$$

$$= s\langle -\cos\theta, -\sin\theta, 1 \rangle$$

$$\| \text{''} \quad \text{''} \| = s\sqrt{\cos^2\theta + \sin^2\theta + 1}$$

$$= \sqrt{2}\, s$$

$$\text{Surface area} = \iint_D \sqrt{2}\, s \, dA$$

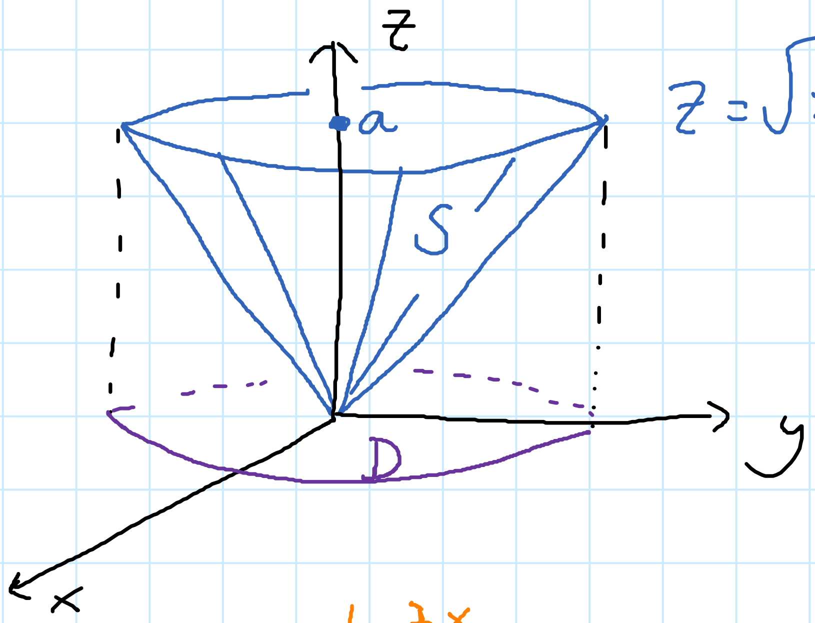
$$= \int_0^{2\pi} \int_0^a \sqrt{2}\, s \, ds \, d\theta$$

$$= \sqrt{2} \cdot 2\pi \left. \frac{s^2}{2} \right|_0^a$$

$$= \sqrt{2}\, \pi a^2$$

Surface area example (2)

Method B: Cone give as the graph of a function.



$$z = \sqrt{x^2+y^2}$$

$$= \tfrac{1}{2} \cdot 2x$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

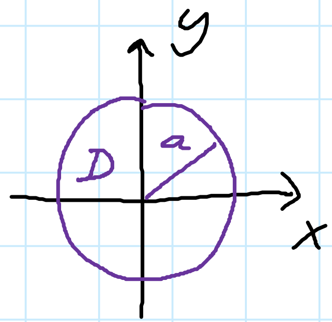
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1$$

$$= 2$$

Surface area

$$= \iint_D \sqrt{z_x^2 + z_y^2 + 1} \, dA$$

$$= \iint_D \sqrt{2} \, dA$$



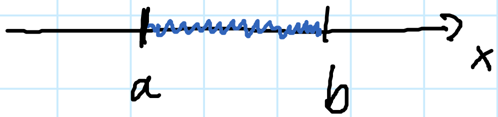
$$= \sqrt{2} \iint_D 1 \, dA$$

$$= \sqrt{2} \text{ Area of } D$$

$$= \sqrt{2}\, \pi a^2$$ as before :)

Triple Integrals

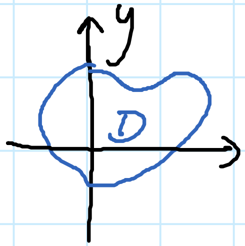
$$\int_a^b f(x)\,dx \qquad \Delta x = \vdash\!\dashv$$



$$\iint_D f(x,y)\,dA$$

$$\iint$$

$$\Delta A = \square = \Delta x \Delta y$$

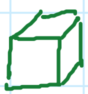


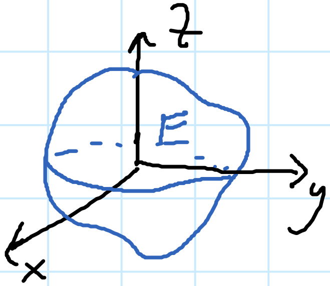
$$\iiint_E f(x,y,z)\,dV$$

$$\iiint$$

$$\Delta V = \Delta x \Delta y \Delta z$$

=





E = solid 3D region

The triple integral is defined in terms of the limit of Riemann sums just as for the single and double integral (no new ideas). See the textbooks for details.

Interpretations:

① $\iiint_E 1\,dV$ = volume of E

② $\iiint_E \overbrace{\rho(x,y,z)}^{=\text{mass density}}\,dV$ = mass of E

③ $$\oiint_{\partial\Omega} \underbrace{\mathbf{E}}_{\text{Electric Field}} \cdot d\mathbf{S} = 4\pi \iiint_\Omega \underbrace{\rho}_{\text{charge density}}\, dV$$ Gauss's Law

Fubini's Theorem.

E = Rectangular Box $= [a,b]\times[c,d]\times[e,f]$

$g(x,y,z)$ is a continuous function defined on E.

$$\iiint_E g(x,y,z)\,dV = \int_a^b\int_c^d\int_e^f g(x,y,z)\,dz\,dy\,dx$$

= and $\underline{\ 5\ }$ other orders

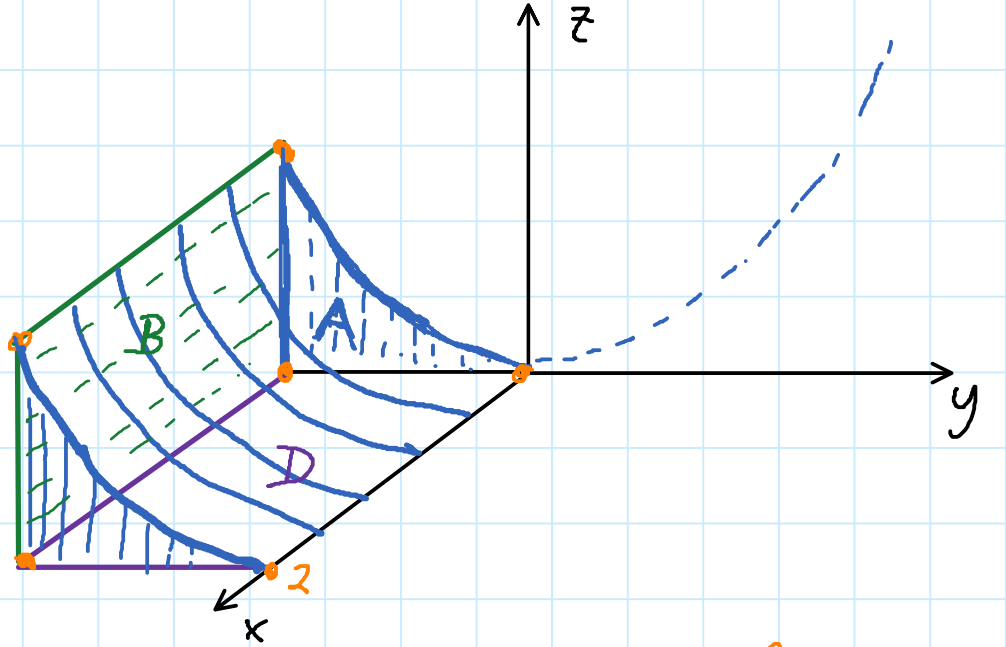
Triple integral example

Write $\int_0^2\int_{-1}^0\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx$ in the 5 other orders.

(We will just do 2 in lecture

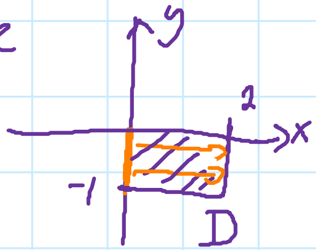
① $dz\,dx\,dy$

② $dx\,dy\,dz$

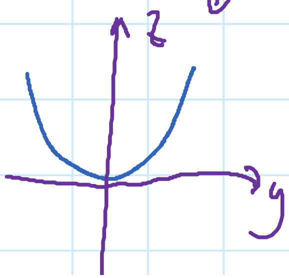


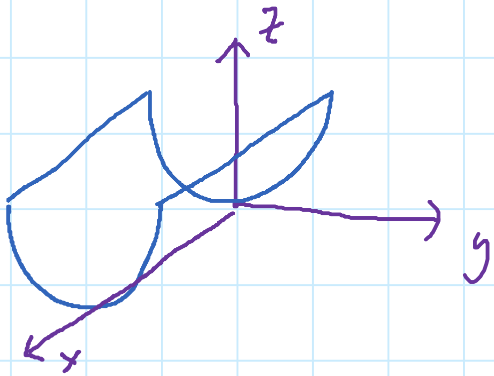
Solution. First we sketch the region and the projections onto the coordinate planes.

$\left.\begin{array}{l} 0 \le x \le 2 \\ -1 \le y \le 0 \end{array}\right\}$ rectangle



$0 \le z \le y^2$





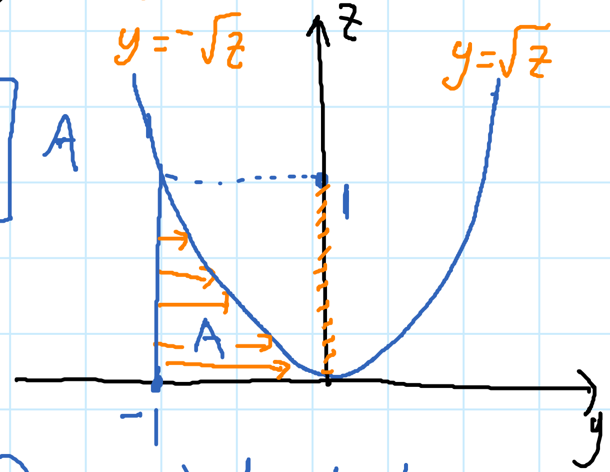
① $\left.\begin{array}{l} -1 \le y \le 0 \\ 0 \le x \le 2 \end{array}\right]$ D $\qquad \int_{-1}^{0}\int_0^2\int_0^{y^2} f\,dz\,dx\,dy$

$0 \le z \le y^2$

$\left(z = y^2 \Rightarrow y = \pm\sqrt{z}\right)$

② $\left.\begin{array}{l} 0 \le z \le 1 \\ -1 \le y \le -\sqrt{z} \end{array}\right]$ A

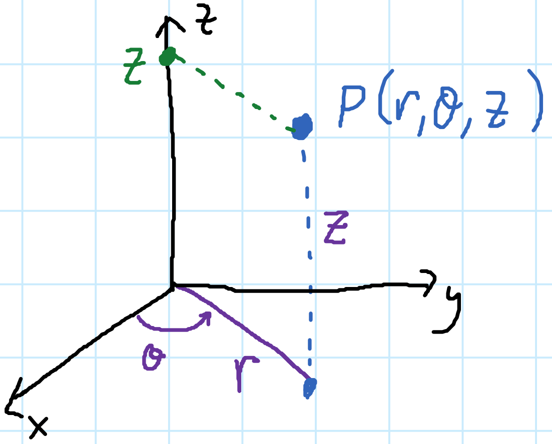
$0 \le x \le 2$



$$\int_0^1\int_{-1}^{-\sqrt{z}}\int_0^2 f(x,y,z)\,dx\,dy\,dz$$

Cylindrical coordinates (r, θ, z)

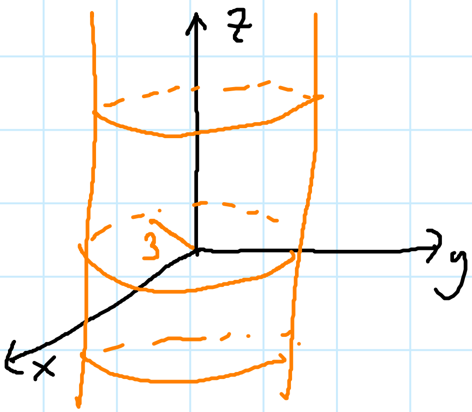
Cylindrical coords = "polar + z"



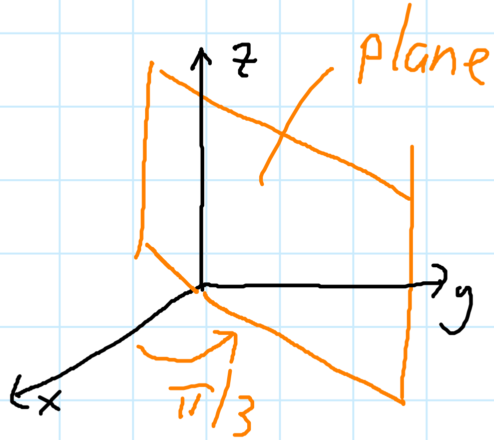
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Some surfaces in cylindrical coords

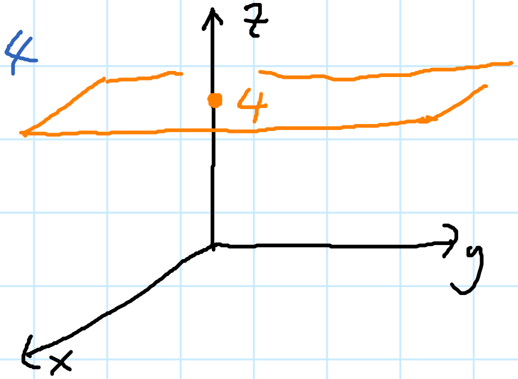
① $r = 3$ (θ, z are free)



② $\theta = \pi/3$



③ $z = 4$

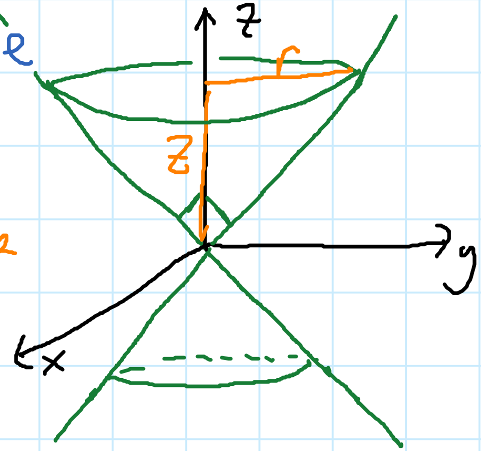


④ Right-angled cone

What is the equation in cylindrical coords

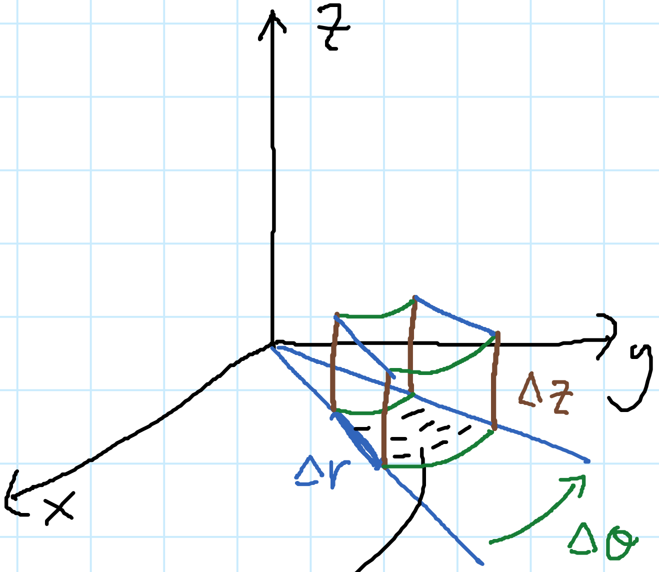
Cartesian: $z^2 = x^2 + y^2$

$$\boxed{z = r}$$



Integration in cylindrical

What is the volume of a small "box" determined by $\Delta r, \Delta\theta, \Delta z$



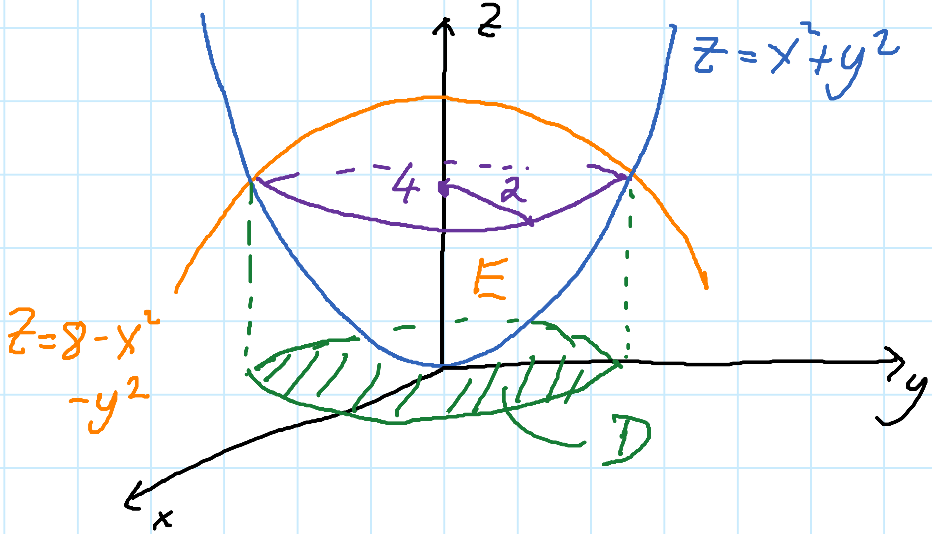
$\Delta A = r\Delta r\Delta\theta$ (polar)

$\Delta V = \Delta A \, \Delta z$

$\approx r\Delta z\Delta r\Delta\theta$

"$dV = r\,dz\,dr\,d\theta$"

Example Find the volume bounded by $z = x^2+y^2$ and $z = 8-x^2-y^2$



In cylindrical coords: $z = r^2$, $z = 8-r^2$

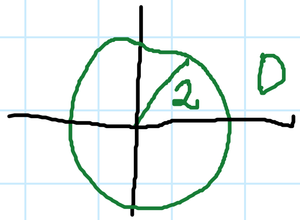
Curve of intersection:

$z = r^2$, $z = 8-r^2$

Intersection: $r^2 = 8-r^2 \Rightarrow r^2 = 4$

$\Rightarrow r = 2$

when $r = 2$, $z = 4$



D $0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

Cylindrical coord example (2)

$$\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} 1\ \overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$$= \int_0^{2\pi}\int_0^2 z\Big|_{r^2}^{8-r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \underbrace{(8-r^2)}_{\text{TOP}} - \underbrace{(r^2)}_{\text{volume}}\ \underbrace{r\,dr\,d\theta}_{dA}$$

$$\left(= \iint_D (\text{TOP} - \text{BOTTOM})\, dA \right)$$

$$= 2\pi \int_0^2 8r - 2r^3\, dr$$

$$= 2\pi \left(4r^2 - \tfrac{1}{2}r^4\right)\Big|_0^2$$

$$= 2\pi(16 - 8) = 16\pi \text{ units}^3$$