Lecture 12

Spherical coordinates

- Defined spherical coordinates. Noted that θ and ϕ correspond to longitude and latitude on the surface of the earth. Also discussed that any point in $\mathbb{R}^3$ can be specified with $\rho > 0, 0 \le \theta < 2\pi, 0 \le \emptyset \le \pi$.

- Sketched some surface in spherical coordinates. For example, ρ=constant, θ= constant, and ϕ= constant.

- Used geometry to figure out that $\Delta V \approx \rho^2 \sin(\phi)\Delta\rho\Delta\phi\Delta\,\theta$ Hpefully this agree with what you obtained in homework using the Jacobian.

- Found the volume: (1) The right circular cone (noting that the integral looks very different than above when we used cylindrical coords). (2) The "cap" of a sphere using a triple integral in spherical coordinates.

- In both these example we converted the plane $z = c$ to spherical coords, obtaining $\rho = c/\cos(\phi)$.

- Go over the midterm solutons if time.

Where to find this material

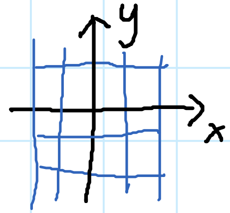
- Adams and Essex 14.6
- Corral, 3.5 (mostly about general changes of variables)
- Guichard, 15.6
- Active Calculus.11.8

Definition of sphereical coordinates

So far we have learned about the following coordinate systems

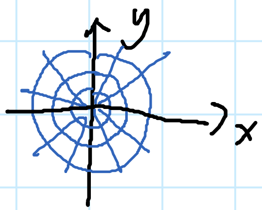
$\mathbb{R}^2$

Cartesian



(x,y)

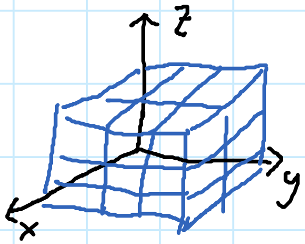
Polar



(r,θ)

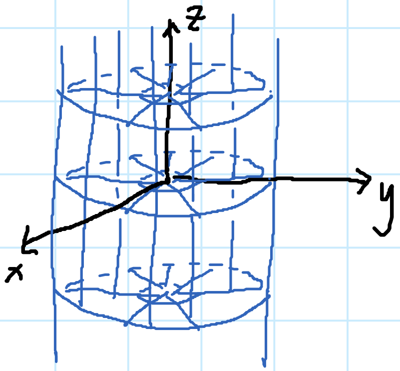
$\mathbb{R}^3$

Cartesian

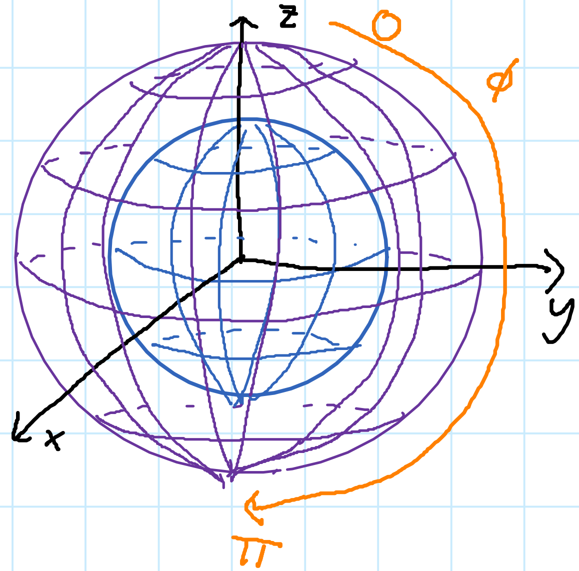


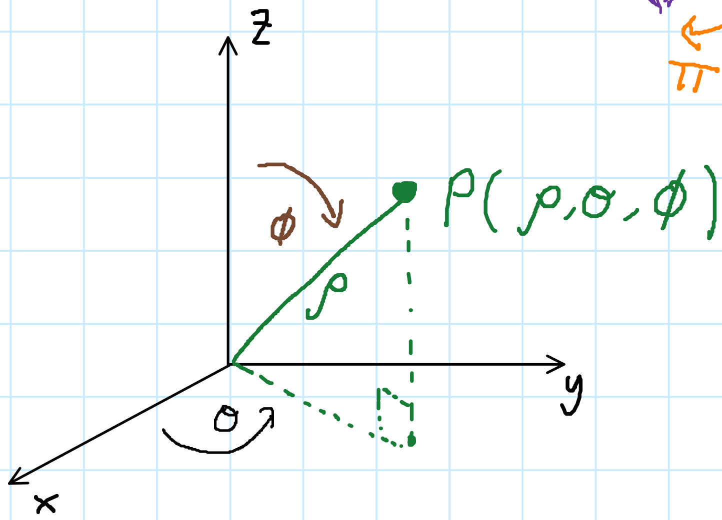
(x,y,z)

Cylindrical



SPHERICAL





θ = (usual) angle from the positive x-axis

$0 \le \theta < 2\pi$ Longitude

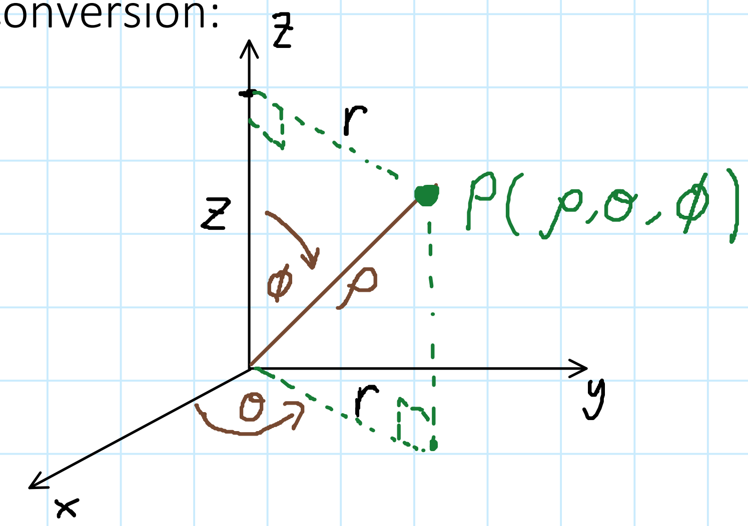
ϕ = angle from the positive z-axis

$0 \le \phi \le \pi$ Latitude

ρ = distance from the origin

$\rho \ge 0$

Conversion:



$$\cos\phi = \frac{z}{\rho} \Rightarrow \boxed{z = \rho\cos\phi}$$

$$\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho\sin\phi$$

From polar we know $x = r\cos\theta$

$y = r\sin\theta$

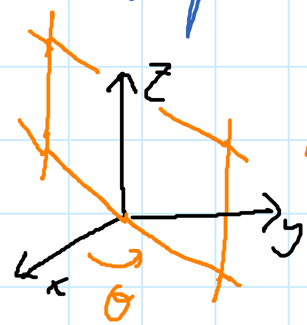
Then,

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \end{aligned}}$$

Also: $x^2 + y^2 + z^2 = \rho^2$

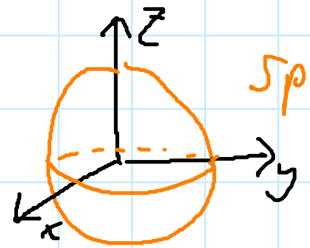
Some simple surfaces in spherical coords

① $\theta = \text{costant}$



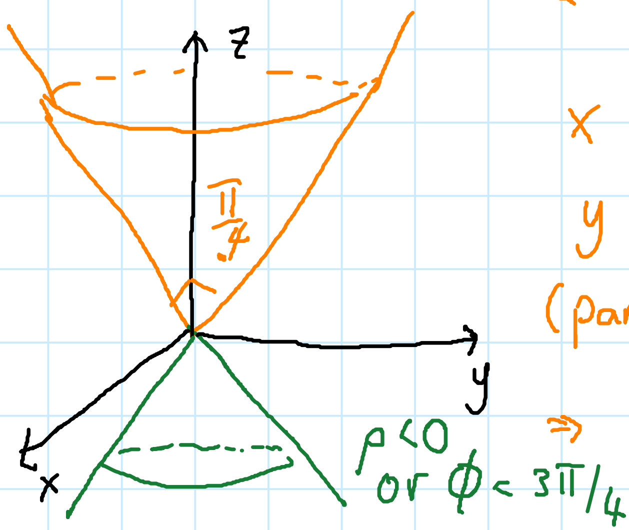
plane

② $\rho = \text{costant}$



sphere

③ $\phi = \pi/4$



$z = \rho/\sqrt{2}$

$x = (\rho/\sqrt{2})\cos\theta$

$y = (\rho/\sqrt{2})\sin\theta$

(parametric equation)

$\Rightarrow x^2 + y^2 = z^2$

Integration in spherical coords

We need to find what ΔV is in term of $\Delta\rho, \Delta\phi, \Delta\theta$

Method 1 - Using the general change of variable formula involving the Jacobian
(this is mathematically rigorous but there is no intuition for the formula)

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$F\begin{bmatrix}\rho\\ \phi\\ \theta\end{bmatrix} = \begin{bmatrix}\rho\sin\phi\cos\theta\\ \rho\sin\phi\sin\theta\\ \rho\cos\phi\end{bmatrix}$$

$J_F = $

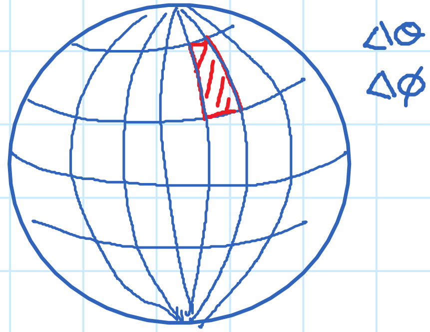
$\det(J_F) = \rho^2\sin\phi$

Homework #5

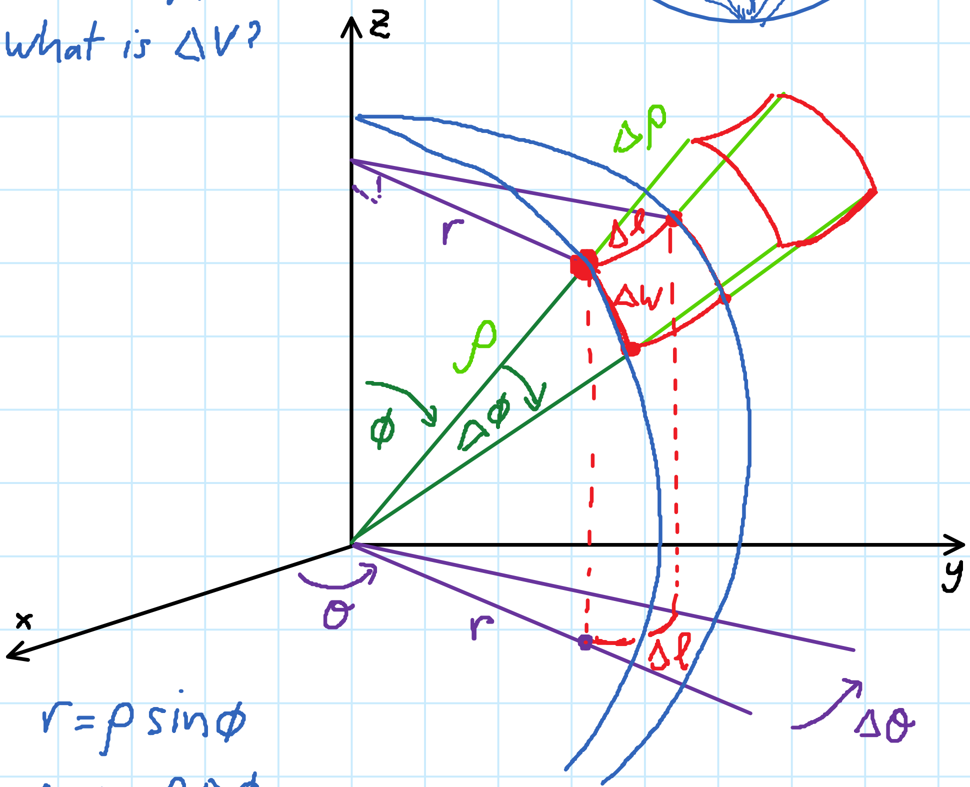
Length^2 Length dimensionless

So $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Method 2 - Geometrically (this is only an approximation and we do not determine the error, but it gives some intuition)



Given $\Delta\rho, \Delta\phi, \Delta\theta$
what is ΔV?



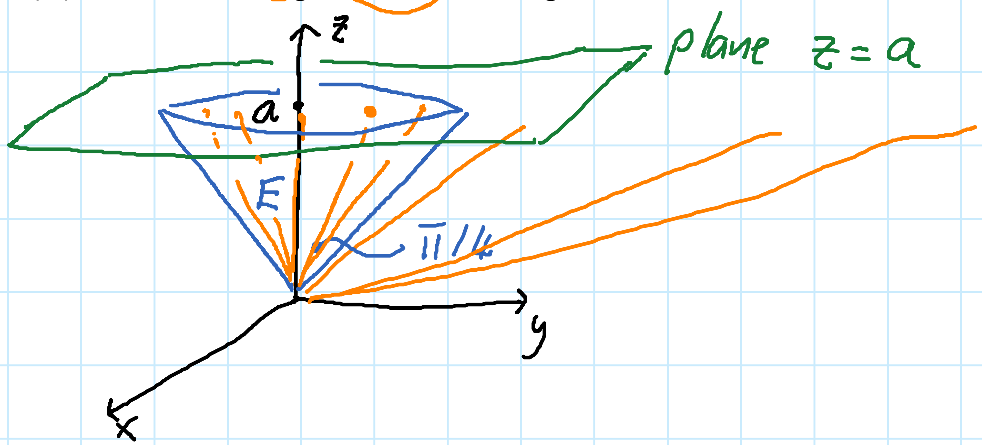
$r = \rho\sin\phi$
$\Delta w = \rho\Delta\phi$
$\Delta\ell = r\Delta\theta$

$$\Delta V \approx \Delta\rho\,\Delta w\,\Delta\ell = \Delta\rho(\rho\Delta\phi)(\rho\sin\phi\,\Delta\theta)$$
$$= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

Examples

(1) Volume of right cone of height a

angle $= \pi/2$



Describe the solid cone E in spherical coords

$0 \leq \theta < 2\pi$

$0 \leq \phi \leq \frac{\pi}{4}$

$0 \leq \rho \leq$ plane

Equation in spherical coords

$z = \rho \cos\phi$

$z = a$

$\Rightarrow a = \rho\cos\phi$

$\Rightarrow \rho = a/\cos\phi$

$$\text{Volume} = \iiint_E 1 \, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{a/\cos\phi} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3 \Big|_0^{a/\cos\phi} \sin\phi \, d\phi$$

$$= \frac{2\pi a^3}{3}\int_0^{\frac{\pi}{4}} \frac{\sin\phi}{\cos^3\phi} \, d\phi$$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$$= \frac{2\pi a^3}{3}\left(\frac{1}{2}\right)\cos^{-2}\phi \Big|_0^{\pi/4}$$

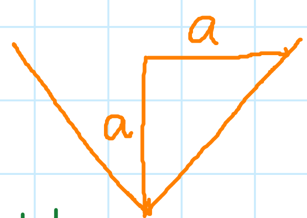
$$= \frac{1}{3}\pi a^3 \left(\sqrt{2}^2 - 1\right)$$

$$= \frac{1}{3}\pi a^3$$

$a^3 = a^2 \cdot a$

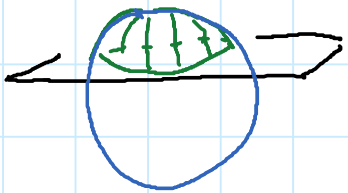
a^2 = radius, a = height

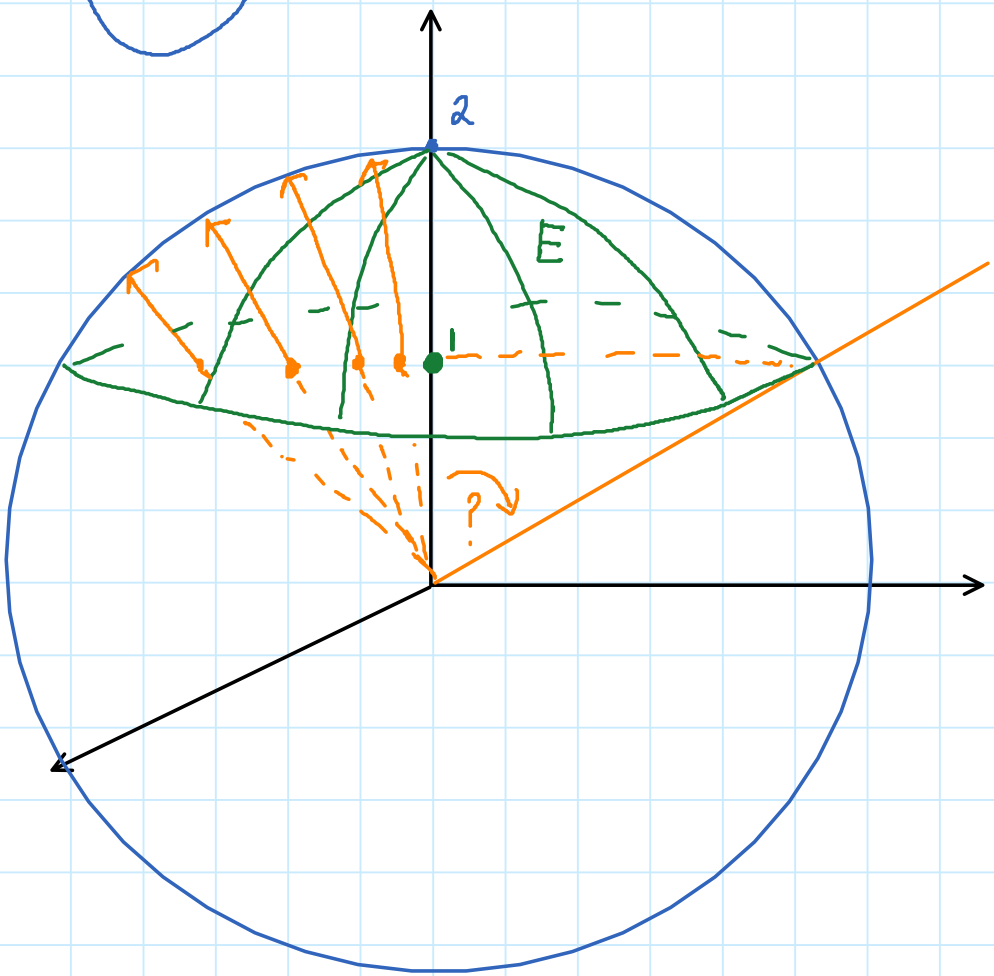
$= \frac{1}{3}$ cylinder



Examples (2)

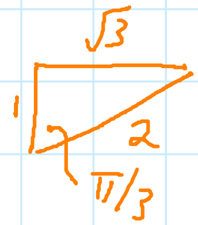
(2) Find the volume of the cap of a sphere of radius 2 cut off by a plane of distance 1 from the center.





E: $0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/3$



$$\text{plane} \le \rho \le \text{sphere}$$

$$\underbrace{z=1}_{1/\cos\phi} \qquad \underbrace{\text{sphere}}_{2}$$

$$\frac{1}{\cos\phi} \le \rho \le 2$$

$$\text{Volume} = \iiint_E 1\, dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{1/\cos\phi}^{2} 1\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \frac{1}{3} \rho^3 \Big|_{1/\cos\phi}^{2} \sin\phi\, d\phi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{3}} 8\sin\phi - \frac{\sin\phi}{\cos^3\phi}\, d\phi$$

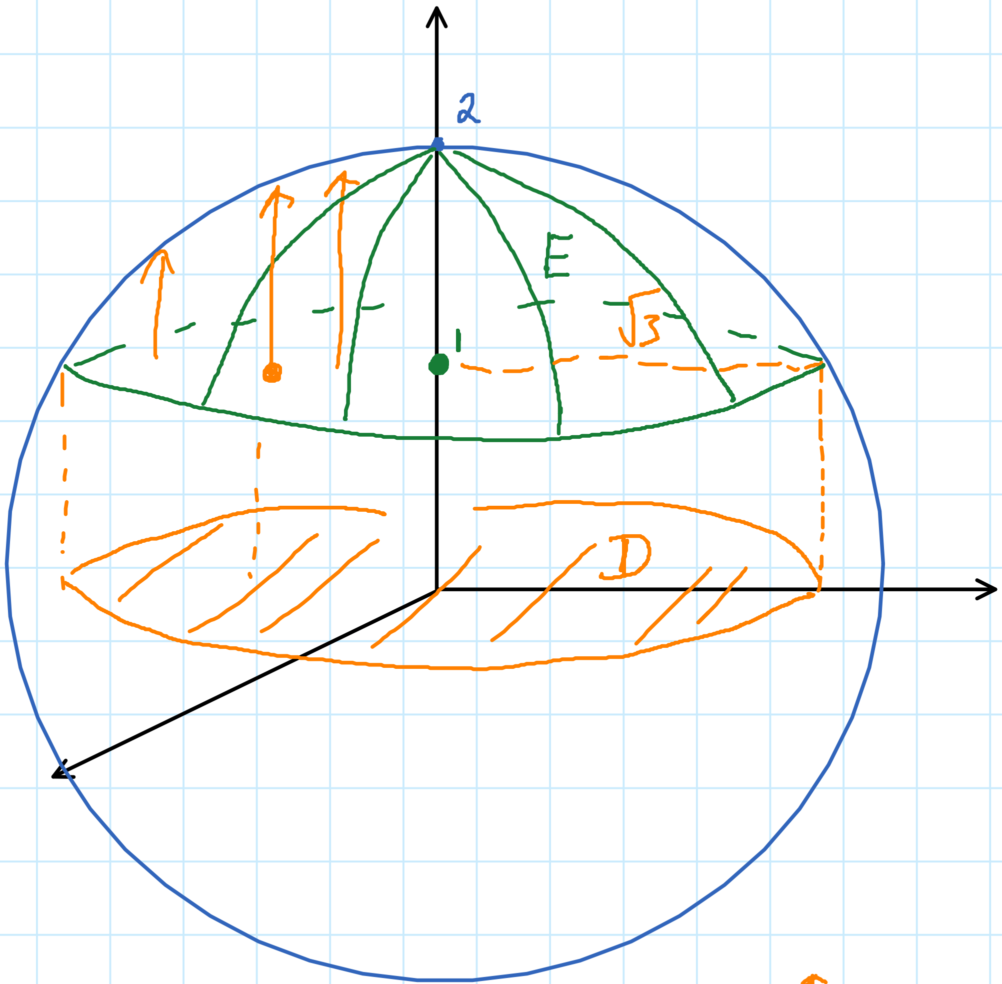
$$= \frac{2\pi}{3} \left(-8\cos\phi - \frac{\cos^{-2}\phi}{2} \right) \Big|_0^{\pi/3}$$

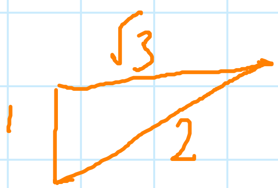
$$= \frac{2\pi}{3} \left(\underbrace{-8\left(\tfrac{1}{2} - 1\right)}_{=4} - \underbrace{\tfrac{1}{2}\left(4 - 1\right)}_{3/2} \right)$$

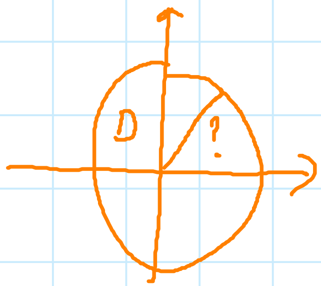
$$= 5\pi/3$$

Example (3)

Redo the previous example in cylindrical coords







E: $0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{3}$] region in the xy plane

plane $\le z \le$ sphere

$z = 1$

$\overbrace{x^2+y^2}^{r^2} + z^2 = 4$

$z = +\sqrt{4-r^2}$

$$\text{Volume} = \iiint 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^{\sqrt{3}} r z \Big|_{z=1}^{z=\sqrt{4-r^2}} dr$$

$$= 2\pi \int_0^{\sqrt{3}} r\sqrt{4-r^2} - r\, dr \qquad \text{sub } u = r^2$$

$$= 2\pi \left(-\frac{1}{2}\left(\frac{2}{3}\right)(4-r^2)^{3/2} - \frac{r^2}{2} \right)\Bigg|_0^{\sqrt{3}}$$

$$= 2\pi \left(-\frac{1}{3}(1-8) - \frac{3}{2} \right)$$

$$7/3 - 3/2 = \frac{14-9}{6}$$

$$= \frac{5\pi}{3} \quad \text{as before :)}$$