MEC-E8003 Beam, plate and shell models, week 10/2021

1. Given $\vec{a}=a_{xx}\vec{i}\vec{i}+a_{xy}\vec{i}\vec{j}+a_{yx}\vec{j}\vec{i}+a_{yy}\vec{j}\vec{j}$ and $\vec{b}=b_x\vec{i}+b_y\vec{j}$, determine $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{b}\cdot\vec{a}$, and $\vec{b}\times\vec{a}$.

   Answer

   $$\vec{a}\cdot\vec{b}=(a_{xx}b_x+a_{xy}b_y)\vec{i}+(a_{yx}b_x+a_{yy}b_y)\vec{j}\,,\quad \vec{a}\times\vec{b}=(a_{xx}b_y-a_{xy}b_x)\vec{i}\vec{k}+(a_{yx}b_y-a_{yy}b_x)\vec{j}\vec{k}$$

   $$\vec{b}\cdot\vec{a}=(b_x a_{xx}+b_y a_{yx})\vec{i}+(b_x a_{xy}+b_y a_{yy})\vec{j}\,,\quad \vec{b}\times\vec{a}=(b_x a_{yx}-b_y a_{xx})\vec{k}\vec{i}+(b_x a_{yy}-b_y a_{xy})\vec{k}\vec{j}$$

2. Determine $tr([S])\equiv\vec{I}:\vec{S}$, $\vec{S}:\vec{S}$, $\vec{S}:\vec{S}_{\mathrm{c}}$, $\vec{A}\cdot\vec{S}$ and $\vec{S}\cdot\vec{T}$ when the components of tensors $\vec{A}$, $\vec{S}$ and $\vec{T}$ in the orthonormal $(\vec{i},\vec{j},\vec{k})$ basis are

   $$\{A\}=\begin{Bmatrix}-1\\1\\0\end{Bmatrix},\quad [S]=\begin{bmatrix}1&-1&0\\0&1&0\\0&0&0\end{bmatrix} \text{ and } [T]=\begin{bmatrix}1&0&0\\1&-1&0\\0&0&0\end{bmatrix}.$$

   Answer $tr([S])=2$, $\vec{S}:\vec{S}=2$, $\vec{S}:\vec{S}_{\mathrm{c}}=3$, $\vec{A}\cdot\vec{S}=-\vec{i}+2\vec{j}$ and $\vec{S}\cdot\vec{T}=\vec{i}\vec{j}+\vec{j}\vec{i}-\vec{j}\vec{j}$

3. Given the strain components ε_{xx}, ε_{yy}, ε_{xy}, and ε_{yx} in a Cartesian $(x,y)-$coordinate system, derive the strain components ε_{rr}, $\varepsilon_{\phi\phi}$, $\varepsilon_{r\phi}$, and $\varepsilon_{\phi r}$ of the polar $(r,\phi)-$coordinate system (in terms of the Cartesian components). Use the invariance of tensor quantities.

   Answer $$\begin{bmatrix}\varepsilon_{rr}&\varepsilon_{r\phi}\\\varepsilon_{\phi r}&\varepsilon_{\phi\phi}\end{bmatrix}=\begin{bmatrix}\cos\phi&\sin\phi\\-\sin\phi&\cos\phi\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}\\\varepsilon_{yx}&\varepsilon_{yy}\end{bmatrix}\begin{bmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{bmatrix}$$

4. Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

   $$\vec{r}=\begin{Bmatrix}r\\0\\z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\\\vec{e}_z\end{Bmatrix}=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\\\vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}r\\0\\z\end{Bmatrix},\quad \nabla=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\\\vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\\partial/(r\partial\phi)\\\partial/\partial z\end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\\\vec{e}_z\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\-\vec{e}_r\\0\end{Bmatrix} \text{ (zeros otherwise).}$$

   Answer $\nabla\vec{r}=\vec{I}$, $\nabla\cdot\vec{r}=3$, $\nabla\times\vec{r}=0$

5. Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and ∇^2u in the polar coordinate system. Vector $\vec{u}(r,\phi)=u_r\vec{e}_r+u_\phi\vec{e}_\phi$ and scalar $u(r,\phi)$ depend on the polar coordinates r and ϕ. In the polar coordinate system

   $$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi},\quad \frac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi,\quad \frac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r,\quad (\text{and } \vec{e}_r\times\vec{e}_\phi=\vec{k}\,).$$

Answer

$$\nabla\cdot\vec{u}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi},\quad \nabla\times\vec{u}=(\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)-\frac{1}{r}\frac{\partial u_r}{\partial\phi})\vec{k},\ \nabla^2 u=(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2})u$$

6. In the beam coordinate system and planar case, the displacement assumption of a curved Timoshenko beam model is $\vec{u}=\vec{u}_0+\vec{\theta}_0\times\vec{\rho}$, where $\vec{u}_0=u(s)\vec{e}_s+v(s)\vec{e}_n$, $\vec{\theta}_0=\psi(s)\vec{e}_b$, and $\vec{\rho}=n\vec{e}_n$. Derive the small strain component expressions ε_{ss} and $\varepsilon_{sn}=\varepsilon_{ns}$ using

$$\vec{\varepsilon}=\frac{1}{2}\left[\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}\right],\ \nabla=\vec{e}_s\frac{1}{1-n/R}\frac{\partial}{\partial s}+\vec{e}_n\frac{\partial}{\partial n},\ \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=\frac{1}{R}\begin{Bmatrix}\vec{e}_n\\ -\vec{e}_s\end{Bmatrix},\ \text{and}\ \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=0.$$

Assume that curvature $\kappa=1/R$ is constant.

Answer $\varepsilon_{ss}=\frac{R}{R-n}(u'-\psi'n-\frac{v}{R})$, $\varepsilon_{sn}=\varepsilon_{ns}=\frac{1}{2}\frac{R}{R-n}(\frac{1}{R}u+v'-\psi)$

7. Mapping $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ defines the cylindrical $(r,\phi,z)-$coordinate system. Use (in detail) the generic formula

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix},\ \text{where}\ \eta\in\{r,\phi,z\}$$

to find the derivatives of the basis vectors.

Answer $\frac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi$, $\frac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r$ (zeros otherwise)

8. Derive the gradient expression of the spherical $(\theta,\phi,r)-$coordinate system, when the mapping defining the coordinate system is given by $\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$.

Answer $\nabla=\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}+\vec{e}_r\frac{\partial}{\partial r}$

9. Compute the derivatives of the basis vectors, gradient operator, and curvature for the cylindrical shell geometry with the mid-surface representation $\vec{r}_0(\phi,z)=R(\cos\phi\vec{i}+\sin\phi\vec{j})+z\vec{k}$ in terms of coordinates (ϕ,z). Notice that the order of the coordinates differs from that of the lecture notes, which affects, e.g., direction of $\vec{e}_n$.

Answer $\frac{\partial\vec{e}_\phi}{\partial\phi}=-\vec{e}_n$, $\frac{\partial\vec{e}_n}{\partial\phi}=\vec{e}_\phi$, $\nabla=(\frac{R}{R+n})\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_n\frac{\partial}{\partial n}$, $\vec{\kappa}=\vec{e}_\phi\vec{e}_\phi\frac{1}{R+n}$

10. Consider the mid-surface mapping $\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Answer $\dfrac{\partial \vec{e}_r}{\partial \phi} = 2\vec{e}_\phi$, $\dfrac{\partial \vec{e}_\phi}{\partial \phi} = -2\vec{e}_r$, $\nabla = \vec{e}_r \dfrac{1}{2r}\dfrac{\partial}{\partial r} + \vec{e}_\phi \dfrac{1}{2r^2}\dfrac{\partial}{\partial \phi} + \vec{e}_n \dfrac{\partial}{\partial n}$

Given $\vec{a} = a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j}$, determine $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{b}\cdot\vec{a}$, and $\vec{b}\times\vec{a}$.

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply.

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_y\vec{j} \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i}\cdot\vec{i} + a_{xy}\vec{i}\vec{j}\cdot\vec{i} + a_{yx}\vec{j}\vec{i}\cdot\vec{i} + a_{yy}\vec{j}\vec{j}\cdot\vec{i})b_x + (a_{xx}\vec{i}\vec{i}\cdot\vec{j} + a_{xy}\vec{i}\vec{j}\cdot\vec{j} + a_{yx}\vec{j}\vec{i}\cdot\vec{j} + a_{yy}\vec{j}\vec{j}\cdot\vec{j})b_y \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = a_{xx}b_x\vec{i} + a_{yx}b_x\vec{j} + a_{xy}b_y\vec{i} + a_{yy}b_y\vec{j} = (a_{xx}b_x + a_{xy}b_y)\vec{i} + (a_{yx}b_x + a_{yy}b_y)\vec{j}\,.$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_y\vec{j} \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i}\times\vec{i} + a_{xy}\vec{i}\vec{j}\times\vec{i} + a_{yx}\vec{j}\vec{i}\times\vec{i} + a_{yy}\vec{j}\vec{j}\times\vec{i})b_x +$$

$$(a_{xx}\vec{i}\vec{i}\times\vec{j} + a_{xy}\vec{i}\vec{j}\times\vec{j} + a_{yx}\vec{j}\vec{i}\times\vec{j} + a_{yy}\vec{j}\vec{j}\times\vec{j})b_y \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = -a_{xy}b_x\vec{i}\vec{k} - a_{yy}b_x\vec{j}\vec{k} + a_{xx}b_y\vec{i}\vec{k} + a_{yx}b_y\vec{j}\vec{k} = (a_{xx}b_y - a_{xy}b_x)\vec{i}\vec{k} + (a_{yx}b_y - a_{yy}b_x)\vec{j}\vec{k}\,.$$

$$\vec{b}\cdot\vec{a} = (b_x\vec{i} + b_y\vec{j})\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x\vec{i}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y\vec{j}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x(a_{xx}\vec{i}\cdot\vec{i}\vec{i} + a_{xy}\vec{i}\cdot\vec{i}\vec{j} + a_{yx}\vec{i}\cdot\vec{j}\vec{i} + a_{yy}\vec{i}\cdot\vec{j}\vec{j}) + b_y(a_{xx}\vec{j}\cdot\vec{i}\vec{i} + a_{xy}\vec{j}\cdot\vec{i}\vec{j} + a_{yx}\vec{j}\cdot\vec{j}\vec{i} + a_{yy}\vec{j}\cdot\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x a_{xx}\vec{i} + b_x a_{xy}\vec{j} + b_y a_{yx}\vec{i} + b_y a_{yy}\vec{j} = (b_x a_{xx} + b_y a_{yx})\vec{i} + (b_x a_{xy} + b_y a_{yy})\vec{j}\,.$$

$$\vec{b}\times\vec{a} = (b_x\vec{i} + b_y\vec{j})\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b} \times \vec{a} = b_x \vec{i} \times (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y \vec{j} \times (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b} \times \vec{a} = b_x (a_{xx}\vec{i} \times \vec{i}\vec{i} + a_{xy}\vec{i} \times \vec{i}\vec{j} + a_{yx}\vec{i} \times \vec{j}\vec{i} + a_{yy}\vec{i} \times \vec{j}\vec{j}) +$$

$$b_y (a_{xx}\vec{j} \times \vec{i}\vec{i} + a_{xy}\vec{j} \times \vec{i}\vec{j} + a_{yx}\vec{j} \times \vec{j}\vec{i} + a_{yy}\vec{j} \times \vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b} \times \vec{a} = b_x a_{yx}\vec{k}\vec{i} + b_x a_{yy}\vec{k}\vec{j} - b_y a_{xx}\vec{k}\vec{i} - b_y a_{xy}\vec{k}\vec{j} = (b_x a_{yx} - b_y a_{xx})\vec{k}\vec{i} + (b_x a_{yy} - b_y a_{xy})\vec{k}\vec{j} \ .$$

Determine $tr([S]) \equiv \vec{I} : \vec{S}$, $\vec{S} : \vec{S}$, $\vec{S} : \vec{S}_{\mathrm{c}}$, $\vec{A} \cdot \vec{S}$ and $\vec{S} \cdot \vec{T}$ when the components of tensors $\vec{A}$, $\vec{S}$ and $\vec{T}$ in the orthonormal $(\vec{i}, \vec{j}, \vec{k})$ basis are

$$\{A\} = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}, \ [S] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution

The inner products of the basis vectors $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$ and $\vec{k} \cdot \vec{k} = 1$ all the other combinations giving zeros. The double inner product should be treated just as two inner products by keeping the positions of the multiplication operator with respect to vectors. Therefore, in the vector identity $\vec{a}\vec{b} : \vec{c}\vec{d} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$, the first inner product between $\vec{b}$ and $\vec{c}$ produces a scalar which can be moved in front of the expression. What remains is the inner product between $\vec{a}$ and $\vec{d}$.

In conjugate $\vec{S}_{\mathrm{c}}$ to $\vec{S}$ the component matrix is transposed which corresponds to order change in all the dyads $\vec{a}\vec{b} \to \vec{b}\vec{a} \quad \vec{a}, \vec{b} \in \{\vec{i}, \vec{j}, \vec{k}\}$ of the tensor representation. Representations of $\vec{A}$, $\vec{S}$, $\vec{S}_{\mathrm{c}}$, $\vec{T}$ and the second order unit tensor $\vec{I}$ in $(\vec{i}, \vec{j}, \vec{k})-$basis

$$\vec{A} = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = -\vec{i} + \vec{j} \ ,$$

$$\vec{S} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j} \ ,$$

$$\vec{S}_{\mathrm{c}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} - \vec{j}\vec{i} + \vec{j}\vec{j} \ ,$$

$$\vec{T} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} + \vec{j}\vec{i} - \vec{j}\vec{j} \ ,$$

$$\vec{I} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k} \ .$$

Evaluation of a tensor product expressions consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Double inner product $tr([S]) \equiv \vec{\vec{I}} : \vec{\vec{S}}$ produces a scalar

(I) $tr([S]) = \vec{\vec{I}} : \vec{\vec{S}} = (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) : (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) \quad \Leftrightarrow$

(II) $tr([S]) = \vec{i}\vec{i} : \vec{i}\vec{i} - \vec{i}\vec{i} : \vec{i}\vec{j} + \vec{i}\vec{i} : \vec{j}\vec{j} + \vec{j}\vec{j} : \vec{i}\vec{i} - \vec{j}\vec{j} : \vec{i}\vec{j} + \vec{j}\vec{j} : \vec{j}\vec{j} + \vec{k}\vec{k} : \vec{i}\vec{i} - \vec{k}\vec{k} : \vec{i}\vec{j} + \vec{k}\vec{k} : \vec{j}\vec{j} \quad \Leftrightarrow$

(III) $tr([S]) = 1 - 0 + 0 + 0 - 0 + 1 + 0 + 0 + 0 \quad \Leftrightarrow$

(IV) $tr([S]) = 2$.

Double inner product $\vec{\vec{S}} : \vec{\vec{S}}$ produces a scalar

(I) $\vec{\vec{S}} : \vec{\vec{S}} = (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) : (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) \quad \Leftrightarrow$

(II) $\vec{\vec{S}} : \vec{\vec{S}} = \vec{i}\vec{i} : \vec{i}\vec{i} - \vec{i}\vec{i} : \vec{i}\vec{j} + \vec{i}\vec{i} : \vec{j}\vec{j} - \vec{i}\vec{j} : \vec{i}\vec{i} + \vec{i}\vec{j} : \vec{i}\vec{j} - \vec{i}\vec{j} : \vec{j}\vec{j} + \vec{j}\vec{j} : \vec{i}\vec{i} - \vec{j}\vec{j} : \vec{i}\vec{j} + \vec{j}\vec{j} : \vec{j}\vec{j} \quad \Leftrightarrow$

(III) $\vec{\vec{S}} : \vec{\vec{S}} = 1 - 0 + 0 - 0 + 0 - 0 + 0 - 0 + 1 \quad \Leftrightarrow$

(IV) $\vec{\vec{S}} : \vec{\vec{S}} = 2$.

Double inner product $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}}$ produces a scalar

(I) $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) : (\vec{i}\vec{i} - \vec{j}\vec{i} + \vec{j}\vec{j}) \quad \Leftrightarrow$

(II) $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = \vec{i}\vec{i} : \vec{i}\vec{i} - \vec{i}\vec{i} : \vec{j}\vec{i} + \vec{i}\vec{i} : \vec{j}\vec{j} - \vec{i}\vec{j} : \vec{i}\vec{i} + \vec{i}\vec{j} : \vec{j}\vec{i} - \vec{i}\vec{j} : \vec{j}\vec{j} + \vec{j}\vec{j} : \vec{i}\vec{i} - \vec{j}\vec{j} : \vec{j}\vec{i} + \vec{j}\vec{j} : \vec{j}\vec{j} \quad \Leftrightarrow$

(III) $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = 1 - 0 + 0 - 0 + 1 - 0 + 0 - 0 + 1 \quad \Leftrightarrow$

(IV) $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = 3$.

Inner product $\vec{A} \cdot \vec{\vec{S}}$ produces a vector

(I) $\vec{A} \cdot \vec{\vec{S}} = (-\vec{i} + \vec{j}) \cdot (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) \quad \Leftrightarrow$

(II) $\vec{A} \cdot \vec{\vec{S}} = -\vec{i} \cdot \vec{i}\vec{i} + \vec{i} \cdot \vec{i}\vec{j} - \vec{i} \cdot \vec{j}\vec{j} + \vec{j} \cdot \vec{i}\vec{i} - \vec{j} \cdot \vec{i}\vec{j} + \vec{j} \cdot \vec{j}\vec{j} \quad \Leftrightarrow$

(III) $\vec{A} \cdot \vec{\vec{S}} = -\vec{i} + \vec{j} - 0\vec{j} + 0\vec{i} - 0\vec{j} + \vec{j} \quad \Leftrightarrow$

(IV) $\vec{A} \cdot \vec{\vec{S}} = -\vec{i} + 2\vec{j}$.

Inner product $\vec{\vec{S}} \cdot \vec{\vec{T}}$ produces a second order tensor

(I) $\vec{\vec{S}} \cdot \vec{\vec{T}} = (\vec{i}\vec{i} - \vec{i}\vec{j} + \vec{j}\vec{j}) \cdot (\vec{i}\vec{i} + \vec{j}\vec{i} - \vec{j}\vec{j}) \quad \Leftrightarrow$

(II) $\vec{\vec{S}} \cdot \vec{\vec{T}} = \vec{i}\vec{i} \cdot \vec{i}\vec{i} + \vec{i}\vec{i} \cdot \vec{j}\vec{i} - \vec{i}\vec{i} \cdot \vec{j}\vec{j} - \vec{i}\vec{j} \cdot \vec{i}\vec{i} - \vec{i}\vec{j} \cdot \vec{j}\vec{i} + \vec{i}\vec{j} \cdot \vec{j}\vec{j} + \vec{j}\vec{j} \cdot \vec{i}\vec{i} + \vec{j}\vec{j} \cdot \vec{j}\vec{i} - \vec{j}\vec{j} \cdot \vec{j}\vec{j} \quad \Leftrightarrow$

(III) $\overleftrightarrow{S} \cdot \overleftrightarrow{T} = \vec{i}\vec{i} + 0 - 0 + 0 - \vec{i}\vec{i} + \vec{i}\vec{j} + 0 + \vec{j}\vec{i} - \vec{j}\vec{j} \quad \Leftrightarrow$

(IV) $\overleftrightarrow{S} \cdot \overleftrightarrow{T} = \vec{i}\vec{j} + \vec{j}\vec{i} - \vec{j}\vec{j}$.

Given the strain components ε_{xx}, ε_{yy}, ε_{xy}, and ε_{yx} in a Cartesian $(x,y)-$coordinate system, derive the strain components ε_{rr}, $\varepsilon_{\phi\phi}$, $\varepsilon_{r\phi}$, and $\varepsilon_{\phi r}$ of the polar $(r,\phi)-$coordinate system (in terms of the Cartesian components). Use the invariance of tensor quantities.

Solution

In mechanics tensors (vectors etc.) represent physical quantities which can be expressed in terms of any basis vector set. Components depend on the selection of the basis vectors but the quantity itself does not. With notation $\vec{e}_x \equiv \vec{i}$ and $\vec{e}_y \equiv \vec{j}$, the relationship between the Cartesian and polar basis is

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix} = [F] \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}$$

and invariance of $\vec{\vec{\varepsilon}}$ with respect to the coordinate system means that

$$\vec{\vec{\varepsilon}} = \varepsilon_{xx}\vec{e}_x\vec{e}_x + \varepsilon_{xy}\vec{e}_x\vec{e}_y + \varepsilon_{yx}\vec{e}_y\vec{e}_x + \varepsilon_{yy}\vec{e}_y\vec{e}_y = \varepsilon_{rr}\vec{e}_r\vec{e}_r + \varepsilon_{r\phi}\vec{e}_r\vec{e}_\phi + \varepsilon_{\phi r}\vec{e}_\phi\vec{e}_r + \varepsilon_{\phi\phi}\vec{e}_\phi\vec{e}_\phi .$$

By substituting $\vec{e}_x = \mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi$ and $\vec{e}_y = \mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi$ into the representation in the Cartesian $(x,y)-$system

$$\vec{\vec{\varepsilon}} = \varepsilon_{xx}\vec{e}_x\vec{e}_x + \varepsilon_{xy}\vec{e}_x\vec{e}_y + \varepsilon_{yx}\vec{e}_y\vec{e}_x + \varepsilon_{yy}\vec{e}_y\vec{e}_y \quad \Rightarrow$$

$$\vec{\vec{\varepsilon}} = \varepsilon_{xx}(\mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi)(\mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi) + \varepsilon_{xy}(\mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi)(\mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi) +$$

$$\varepsilon_{yx}(\mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi)(\mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi) + \varepsilon_{yy}(\mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi)(\mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi) \quad \Leftrightarrow$$

$$\vec{\vec{\varepsilon}} = \varepsilon_{xx}(\mathrm{c}^2\phi\vec{e}_r\vec{e}_r - \mathrm{c}\phi\mathrm{s}\phi\vec{e}_r\vec{e}_\phi - \mathrm{s}\phi\mathrm{c}\phi\vec{e}_\phi\vec{e}_r + \mathrm{s}^2\phi\vec{e}_\phi\vec{e}_\phi) + \varepsilon_{xy}(\mathrm{c}\phi\mathrm{s}\phi\vec{e}_r\vec{e}_r + \mathrm{c}^2\phi\vec{e}_r\vec{e}_\phi - \mathrm{s}^2\phi\vec{e}_\phi\vec{e}_r - \mathrm{s}\phi\mathrm{c}\phi\vec{e}_\phi\vec{e}_\phi) +$$

$$\varepsilon_{yx}(\mathrm{s}\phi\mathrm{c}\phi\vec{e}_r\vec{e}_r - \mathrm{s}^2\phi\vec{e}_r\vec{e}_\phi + \mathrm{c}^2\phi\vec{e}_\phi\vec{e}_r - \mathrm{c}\phi\mathrm{s}\phi\vec{e}_\phi\vec{e}_\phi) + \varepsilon_{yy}(\mathrm{s}^2\phi\vec{e}_r\vec{e}_r + \mathrm{s}\phi\mathrm{c}\phi\vec{e}_r\vec{e}_\phi + \mathrm{c}\phi\mathrm{s}\phi\vec{e}_\phi\vec{e}_r + \mathrm{c}^2\phi\vec{e}_\phi\vec{e}_\phi) .$$

and after collecting the components

$$\vec{\vec{\varepsilon}} = (\varepsilon_{xx}\mathrm{c}^2\phi + \varepsilon_{xy}\mathrm{c}\phi\mathrm{s}\phi + \varepsilon_{yx}\mathrm{s}\phi\mathrm{c}\phi + \varepsilon_{yy}\mathrm{s}^2\phi)\vec{e}_r\vec{e}_r + (-\varepsilon_{xx}\mathrm{c}\phi\mathrm{s}\phi + \varepsilon_{xy}\mathrm{c}^2\phi - \varepsilon_{yx}\mathrm{s}^2\phi + \varepsilon_{yy}\mathrm{s}\phi\mathrm{c}\phi)\vec{e}_r\vec{e}_\phi +$$

$$(-\varepsilon_{xx}\mathrm{s}\phi\mathrm{c}\phi - \varepsilon_{xy}\mathrm{s}^2\phi + \varepsilon_{yx}\mathrm{c}^2\phi + \varepsilon_{yy}\mathrm{c}\phi\mathrm{s}\phi)\vec{e}_\phi\vec{e}_r + (\varepsilon_{xx}\mathrm{s}^2\phi - \varepsilon_{xy}\mathrm{s}\phi\mathrm{c}\phi - \varepsilon_{yx}\mathrm{c}\phi\mathrm{s}\phi + \varepsilon_{yy}\mathrm{c}^2\phi)\vec{e}_\phi\vec{e}_\phi$$

$$= \varepsilon_{rr}\vec{e}_r\vec{e}_r + \varepsilon_{r\phi}\vec{e}_r\vec{e}_\phi + \varepsilon_{\phi r}\vec{e}_\phi\vec{e}_r + \varepsilon_{\phi\phi}\vec{e}_\phi\vec{e}_\phi$$

Comparison of the representations gives

$$
\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{r\phi} \\ \varepsilon_{\phi r} \\ \varepsilon_{\phi\phi} \end{Bmatrix} = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi & \cos\phi\sin\phi & \sin^2\phi \\ -\cos\phi\sin\phi & \cos^2\phi & -\sin^2\phi & \cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin^2\phi & \cos^2\phi & \cos\phi\sin\phi \\ \sin^2\phi & -\cos\phi\sin\phi & -\cos\phi\sin\phi & \cos^2\phi \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \\ \varepsilon_{yy} \end{Bmatrix}.
$$

Alternatively, one may work with matrices (a more convenient way for vectors and second order tensors)

$$
\vec{\varepsilon} = \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} [\mathrm{F}]^{-\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}.
$$

Therefore, components of the two systems are related by

$$
\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} = [\mathrm{F}]^{-\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{-1} \quad ([\mathrm{F}]^{-\mathrm{T}} = [\mathrm{F}] \text{ and } [\mathrm{F}]^{-1} = [\mathrm{F}]^{\mathrm{T}} \text{ here})
$$

giving

$$
\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} = [\mathrm{F}] \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{\mathrm{T}} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.
$$

Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

$$\vec{r}=\begin{Bmatrix} r\\ 0\\ z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r\\ 0\\ z\end{Bmatrix},\quad \nabla=\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r\\ \partial/(r\partial\phi)\\ \partial/\partial z\end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix} \vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix} \text{ (zeros otherwise).}$$

Solution

In a term, gradient operator ∇ acts on everything on its right hand side. Otherwise the operator is treated like a vector (if the basis vectors are not constants, the derivative operators should be after the basis vectors)

$$\vec{r}=\begin{Bmatrix} r\\ 0\\ z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r\\ 0\\ z\end{Bmatrix}=r\vec{e}_r+z\vec{e}_z\,,$$

$$\nabla=\begin{Bmatrix} \vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r\\ \partial/(r\partial\phi)\\ \partial/\partial z\end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}\,.$$

Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term by term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome. Gradient of the position vector is a second order tensor

(I) $\quad \nabla\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\quad \nabla\vec{r}=\vec{e}_r\frac{\partial}{\partial r}r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}z\vec{e}_z\quad\Leftrightarrow$

(III) $\quad \nabla\vec{r}=\vec{e}_r\vec{e}_r+0+\vec{e}_\phi\vec{e}_\phi+0+0+\vec{e}_z\vec{e}_z\quad\Leftrightarrow$

(IV) $\quad \nabla\vec{r}=\vec{e}_r\vec{e}_r+\vec{e}_\phi\vec{e}_\phi+\vec{e}_z\vec{e}_z=\vec{I}\,.$

Divergence of the position vector is a scalar

(I) $\quad \nabla\cdot\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})\cdot(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\quad \nabla\cdot\vec{r}=\vec{e}_r\frac{\partial}{\partial r}\cdot r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\cdot z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}\cdot r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}\cdot z\vec{e}_z\quad\Leftrightarrow$

(III) $\quad \nabla\cdot\vec{r}=1+0+1+0+0+1\quad\Leftrightarrow$

(IV) $\nabla \cdot \vec{r} = 3$.

Curl of the position vector is a vector

(I) $\nabla \times \vec{r} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) \times (r\vec{e}_r + z\vec{e}_z) \quad \Leftrightarrow$

(II) $\nabla \times \vec{r} = \vec{e}_r \frac{\partial}{\partial r} \times r\vec{e}_r + \vec{e}_r \frac{\partial}{\partial r} \times z\vec{e}_z + \vec{e}_\phi \frac{\partial}{r\partial\phi} \times r\vec{e}_r + \vec{e}_\phi \frac{\partial}{r\partial\phi} \times z\vec{e}_z + \vec{e}_z \frac{\partial}{\partial z} \times r\vec{e}_r + \vec{e}_z \frac{\partial}{\partial z} \times z\vec{e}_z \quad \Leftrightarrow$

(III) $\nabla \times \vec{r} = 0 + 0 + 0 + 0 + 0 + 0 \quad \Leftrightarrow$

(IV) $\nabla \times \vec{r} = 0$.

Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in the polar coordinate system. Vector $\vec{u}(r,\phi)=u_r\vec{e}_r+u_\phi\vec{e}_\phi$ and scalar $u(r,\phi)$ depend on the polar coordinates r and ϕ. In the polar coordinate system

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi},\ \ \frac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi,\ \frac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r,\ \ \text{(and } \vec{e}_r\times\vec{e}_\phi=\vec{k}\text{).}$$

Solution

In manipulation of vector expression containing vectors and dyads, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is part of the expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Divergence of a vector

(I) $$\nabla\cdot\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(II) $$\nabla\cdot\vec{u}=\vec{e}_r\cdot\frac{\partial}{\partial r}(u_r\vec{e}_r)+\vec{e}_r\cdot\frac{\partial}{\partial r}(u_\phi\vec{e}_\phi)+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(u_r\vec{e}_r)+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(III) $$\nabla\cdot\vec{u}=\vec{e}_r\cdot(\frac{\partial u_r}{\partial r}\vec{e}_r+u_r\frac{\partial\vec{e}_r}{\partial r})+\vec{e}_r\cdot(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+u_\phi\frac{\partial\vec{e}_\phi}{\partial r})+\vec{e}_\phi\cdot(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\frac{\partial\vec{e}_r}{\partial\phi})+$$

$$\vec{e}_\phi\cdot(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi+\frac{u_\phi}{r}\frac{\partial\vec{e}_\phi}{\partial\phi})\quad\Leftrightarrow$$

$$\nabla\cdot\vec{u}=\vec{e}_r\cdot\frac{\partial u_r}{\partial r}\vec{e}_r+\vec{e}_r\cdot\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+\vec{e}_\phi\cdot(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\vec{e}_\phi)+\vec{e}_\phi\cdot(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi-\frac{u_\phi}{r}\vec{e}_r)\quad\Leftrightarrow$$

$$\nabla\cdot\vec{u}=\frac{\partial u_r}{\partial r}+u_r\frac{1}{r}+\frac{\partial u_\phi}{r\partial\phi}\quad\Leftrightarrow$$

(IV) $$\nabla\cdot\vec{u}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}.$$

Curl of a vector

(I) $$\nabla\times\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\times(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(II) $$\nabla\times\vec{u}=\vec{e}_r\frac{\partial}{\partial r}\times u_r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\times u_\phi\vec{e}_\phi+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times u_r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times u_\phi\vec{e}_\phi\quad\Leftrightarrow$$

(III) $$\nabla\times\vec{u}=\vec{e}_r\times(\frac{\partial u_r}{\partial r}\vec{e}_r+u_r\frac{\partial\vec{e}_r}{\partial r})+\vec{e}_r\times(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+u_\phi\frac{\partial\vec{e}_\phi}{\partial r})+\vec{e}_\phi\times(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\frac{\partial\vec{e}_r}{\partial\phi})+$$

$$\vec{e}_\phi\times(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi+\frac{u_\phi}{r}\frac{\partial\vec{e}_\phi}{\partial\phi})\quad\Leftrightarrow$$

$$\nabla\times\vec{u}=\vec{e}_r\times\frac{\partial u_r}{\partial r}\vec{e}_r+\vec{e}_r\times\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+\vec{e}_\phi\times(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\vec{e}_\phi)+\vec{e}_\phi\times(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi-\frac{u_\phi}{r}\vec{e}_r)\quad\Leftrightarrow$$

$$\nabla\times\vec{u}=\frac{\partial u_\phi}{\partial r}\vec{k}-\frac{\partial u_r}{r\partial\phi}\vec{k}+\frac{u_\phi}{r}\vec{k}\quad\Leftrightarrow$$

(IV) $$\nabla\times\vec{u}=(\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)-\frac{1}{r}\frac{\partial u_r}{\partial\phi})\vec{k}\,.$$

Laplacian of a scalar

(I) $$\nabla^2u=(\nabla\cdot\nabla)u=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})u\quad\Leftrightarrow$$

(II) $$\nabla^2u=\vec{e}_r\cdot\frac{\partial}{\partial r}(\vec{e}_r\frac{\partial}{\partial r})u+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(\vec{e}_r\frac{\partial}{\partial r})u+\vec{e}_r\cdot\frac{\partial}{\partial r}(\vec{e}_\phi\frac{\partial}{r\partial\phi})u+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(\vec{e}_\phi\frac{\partial}{r\partial\phi})u\quad\Leftrightarrow$$

$$\nabla^2u=\vec{e}_r\cdot(\frac{\partial\vec{e}_r}{\partial r}\frac{\partial}{\partial r}+\vec{e}_r\frac{\partial^2}{\partial r^2})u+\vec{e}_\phi\cdot(\frac{\partial\vec{e}_r}{r\partial\phi}\frac{\partial}{\partial r}+\vec{e}_r\frac{1}{r}\frac{\partial^2}{\partial\phi\partial r})u+$$

$$\vec{e}_r\cdot(\frac{\partial\vec{e}_\phi}{\partial r}\frac{\partial}{r\partial\phi}-\vec{e}_\phi\frac{1}{r^2}\frac{\partial}{\partial\phi}+\vec{e}_\phi\frac{1}{r}\frac{\partial^2}{\partial r\partial\phi})u+\vec{e}_\phi\cdot(\frac{\partial\vec{e}_\phi}{r\partial\phi}\frac{\partial}{r\partial\phi}+\vec{e}_\phi\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2})u\quad\Leftrightarrow$$

$$\nabla^2u=\vec{e}_r\cdot(\vec{e}_r\frac{\partial^2}{\partial r^2})u+\vec{e}_\phi\cdot(\frac{1}{r}\vec{e}_\phi\frac{\partial}{\partial r}+\vec{e}_r\frac{1}{r}\frac{\partial^2}{\partial\phi\partial r})u+\vec{e}_r\cdot(-\vec{e}_\phi\frac{1}{r^2}\frac{\partial}{\partial\phi}+\vec{e}_\phi\frac{1}{r}\frac{\partial^2}{\partial r\partial\phi})u+$$

$$\vec{e}_\phi\cdot(-\frac{1}{r}\vec{e}_r\frac{\partial}{r\partial\phi}+\vec{e}_\phi\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2})u\quad\Leftrightarrow$$

$$\nabla^2u=\frac{\partial^2}{\partial r^2}u+\frac{1}{r}\frac{\partial}{\partial r}u+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}u\quad\Leftrightarrow$$

(IV) $$\nabla^2u=(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2})u\,.$$

In the beam coordinate system and planar case, the displacement assumption of a curved Timoshenko beam model is $\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho}$, where $\vec{u}_0 = u(s)\vec{e}_s + v(s)\vec{e}_n$, $\vec{\theta}_0 = \psi(s)\vec{e}_b$, and $\vec{\rho} = n\vec{e}_n$. Derive the small strain component expressions ε_{ss} and $\varepsilon_{sn} = \varepsilon_{ns}$ using

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}],\ \nabla = \vec{e}_s \frac{1}{1-n/R}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n},\ \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix} = \frac{1}{R}\begin{Bmatrix}\vec{e}_n\\ -\vec{e}_s\end{Bmatrix},\ \text{and}\ \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix} = 0.$$

Assume that curvature $\kappa = 1/R$ is constant.

Solution

The curved beam coordinates are distance s measured along the mid-curve (identifying the particles along the mid-curve), and n,b identifying the particles away from the mid-curve. The $(s,n,b)-$ system is orthonormal and right-handed. The gradient expression of the coordinate system

$$\nabla = \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n}$$

is available in the formula collection. When the expressions of the cross-section translation, cross-section rotation and the relative position vector are substituted there, the displacement expression takes the form

$$\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho} = u(s)\vec{e}_s + v(s)\vec{e}_n + \psi(s)\vec{e}_b \times n\vec{e}_n = (u - \psi n)\vec{e}_s + v\vec{e}_n .$$

Displacement gradient is given by (Lagrange's notation for derivative with respect to s)

$$\nabla\vec{u} = (\vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n})[(u-\psi n)\vec{e}_s + v\vec{e}_n] \quad \Leftrightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s}(u-\psi n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s}(v\vec{e}_n) + (\vec{e}_n \frac{\partial}{\partial n})(u-\psi n)\vec{e}_s + \vec{e}_n \frac{\partial}{\partial n} v\vec{e}_n \quad \Leftrightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}(u' - \psi' n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}(u-\psi n)\vec{e}_s' + \vec{e}_s \frac{R}{R-n} v'\vec{e}_n + \vec{e}_s \frac{R}{R-n} v\vec{e}_n' - \vec{e}_n \psi \vec{e}_s \quad \Rightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}(u' - \psi' n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}(u-\psi n)\frac{1}{R}\vec{e}_n + \vec{e}_s \frac{R}{R-n} v\vec{e}_n - \vec{e}_s \frac{R}{R-n} v \frac{1}{R}\vec{e}_s - \vec{e}_n \psi \vec{e}_s \quad \Leftrightarrow$$

$$\nabla\vec{u} = [\frac{R}{R-n}(u' - \psi' n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + [\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_s\vec{e}_n - \psi\vec{e}_n\vec{e}_s .$$

In conjugate tensor, order of the basis vectors is changed

$$(\nabla\vec{u})_{\mathrm{c}} = [\frac{R}{R-n}(u' - \psi' n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + [\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_n\vec{e}_s - \psi\vec{e}_s\vec{e}_n .$$

Definition of the small (linear) strains $\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$ gives first

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_s\vec{e}_n - \frac{1}{2}\psi\vec{e}_n\vec{e}_s +$$

$$\frac{1}{2}[\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_n\vec{e}_s - \frac{1}{2}\psi\vec{e}_s\vec{e}_n \quad \Leftrightarrow$$

$$\vec{\vec{\varepsilon}} = [\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v' - \psi](\vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s)\,.$$

Hence as the components of strain are the multipliers of the basis vector pairs with the same order of indices

$$\varepsilon_{ss} = \frac{R}{R-n}(u'-\psi'n - \frac{1}{R}v) \quad \text{and} \quad \varepsilon_{sn} = \varepsilon_{ns} = \frac{1}{2}\frac{R}{R-n}(\frac{1}{R}u + v' - \psi)\,.$$

Mapping $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ defines the cylindrical $(r,\phi,z)-$coordinate system. Use (in detail) the generic formula

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}\text{, where } \eta\in\{r,\phi,z\}$$

to find the derivatives of the basis vectors.

Solution

Basis vectors of the $(r,\phi,z)-$coordinate system are obtained as derivatives of the position vector $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ with respect to $(r,\phi,z)-$coordinates. As the position vector is expressed in terms of the constants basis vectors of a Cartesian system, the outcome is a relationship between the basis vectors of the spherical and Cartesian systems (i.e. the matrix $[F]$ needed in the generic expression for the basis vector derivatives)

$$\frac{\partial}{\partial r}\vec{r}=\cos\phi\vec{i}+\sin\phi\vec{j}\quad\Rightarrow\quad\left|\frac{\partial}{\partial r}\vec{r}\right|=1,$$

$$\frac{\partial}{\partial\phi}\vec{r}=-r\sin\phi\vec{i}+r\cos\phi\vec{j}\quad\Rightarrow\quad\left|\frac{\partial}{\partial\phi}\vec{r}\right|=r,$$

$$\frac{\partial}{\partial z}\vec{r}=\vec{k}\quad\Rightarrow\quad\left|\frac{\partial}{\partial z}\vec{r}\right|=1.$$

The relationship between the basis vectors can be written as

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial r)/|\partial\vec{r}/\partial r|\\ (\partial\vec{r}/\partial\phi)/|\partial\vec{r}/\partial\phi|\\ (\partial\vec{r}/\partial z)/|\partial\vec{r}/\partial z|\end{Bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

in which the matrix satisfies $[F]^{-1}=[F]^{\mathrm{T}}$. The generic formula for the partial derivatives of the basis vectors gives now

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix},\ \leftarrow$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}-\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix},\ \leftarrow$$

and

$$\frac{\partial}{\partial z}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos\phi&-\sin\phi&0\\ \sin\phi&\cos\phi&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}.$$

Derive the gradient expression of the spherical $(\theta,\phi,r)-$coordinate system, when the mapping defining the coordinate system is given by $\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$.

Solution

According to the generic recipe (formulae collection)

$$\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial\alpha)/|\partial\vec{r}/\partial\alpha|\\ (\partial\vec{r}/\partial\beta)/|\partial\vec{r}/\partial\beta|\\ (\partial\vec{r}/\partial\gamma)/|\partial\vec{r}/\partial\gamma|\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\quad \eta\in\{\alpha,\beta,\gamma\},$$

$$\nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial\gamma\end{Bmatrix}\ \text{ where } [H]=\begin{bmatrix}\partial r_x/\partial\alpha & \partial r_y/\partial\alpha & \partial r_z/\partial\alpha\\ \partial r_x/\partial\beta & \partial r_y/\partial\beta & \partial r_z/\partial\beta\\ \partial r_x/\partial\gamma & \partial r_y/\partial\gamma & \partial r_z/\partial\gamma\end{bmatrix},$$

in which $\alpha=\theta$, $\beta=\phi$, and $\gamma=r$ in the present case. Matrices $[F]$ and $[H]$ depend on the mapping

$$\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})=r_x(\theta,\phi,r)\vec{i}+r_y(\theta,\phi,r)\vec{j}+r_z(\theta,\phi,r)\vec{k}\,.$$

By definition

$$\vec{e}_\theta=\frac{\partial\vec{r}}{\partial\theta}/\left|\frac{\partial\vec{r}}{\partial\theta}\right|=\cos\theta\cos\phi\vec{i}+\cos\theta\sin\phi\vec{j}-\sin\theta\vec{k}\,,$$

$$\vec{e}_\phi=\frac{\partial\vec{r}}{\partial\phi}/\left|\frac{\partial\vec{r}}{\partial\phi}\right|=-\sin\phi\vec{i}+\cos\phi\vec{j}\,,$$

$$\vec{e}_r=\frac{\partial\vec{r}}{\partial r}/\left|\frac{\partial\vec{r}}{\partial r}\right|=\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k}$$

and therefore

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\ \text{ so }\ [F]=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}.$$

According to the mapping, the relationship between the components of the position vector in the Cartesian and cylindrical systems are $r_x=r\sin\theta\cos\phi$, $r_y=r\sin\theta\sin\phi$, and $r_z=r\cos\theta$

$$[H]=\begin{bmatrix}\partial r_x/\partial\theta & \partial r_y/\partial\theta & \partial r_z/\partial\theta\\ \partial r_x/\partial\phi & \partial r_y/\partial\phi & \partial r_z/\partial\phi\\ \partial r_x/\partial r & \partial r_y/\partial r & \partial r_z/\partial r\end{bmatrix}=\begin{bmatrix}r\mathrm{c}\theta\mathrm{c}\phi & r\mathrm{c}\theta\mathrm{s}\phi & -r\mathrm{s}\theta\\ -r\mathrm{s}\theta\mathrm{s}\phi & r\mathrm{s}\theta\mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}.$$

Gradient follows now from the generic recipe

$$\nabla=\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\theta\\ \partial/\partial\phi\\ \partial/\partial r\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}^{\mathrm{T}}([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\partial/\partial\theta\\ \partial/\partial\phi\\ \partial/\partial r\end{Bmatrix}.$$

Let us calculate first the matrix inside the parenthesis

$$[H][F]^{\mathrm{T}}=\begin{bmatrix}r\mathrm{c}\theta\mathrm{c}\phi & r\mathrm{c}\theta\mathrm{s}\phi & -r\mathrm{s}\theta\\ -r\mathrm{s}\theta\mathrm{s}\phi & r\mathrm{s}\theta\mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & -\mathrm{s}\phi & \mathrm{s}\theta\mathrm{c}\phi\\ \mathrm{c}\theta\mathrm{s}\phi & \mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi\\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta\end{bmatrix}=\begin{bmatrix}r & 0 & 0\\ 0 & r\mathrm{s}\theta & 0\\ 0 & 0 & 1\end{bmatrix}\Leftrightarrow$$

$$([H][F]^{\mathrm{T}})^{-1}=\begin{bmatrix}r & 0 & 0\\ 0 & r\mathrm{s}\theta & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1/r & 0 & 0\\ 0 & 1/(r\mathrm{s}\theta) & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Substituting into the gradient expression

$$\nabla=\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}1/r & 0 & 0\\ 0 & 1/(r\mathrm{s}\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\partial/\partial\theta\\ \partial/\partial\phi\\ \partial/\partial r\end{Bmatrix}=\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}+\vec{e}_r\frac{\partial}{\partial r}.$$

Compute the derivatives of the basis vectors, gradient operator, and curvature for the cylindrical shell geometry with the mid-surface representation $\vec{r}_0(\phi,z) = R(\cos\phi\vec{i} + \sin\phi\vec{j}) + z\vec{k}$ in terms of coordinates (ϕ, z). Notice that the order of the coordinates differs from that of the lecture notes, which affects, e.g., direction of $\vec{e}_n$.

Solution

Solution of the problems consists of two parts. The aim of the first part is to derive the derivatives of the basis vectors and the gradient operator representation of a curvilinear system (by direct calculation or from the formulae collection). The second part is just application of the curvature tensor definition

$$\vec{\kappa} = (\nabla\vec{e}_n)_{\mathrm{c}}$$

describing in a concise manner the way the coordinate system deviates from being Cartesian. The mid-surface curvature tensor corresponds to $n = 0$. Measures of $\vec{\kappa}$, like Gaussian curvature or mean curvature, describe the geometry of the mid-surface.

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells)

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0 / \partial\phi) / |\partial\vec{r}_0 / \partial\phi| \\ (\partial\vec{r}_0 / \partial z) / |\partial\vec{r}_0 / \partial z| \\ \vec{e}_\phi \times \vec{e}_z \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Derivatives of the basis vector follow from the generic formula

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix}$$

the remaining being zeros. Mapping for the mid-surface is kind of labelling system for the particles (points) on that surface. To have a labelling system for all the particles (points) of a thin body, a relative position vector is also needed, so

$$\vec{r}(\phi,z,n) = \vec{r}_0(\phi,z) + n\vec{e}_n(\phi,z).$$

The relative position vector defines also the line segments perpendicular to the mid-surface (an important concept in plate theory). The Hessian of the mapping between the Cartesian and thin-body (ϕ,z,n) – coordinates takes the form

$$[H]=\begin{bmatrix}\partial r_x/\partial\phi & \partial r_y/\partial\phi & \partial r_z/\partial\phi\\ \partial r_x/\partial z & \partial r_y/\partial z & \partial r_z/\partial z\\ \partial r_x/\partial n & \partial r_y/\partial n & \partial r_z/\partial n\end{bmatrix}=\begin{bmatrix}-(n+R)\sin\phi & (n+R)\cos\phi & 0\\ 0 & 0 & 1\\ \cos\phi & \sin\phi & 0\end{bmatrix}$$

having the inverse

$$[H]^{-1}=\begin{bmatrix}-\sin\phi/(n+R) & 0 & \cos\phi\\ \cos\phi/(n+R) & 0 & \sin\phi\\ 0 & 1 & 0\end{bmatrix}.$$

The generic form of the gradient expression gives now (formulae collection)

$$\nabla=\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_z\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}-\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\\ \cos\phi & \sin\phi & 0\end{bmatrix}\begin{bmatrix}-\dfrac{\sin\phi}{n+R} & 0 & \cos\phi\\ \dfrac{\cos\phi}{n+R} & 0 & \sin\phi\\ 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\partial/\partial\phi\\ \partial/\partial z\\ \partial/\partial n\end{Bmatrix}\Leftrightarrow$$

$$\nabla=\vec{e}_\phi\frac{1}{n+R}\frac{\partial}{\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_n\frac{\partial}{\partial n}.$$

Curvature is obtained from the gradient of the normal vector

$$\vec{\kappa}_{\mathrm{c}}=\vec{e}_\phi\frac{1}{n+R}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_z\frac{\partial\vec{e}_n}{\partial z}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=\vec{e}_\phi\vec{e}_\phi\frac{1}{n+R}$$

giving at the mid-surface ($n=0$) $\vec{\kappa}=\vec{e}_\phi\vec{e}_\phi\dfrac{1}{R}$.

Consider the mid-surface mapping $\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Solution

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells). With the present mid-surface $(r,\phi)-$coordinates

$$\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j} \text{ and } \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0 / \partial r)/\left|\partial\vec{r}_0 / \partial r\right| \\ (\partial\vec{r}_0 / \partial\phi)/\left|\partial\vec{r}_0 / \partial\phi\right| \\ \vec{e}_r \times \vec{e}_\phi \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Expressions of the basis vectors of the curvilinear system are

$$\frac{\partial}{\partial r}\vec{r}_0 = 2r\cos(2\phi)\vec{i} + 2r\sin(2\phi)\vec{j} \quad \Rightarrow \quad \vec{e}_r = (\frac{\partial}{\partial r}\vec{r}_0)/|\frac{\partial}{\partial r}\vec{r}_0| = \cos(2\phi)\vec{i} + \sin(2\phi)\vec{j},$$

$$\frac{\partial}{\partial\phi}\vec{r}_0 = -2r^2\sin(2\phi)\vec{i} + 2r^2\cos(2\phi)\vec{j} \quad \Rightarrow \quad \vec{e}_\phi = (\frac{\partial}{\partial\phi}\vec{r}_0)/|\frac{\partial}{\partial\phi}\vec{r}_0| = -\sin(2\phi)\vec{i} + \cos(2\phi)\vec{j},$$

$$\vec{e}_n = \vec{e}_r \times \vec{e}_\phi = [\cos(2\phi)\vec{i} + \sin(2\phi)\vec{j}] \times [-\sin(2\phi)\vec{i} + \cos(2\phi)\vec{j}] = \vec{k}.$$

In a more compact form

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ -\sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \text{in which } [F]^{-1} = [F]^{\mathrm{T}}.$$

Direct use of the definition gives (just take the derivatives on both sides of the relationship above and use inverse of the same relationship to replace the basis vectors of the Cartesian system by the basis vectors of the $(r,\phi,n)-$system)

$$\frac{\partial}{\partial r}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) & 0 \\ \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -2\sin(2\phi) & 2\cos(2\phi) & 0 \\ -2\cos(2\phi) & -2\sin(2\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) & 0 \\ \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix},$$

$$\frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}. \leftarrow$$

Gradient in the $(r,\phi,n)-$ system follows from the mapping

$$\vec{r}(r,\phi,n)=\vec{r}_0+\vec{\rho}=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}+n\vec{k}$$

and the generic formula in terms of $[F]$ and $[H]$ with

$$\begin{Bmatrix}\partial\vec{r}/\partial r\\ \partial\vec{r}/\partial\phi\\ \partial\vec{r}/\partial n\end{Bmatrix}=\begin{bmatrix}2r\cos(2\phi)&2r\sin(2\phi)&0\\-2r^2\sin(2\phi)&2r^2\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\quad\Rightarrow$$

$$[H][F]^{\mathrm{T}}=\begin{bmatrix}2r\cos(2\phi)&2r\sin(2\phi)&0\\-2r^2\sin(2\phi)&2r^2\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2r&0&0\\0&2r^2&0\\0&0&1\end{bmatrix}\Rightarrow$$

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}2r&0&0\\0&2r^2&0\\0&0&1\end{bmatrix}^{-1}\begin{Bmatrix}\partial/\partial r\\ \partial/\partial\phi\\ \partial/\partial n\end{Bmatrix}=\vec{e}_r\frac{1}{2r}\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}\ . \leftarrow$$

Curvature of the mid-surface ($n=0$)

$$\vec{\kappa}_{\mathrm{c}}=\nabla\vec{e}_n=\vec{e}_r\frac{1}{2r}\frac{\partial\vec{e}_n}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=0$$

which indicates that the mid-surface is flat.