Quantum Circuits
Week 1

5/3/2021

1 Key concepts from quantum mechanics

Here we review briefly the basic concepts from quantum theory that are most relevant to this course.

1.1 States, observables, and the Born rule

Throughout this lecture, we assume closed system dynamics. These are the type of problems where the system's state evolve according to the Schrödinger's equation

$$\Big[\ \overbrace{\frac{-\hbar^2}{2m}P^2}^{\text{Kinetic energy}} + \underbrace{V(x)}_{\text{Potential energy}}\ \Big]\,|\psi\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi\rangle$$

A more general formulation of quantum theory defines states as *density operators*, and measurements as generalized *POVMs* (positive operator valued measures). However, in this lecture these topics are not addressed.

1.1.1 States

Every quantum system is associated with a space over the complex numbers $\mathbb{C}$ equipped with an inner product operation. Such a space is called the Hilbert space. All Hilbert spaces are in general infinite dimensional, however we will mostly deal with finite dimensional ones. They have some nice properties such as the existence of a set *basis* vectors and they admit a matrix representation. In the **Schrödinger** picture states are *square integrable* functions. The inner product of any two elements $\in \mathcal{H}$ is

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f(x)} \cdot g(x)\ dx$$

where we have assumed that $f(x)$, and $g(x)$ are functions of the position of a particle and the overbar means the complex conjugate.

The inner product has some properties. Firstly, it defines a state norm $\sqrt{\langle f, f\rangle} = ||f||$, where $0 \leq ||f|| < \infty$, such that equality holds only when $f = 0$. Secondly, it is linear in its second entry[1] $\langle f, \alpha g + \beta k\rangle = \alpha \langle f, g\rangle + \beta \langle f, k\rangle$,whereas anti-linear in the first one $\langle \alpha f + \beta h, g\rangle = \bar{\alpha} \langle f, g\rangle + \bar{\beta} \langle h, g\rangle$ where $\alpha, \beta \in \mathbb{C}$. Finally, it is anti-symmetric $\langle f, g\rangle = \overline{\langle g, f\rangle}$.
Alternatively we may opt for the **Heisenberg** picture to describe our state. In this picture states in a finite dimensional $\mathcal{H}^n$ are represented by column vectors (*kets*), whose entries $\in \mathbb{C}$, $|\psi\rangle = (\alpha_1, \alpha_2, ..., \alpha_2)^\intercal$. The inner product in this case is defined as

$$\langle\psi|\phi\rangle = \sum_n \bar{\psi}_i \phi_i$$

where $\langle\psi|$ is the conjugate transpose (*bra*) of the column vector $|\psi\rangle$.
A finite dimensional Hilbert space always has a set of **linearly independent** elements called *orthonormal* basis (ONB) $\{|i\rangle\}_n$, such that $\langle i|j\rangle = \delta_{ij}$, where $\delta_{ij} = 1$, when $i = j$, and 0 otherwise. Any ONB $\in \mathcal{H}$ resolve the identity operator on $\mathcal{H}$, $\hat{I} = \sum_n |n\rangle \langle n|$, also known as *completeness* relation. Hence any state $|\psi\rangle \in \mathcal{H}$ can be written in terms of these basis elements $|\psi\rangle = \sum_n C_n |n\rangle$, where $C_n = \langle n|\psi\rangle$. Similarly, we can define wavefunctions in the **Schrödinger** picture as $\Psi = \int_{-\infty}^{\infty} \psi(x) |x\rangle \; dx$, where $\psi(x) = \langle x|\psi\rangle$, and provided that $\int_{-\infty}^{\infty} |x\rangle \langle x| = 1$.

1.1.2 Observables

Observables in quantum mechanics are represented by linear operators $L(\mathcal{H})$ on $\mathcal{H}$, such that for $A \in L(\mathcal{H})$, $A(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) = \alpha A |\psi_1\rangle + \beta A |\psi_2\rangle$, where $L(\mathcal{H})$ is the space of all linear operators acting on $\mathcal{H}$, and $\alpha, \beta \in \mathbb{C}$. The conjugate transpose (*adjoint*) $A^\dagger$ of an operator$A$ is defined as $\langle\psi_1|A\psi_2\rangle = \langle\psi_1 A^\dagger|\psi_2\rangle$. Most of the time we are interested in self-adjoint (*Hermitian*) operators $A = A^\dagger$, and unitary operators $A^\dagger = A^{-1}$, such that $AA^\dagger = A^\dagger A = I$. Different ONBs are linked with one another via unitary operators. The spectrum (*possible observed outcomes*) of an observable is either discrete (*such as the allowed energies inside a potential well*) or continuous (*position or momentum of a moving particle*).
In the **Schrödinger picture** the expected value of an observable A with respect to some state function ψ is written as $A_\psi = \int_{-\infty}^{\infty} \psi^\dagger(x) A \psi(x) \, dx$, where, $\psi(x) = \langle x|\psi\rangle$ is the state wavefunction written in position basis $\int_{-\infty}^{\infty} |x\rangle \langle x| \; dx$. Whereas in the **Heisenberg** picture the expected value of an observable A with respect to some ket $|\psi\rangle$ is $\langle\psi|A|\psi\rangle$. Furthermore, the uncertainty of A is defined as $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. The matrix elements of an operator A can be written as

$$\sum_i |i\rangle \langle i| A \sum_j |j\rangle \langle j| = \sum_{i,j} \langle i|A|j\rangle |i\rangle \langle j|$$

where, $\sum_i |i\rangle \langle i| = 1$, $\sum_j |j\rangle \langle j| = 1$, $\langle i|A|j\rangle = A_{ij}$, $|\rangle \langle|$ is called an *outer product*.
The composition of two operators AB is another operator, such that associativity holds when acting on an arbitrary ket $AB |\psi\rangle = A(B |\psi\rangle)$.

[1] This convention is widely used in physics textbooks.

1.1.3 Born Rule

Let us assume that we possess an arbitrary n-dimensional quantum system, its state can be written in terms of a complete ONB as $|\psi\rangle = \sum_n C_n |n\rangle$, where $C_n = \langle n|\psi\rangle$. Suppose now we multiply this expression with $\langle m|$, then by orthogonality and linearity of the inner product map, this operation singles out a specific probability amplitude $C_m = \langle m|\psi\rangle$. The probability of getting this particular outcome is then found to be $|\langle m|\psi\rangle|^2$. In other words, the probability of getting outcome m after projecting (*measuring*) our state on to $\langle m|$ is $|C_m|^2$. Thus the Born rule provides us with a procedure to determine the different probabilities of possible measurement outcomes. Obviously $\sum_m |C_m|^2 = 1$, and the set of *projectors* $\{|m\rangle\}$ constitutes a complete ONB.

2 Hamiltonian, commutators, time evolution,

2.1 Hamiltonian

The Hamiltonian of the system is the Hermitian operator defined as $H = \sum_n E_n |E_n\rangle \langle E_n|$, it entails a complete description of the system's energy spectrum. The energy states of the system are the eigen-states of the Hamiltonian $H|E_n\rangle = E_n |e_n\rangle$. They constitute a complete ONB. By exponentiating the Hamiltonian operator we define a unitary operator $U = e^{-iHt/\hbar}$, then after power expansion of this expression we get $e^{-i(\sum_n E_n|E_n\rangle\langle E_n|)t/\hbar} = \sum_n e^{-i\omega_n t} |n\rangle \langle n|$, where we have used the relation $E = \hbar\omega$.

2.2 Commutator

The commutator of two operators A, and B is defined as $[A, B] = AB - BA$.

2.3 Time evolution

In closed system dynamics, states evolve according to the Schrödinger equation $H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$. Thus the state after time t is written as $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$. On the other hand, operators evolve according to the Heisenberg equation of motion $\frac{dA}{dt} = -i/\hbar[A, H]$.

3 Examples

3.1 Qubits

In the quantum world a qubit is the smallest carrier of information. The space of a single qubit is the 2-dimensional Hilbert space $\mathcal{H}^2$. An example of a basis set in $\mathcal{H}^2$ is the so called computational basis $|0\rangle = (1, 0)^\intercal$, $|1\rangle = (0, 1)^\intercal$. Operators in a qubit space are 2×2 matrices. The Pauli group of matrices is of particular interest to us, especially when we consider different qubit manipulations. It is straight forward to figure out the action of the Pauli matrices on the computational basis, for instance, the Pauli σ_x flips the state of the system $\sigma_x|0\rangle = |1\rangle$, $\sigma_x|1\rangle = |0\rangle$.

Thus by orthogonality of the computational basis, its matrix elements are only off-diagonal $\langle 0|\sigma_x|1\rangle$, and $\langle 1|\sigma_x|0\rangle$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.2 Harmonic oscillator

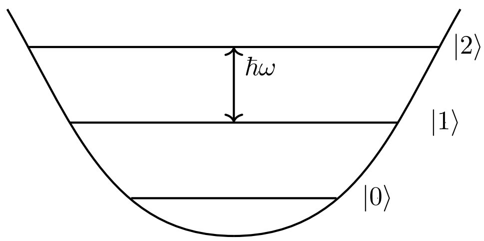

Figure 1

The equally spaced eigen-energies of the harmonic oscillator are a complete ONB, they are called the number states $\{|n\rangle\}_n$. In terms of the ladder operator, the Hamiltonian of a harmonic oscillator can be written as

$$H = \hbar\omega(a^\dagger a + 1/2)$$

where $a = \frac{m\omega}{2\hbar}(x + \frac{i}{m\omega}p)$, $a^\dagger = \frac{m\omega}{2\hbar}(x - \frac{i}{m\omega}p)$, $a^\dagger a = n$ is the number operator, $\hbar$ is the reduced Plank constant, ω of the oscillator's frequency, m is the mass of the oscillator, x is the position operator, and $p = -i\hbar\partial/\partial x$ is the momentum operator.
The position and momentum operators obey a commutation relationship $[x, p] = i\hbar$. Their *complete*, and *normalized* ONB are defined such that $\int_{-\infty}^{\infty} |q\rangle \langle q| = 1$, $\langle q|q'\rangle = \delta(q - q')$, $\int_{-\infty}^{\infty} |p\rangle \langle p| = 1$ $\langle p|p'\rangle = \delta(p - p')$, respectively. Moreover, they can be related via a Fourier transformation $\langle x|p\rangle = \frac{1}{\sqrt{\pi}}e^{ipx}$. Any state of the harmonic oscillator can be written in either the position or the momnetum basis $\Psi(x) = \langle x|\psi\rangle$, $\Psi(p) = \langle p|\psi\rangle$.
Number state are eigenstates of the number operator $\hat{n}\,|n\rangle = n\,|n\rangle$, where n is the number of particles in that particular state. The action of the ladder operators on the number states is defined as $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, and $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Their expectation value with respect to a number state is $\langle n|a|n\rangle = \langle n|a^\dagger|n\rangle = 0$. The ground state of a harmonic oscillator is its lowest possible energy state $a\,|0\rangle = 0$. Any number state can be can be found by successive application of the creation operator on the ground state $|n\rangle = \frac{{a^\dagger}^n}{\sqrt{n!}}\,|0\rangle$.

4 Basic circuit elements and variables

In this lecture we briefly review the fundamental physical concepts required for elementary circuit analysis. Throughout this lecture we adopt the SI system to quantify circuit variables, hence a familiarity with the different orders of magnitude that a circuit variable may vary over is expected.

4.1 Elementary concepts

4.1.1 Electric and displacement currents

Electric charges Q can be either positive or negative, their unit is Coulomb C. For an arbitrary open surface S, the current I through S is defined as

$$I = \frac{dQ}{dt} \;\; C/\text{sec}$$

where C/sec is the Ampere A.
On the other hand, a displacement current I_d is not the result of the physical motion of charged particles. Instead, it occurs due to a variation in the electric field. A good example of this phenomenon is a parallel plate capacitor, where the variation in the density of the charges deposited on the capacitor's plate induces a time-varying electric field. This time-varying electric field is associated with a magnetic field as was established by Maxwell.

4.1.2 Voltage

Voltage V is defined as the work done W on a positive test charge $+Q$ in some electric field E to move it from one point to another.

$$V = \frac{W}{Q} \;\; J/C$$

where J/C is joule (energy units) per Coulomb is one Volt.

4.1.3 Electric power

Electric power P is defined as the rate of energy expenditure (*transfer*).

$$P = \frac{dW}{dt} \;\; J/\text{sec}$$

where J/sec is the Watt W.
Electric power is also related to the voltage and current of a circuit element via the following relation $P = VI$

4.1.4 Magnetic flux

Flux Φ is a quantity associated with inductors (more on inductors and their roles inside a circuit in the upcoming section). The flow of electrons inside a wire creates an associated magnetic field. When the wire is folded into an N-turn coil (inductor), the total flux is defined as

$$\Phi = N\Psi$$

where Ψ is the magnetic flux of one turn.
From Faraday's law, a time varying magnetic flux induces a voltage

$$V = \frac{d\Phi}{dt}$$

4.2 Circuit elements

4.2.1 The resistor

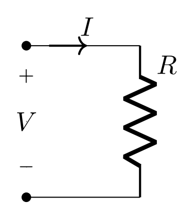

Figure 2

A resistor is an element that impedes the flow of of electric chrges through it. They are responsible for energy dissipation in any circuit by converting electric current into heat. The resistor's value is related to the voltage and the current across it by Ohm's law

$$R = \frac{V}{I}\ \ V/A$$

where V/A is the Ohm Ω
The power dissipated by a resistive element can be calculated using Ohm's law also

$$P = VI = I^2R = \frac{V^2}{R}$$

We can calculate the resistivity of any conductor with a uniform cross-section A via a simple formula

$$R = \frac{\rho L}{A}$$

where ρ is the resistivity, L is the conductor's length, and A is its cross-section area.

4.2.2 The inductor

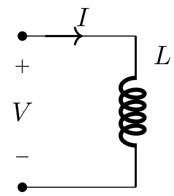

Figure 3

An inductor is a circuit elemnt that stores energy in the magnetic field. It is characterized by its inductance L defined as

$$L = \frac{\Phi}{I} \ \Omega \cdot \text{sec}$$

where $\Omega \cdot \text{sec}$ is the Henry H

The relationship between the voltage and current of an inductor is determined by Faraday's law

$$V = \frac{d\Phi}{dt} = L\frac{dI}{dt}$$

The energy stored or released by an inductor during a time interval can be deduced easily as follows

$$W = \int_{t_1}^{t_2} VI \ dt = \int_{t_1}^{t_2} P \ dt = \frac{1}{2}L(i_2^2 - i_1^2)$$

4.2.3 The capacitor

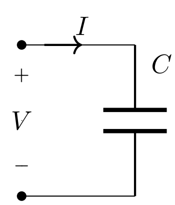

Figure 4

A capacitor is a two parallel plate conductors separated by a a dielectric or insulating material. It stores energy in the electric field. The capacitance unit is Farad F and is defined as

$$C = \frac{Q}{v}F$$

where F is C/V
The current and voltage across a capacitor are related by

$$I = C\frac{dV}{dt}$$

The energy stored or released by a capacitor during a time interval can be deduced easily as follows

$$W = \int_{t_1}^{t_2} VI\ dt = \int_{t_1}^{t_2} P\ dt = \frac{1}{2}C(v_2^2 - v_1^2)$$

5 Circuit laws

Two specific laws are of particular interest to us: Kirchoff's voltage and current laws.

5.1 Kirchoff's voltage law

Simply put, Kirchoff's voltage law states that for any closed loop inside a circuit the algebraic sum of the voltages inside the loop is equal to zero.

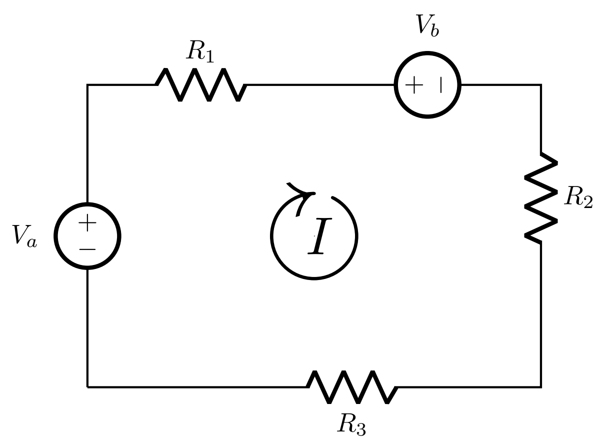

Figure 5

$$\begin{aligned} -V_a + IR_1 + V_b + IR_2 + IR_3 &= 0 \\ V_a + V_b &= I(R_1 + R_2 + R_3) \end{aligned}$$

5.2 Kirchoff's current law

Kirchoff's current law states that the algebraic sum of currents at any node (a point where different circuital elements are connected) is zero.

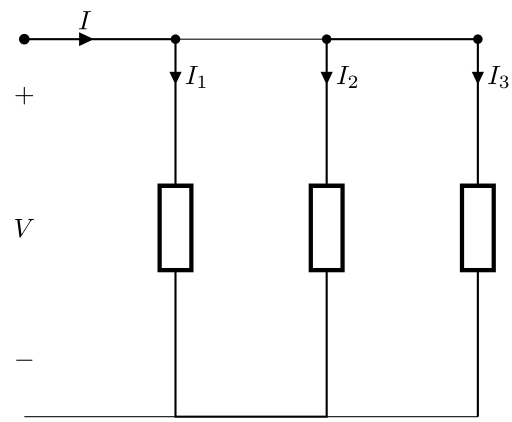

Figure 6

$$I = I_1 + I_2 + I_3$$

where the total current I is equal to the sum of the currents drawn by the element at each node.

5.3 Series and parallel connections of different circuital elements

We just state here the general formulas corresponding to each circuit element.

5.4 Circuit elements in series

$$R_{\text{eq}} = R_1 + R_2 + ... + R_n$$
$$L_{\text{eq}} = L_1 + L_2 + ... + L_n$$
$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_{\text{n}}}$$

5.5 Circuit elements in parallel

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_{\text{n}}}$$
$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + ... + \frac{1}{L_{\text{n}}}$$
$$C_{\text{eq}} = C_1 + C_2 + ... + C_n$$

6 Voltage division

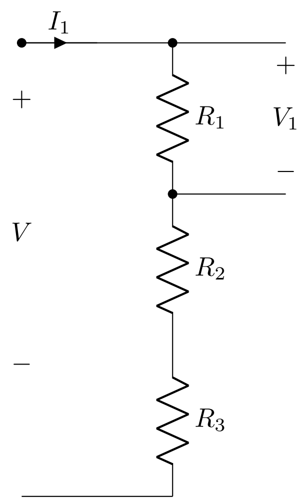

Figure 7

The path created by connecting a set of resistors in series as shown in Fig.7 is what we call a *voltage divider*. It is straight forward to show that

$$V_1 = V\left[\frac{R_1}{R_1 + R_2 + R_3}\right]$$

7 Current division

The parallel arrangement of circuital elements as shown in Fig.6 results in current division. For example, consider the case where the three parallel elements are resistors, the ratio between the current in the first branch and the overall current entering the circuit can be found by applying Ohm's law and KCL.

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

,

$$I_1 = \frac{V}{R_1}$$

$$\frac{I_1}{I} = \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

8 Transmission lines

Transmission lines (TLs) are means to transport energy from one point to another. One particular example of a transmission line is the one presented in Fig.8. It is composed of two parallel plate conductors separated by some dielectric medium. In circuit elements this is modelled as an infinitely long series of inductors and capacitors. They are characterized by impedances (conductances) that have the same value per unit length as the transmission line. The **impedance** of the transmission line at a given frequency defines the relation between the voltage and current of a sinusoidal wave of the same frequency traveling along the line

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R_l + j\omega L_l}{G_l + j\omega C_l}}$$

Signals on a transmission line propagates as waves with a complex **propagation coefficient**

$$\gamma = \sqrt{(R_l + j\omega L_l)(G_l + j\omega C_l)}$$

where The real part of the propagation coefficient represents dissipation, whereas the imaginary part represents the stored power.
Another important characteristic of transmission lines is called the **voltage standing wave ratio** (VSWR). It is a measure of the efficiency of power transfer from a source, through a transmission line, to a load. Ideally a perfect match between the source impedance and that of the transmission line results in lossless transmission. In real systems, some reflections (*destructively interfere with the incident field*) occur leading to voltage maxima and minima. VSWR measures these voltage fluctuations, it is the ratio of the maximum to minimum voltage

$$\text{VSWR} = \frac{|V_{\text{max}}|}{|V_{\text{min}}|} = \frac{1+\Gamma}{1-\Gamma}$$

where Γ is the voltage reflection coefficient near the load. When the load and transmission line are matched no reflection occurs, and the VSWR $= 1$.

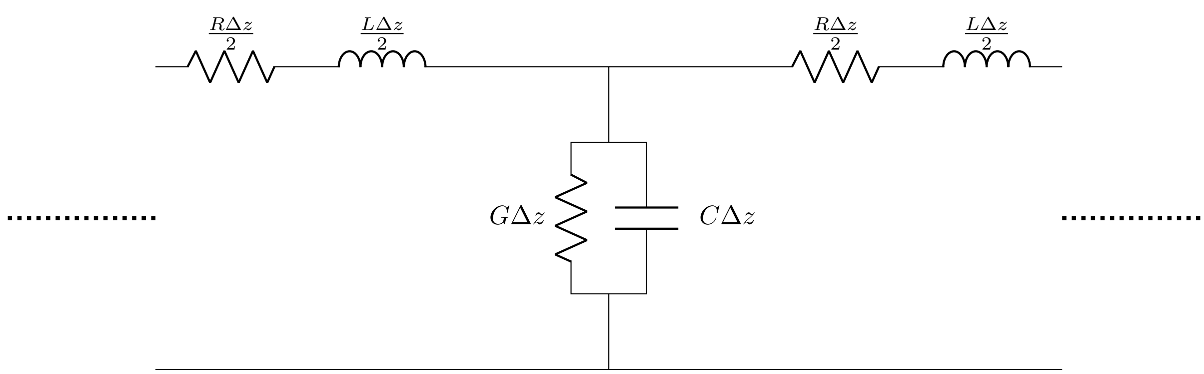

Figure 8 – *An infinitely long transmission line modelling a bath of infinite modes*

9 Prelude: Hamiltonian of a transmission line

In realistic scenarios a quantum harmonic oscillator doesn't operate in absolute isolation of its surroundings as suggested by Schrödinger's closed dynamics. Although we refrained from introducing the complete machinery of open quantum systems, we can still build an intuitive picture of the nature of the coupling between a quantum oscillator (*system*) and an external world *(environment)*. Such situation arises when the oscillator propagates through a lossy wave-guide or being measured by an external device. Heuristically we can write a phenomenological Hamiltonian for the overall system as

$$H_{\text{sys}} + \sum_k \hbar\omega_k b_k^\dagger b_k + \textbf{Coupling}$$

In this picture a lossy waveguide transmission line is modelled as a bath of infinite number of harmonic oscillators, each characterized by a pair of creation and annihilation operators $\{b_k^\dagger, b_k\}$. The nature of the coupling will be determined by the mechanism of energy transfer between the two systems, as we will see in future lectures.