Distributions and tunneling

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I. SHORTCUTS TO FERMI-DIRAC AND BOSE-EINSTEIN DISTRIBUTION

A. Fermi-Dirac distribution

Assume the single-particle state ϵ is a grand canonical ensemble. For this the grand partition function based on Pauli exclusion principle reads

$$\begin{aligned}\mathcal{Z} &= e^{-\beta\times 0} + e^{-\beta(\epsilon-\mu)} \\ &= 1 + e^{-\beta(\epsilon-\mu)},\end{aligned} \tag{1}$$

where $\beta \equiv 1/(k_{\rm B}T)$ and μ is the chemical potential of the system. The grand partition function here is $\mathcal{Z} = e^{-\beta\Phi}$, where Φ is called the grand potential of the system. In general, the grand potential of a system is given by

$$\Phi = \langle E\rangle - TS - \mu\langle N\rangle, \tag{2}$$

where $\langle N\rangle$ is the average occupation in the state ϵ,

$$\langle N\rangle = -\frac{\partial\Phi}{\partial\mu} = \frac{1}{\beta\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = \frac{1}{1+e^{\beta(\epsilon-\mu)}}. \tag{3}$$

Here we identify the population $\langle N\rangle$ with the distribution

$$f(\epsilon) = \frac{1}{1+e^{\beta(\epsilon-\mu)}} \tag{4}$$

i.e. the Fermi-Dirac distribution function.

B. Bose-Einstein distribution

With the same procedure as in the previous subsection, we again consider a single-particle state as a grand canonical system. Unlike in the previous case, here for bosons multiple oocupations are allowed whereby the partition function reads

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{-N\beta(\epsilon-\mu)} = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}, \tag{5}$$

where we summed the geometric series in the second step. Then as before we find the average occupation $\langle N\rangle = \frac{1}{\beta\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$, and we again identify the population $\langle N\rangle$ with the distribution

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1} \tag{6}$$

i.e. the Bose-Einstein distribution function.

II. DENSITY OF STATES

We need some further prerequisites. We deal with a 3D electron gas, since the Fermi wavelength $\lambda_F \ll d$, where d is the smallest dimension of the sample we consider. Typically $\lambda_F < 1$ nm for a metal and dimensions of structures to be presented are $d \sim 100$ nm or larger. In this case the density of states (DOS) is proportional to $\sqrt{\epsilon}$. The Fermi temperature $E_F/k_{\rm B} \sim 10^5$ K for typical metals. This means that both the operating temperature of $T \leq 1$ K,

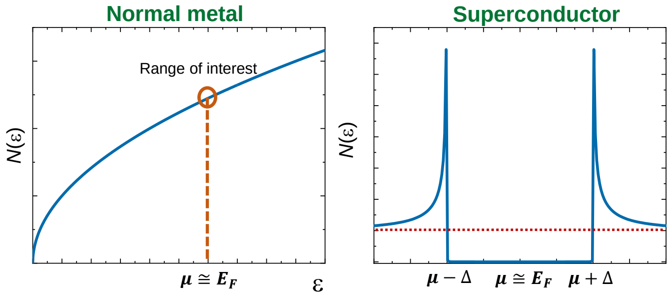

FIG. 1. DOS (a) for normal metal over wide energy range and (b) for superconductor around the Fermi level. Note: In (b) the dashed line corresponds to DOS in the normal state.

and similarly bias voltages $V \sim 1$ mV with $eV/k_{\rm B} \simeq 10$ K are in the range of $\ll E_F/k_{\rm B}$. Therefore we can safely approximate the density of states in the normal state as constant $N(E_F)$ over these relevant energies around the Fermi level (see Fig. 1a). For ordinary superconductors we have BCS DOS $N(\epsilon) = n_S(\epsilon)N(E_F)$ where $n_S(\epsilon) = |\epsilon|/\sqrt{\epsilon^2 - \Delta^2}$ for $|\epsilon| > \Delta$ and $n_S(\epsilon) = 0$ otherwise. Here Δ is the energy gap of the superconductor (Fig. 1b). As an example $\Delta/k_{\rm B} \sim 2$ K for Al.

III. PHENOMENOLOGICAL DERIVATION OF TUNNELING CURRENT

Consider two conductors which are separated by an isulator. This forms the tunnel junction as shown in Fig. 2a. Quantum mechanically, electrons can tunnel through a potential barrier formed by the insulating layer from one electrode to the other, resulting in current flow through the junction. The forward and backward tunneling rates, Γ_f and Γ_b, are given by

$$\Gamma_f = |\mathcal{T}|^2 \int d\epsilon \; n_L(\epsilon) f_L(\epsilon) n_R(\epsilon + eV)[1 - f_R(\epsilon + eV)]$$
$$\Gamma_b = |\mathcal{T}|^2 \int d\epsilon \; n_R(\epsilon + eV) f_R(\epsilon + eV) n_L(\epsilon)[1 - f_L(\epsilon)], \tag{7}$$

where $|\mathcal{T}|^2$ is the "transparency" of the junction and subscript L/R refers to left/right barrier. Here we assume that the bias voltage and temperature are so low as compared to the barrier height (~ 2 eV) that $|\mathcal{T}|^2$ can be assumed constant (and taken out from the integral) for electrons at all relevant states. The two electrodes are shifted with respect to each other by eV, the chemical potential difference, where V is the voltage applied across the junction. The net electrical current through the junction is

$$I = e(\Gamma_f - \Gamma_b). \tag{8}$$

At the end, one obains

$$I = \frac{1}{eR_T} \int d\epsilon \; n_L(\epsilon) n_R(\epsilon + eV)[f_L(\epsilon) - f_R(\epsilon + eV)], \tag{9}$$

where R_T is the resistance of the tunnel barrier.

Next, we specify to an important setup for thermometry, a normal metal-insulator- superconductor NIS tunnel junction, as shown in Fig. 2a. In this case $n_L(\epsilon) = n_S(\epsilon)$ and $n_R(\epsilon) = 1$. We have then

$$I = \frac{1}{eR_T} \int d\epsilon \; n_S(\epsilon)[f_S(\epsilon) - f_N(\epsilon + eV)]. \tag{10}$$

Due to electron-hole symmetry the $I(V)$ is odd, $I(-V) = -I(V)$, and we obtain a symmetric form

$$I = \frac{1}{2eR_T} \int d\epsilon \; n_S(\epsilon)[f_N(\epsilon - eV) - f_N(\epsilon + eV)]. \tag{11}$$

This equation has an important message: the $I - V$ curve depends on the distribution, i.e. temperature in N, but not at all on that in the superconductor.

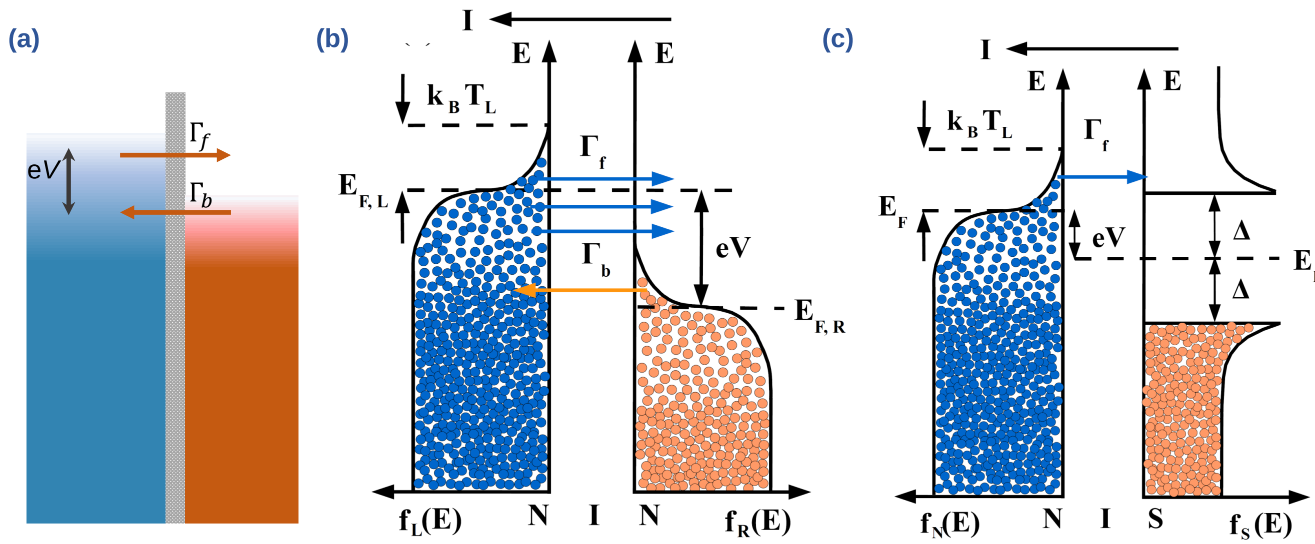

FIG. 2. Scheme of tunneling. (a) Generic with two electrodes seperated by an insulating barrier. (b) Normal to normal tunneling. (c) Normal to superconductor tunneling.

IV. HEAT CURRENT IN TUNNELING

A. Phenomenological derivation

Like for the charge current we can present the two contributions of heat current $L \to R$ and $R \to L$ phenomenologically with the same assumptions as above. In particular, we replace the carried charge e by the carried energy which is $\epsilon - eV$ for the left electrode. Thus the two contributions read

$$\dot{Q}_{L\to R} = |\mathcal{T}|^2 \int d\epsilon(\epsilon - eV)\ n_L(\epsilon - eV) n_R(\epsilon) f_L(\epsilon - eV)[1 - f_R(\epsilon)] \tag{12}$$

$$\dot{Q}_{R\to L} = |\mathcal{T}|^2 \int d\epsilon(\epsilon - eV)\ n_L(\epsilon - eV) n_R(\epsilon) f_R(\epsilon)[1 - f_L(\epsilon - eV)]. \tag{13}$$

Then the net heat current out from L electrode is given by

$$\begin{aligned}\dot{Q}_L &= \dot{Q}_{L\to R} - \dot{Q}_{R\to L} \\ &= \frac{1}{e^2 R_T} \int d\epsilon(\epsilon - eV)\ n_L(\epsilon - eV) n_R(\epsilon)[f_L(\epsilon - eV) - f_R(\epsilon)].\end{aligned} \tag{14}$$

[1] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1996 (2nd edition).
[2] Francesco Giazotto, Tero T. Heikkilä, Arttu Luukanen, Alexander M. Savin, and Jukka P. Pekola, Opportunities for mesoscopics in thermometry and refrigeration: Physics and applications, Rev. Mod. Phys. **78**, 217 (2006).