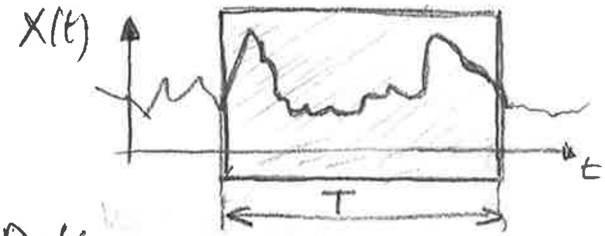
Elements of noise theory for electrical circuits

- Noise: determines the smallest signal that you can measure, so it limits the dynamical range from below



$X(t)$ = random variable, time-dependent

- We will work with ergodic, stationary processes.

Define:

- average of X: $\overline{X} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt$ T = time window

- average power of X:

$$\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)\,dt$$

For example, you have a resistor, then

$$\frac{\overline{V^2}}{R} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \frac{V^2(t)}{R}\,dt$$

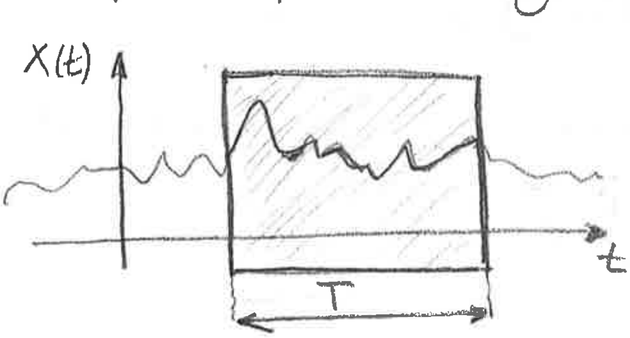
$\frac{V^2(t)}{R}$ → instantaneous power

- autocorrelation function:

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t)\,X(t+\tau)\,dt$$

- obviously:
  - $\overline{X^2} = R_{xx}(\tau=0) \geq |R_{xx}(\tau)|$ for $|\tau|>0$
  - $R_{xx}(\tau) = R_{xx}(-\tau)$ — this is an even function!

- power spectral density (PSD): $S_x(f)$



→ collect data in a window T from your oscilloscope —
— then take the Fourier transform $X(f)$

$$S_x(f) \equiv \lim_{T\to\infty}\frac{|X(f)|^2}{T}$$

$\frac{|X(f)|^2}{T}$ = periodogram (provides an estimate of $S_x(f)$)

Units: if X = Voltage, then $V^2 Hz^{-1}$.

• What is the relation between $S_x(f)$ and the autocorrelation function?

The Wiener-Khinchin theorem

$$S_x(f) = \mathcal{F}[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-2\pi i f \tau} d\tau$$

So the PSD = the Fourier transform of the autocorrelation!

or: $$R_{xx}(\tau) = \int_{-\infty}^{\infty} df\, e^{2\pi i f \tau} S_x(f)$$

– note how interesting this is! if you measure $S_x(f)$ you have access to any correlation (at any τ).

For $\tau = 0$,

$$\overline{x^2} \equiv R_{xx}(0) = \int_{-\infty}^{\infty} df \cdot S_x(f)$$

We will not prove the Wiener-Khinchin theorem (it's not difficult – if you are interested you can find the proof in textbooks)

* but let us give an alternative justification based on Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, x^2(t) = \int_{-\infty}^{\infty} df\, |X(f)|^2$$

power calculated in time-domain $\equiv$ power calculated in frequency-domain

So $$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt\, X^2(t)$$

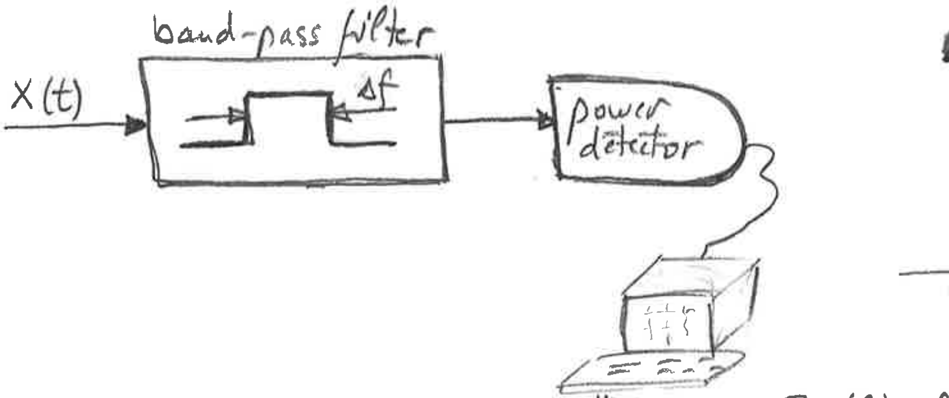
$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} df\, |X(f)|^2$$

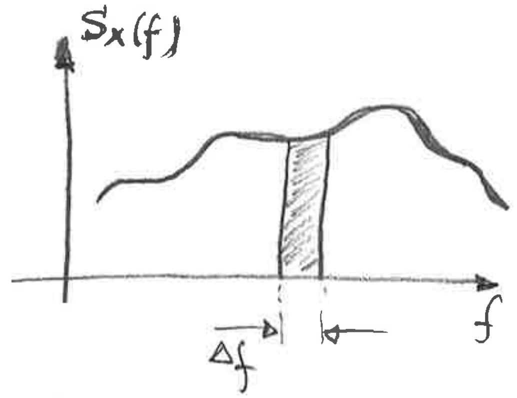
$$= \int_{-\infty}^{\infty} df \left(\lim_{T\to\infty} \frac{|X(f)|^2}{T} \right) = \int_{-\infty}^{\infty} df \cdot S_x(f)$$

So $$\overline{x^2} = \int_{-\infty}^{\infty} df\, S_x(f)$$

• What does this mean physically?

$\mathcal{S}_x(f)\cdot df$ = power in the frequency interval df





will measure $S_x(f)\cdot\Delta f$

However, there is one little inconvenient.
We can always have time taken at least approximately from $-\infty$ to $+\infty$.
But what does it mean to have a negative frequency?
All generators that we have in the lab produce a "positive" frequency.
To bypass this issue it is convenient to restrict ourselves only to positive frequencies, which is what we will do in the rest of the lecture.

How do we do this? Formally, note that for classical fields $X(t)$

$$X(-\omega) = X^*(\omega) \text{ because } X(t) \text{ is real,}$$

therefore $S(f) = S(-f)$.

So
$$\overline{X^2} = \int_{-\infty}^{\infty} df\, S_x(f) = \int_0^{\infty} df\, S_x(f) + \int_0^{\infty} df\, S_x(-f) = \int_0^{\infty} df\, 2S_x(f)$$

Let us introduce $\mathcal{S}_x(f) \equiv 2S_x(f)$ = single-sided spectral density (as opposed to S_x which is double-sided)

$$\overline{X^2} = \int_0^{\infty} df \cdot \mathcal{S}_x(f)$$

A convenient way to express this (but not very rigurous) is

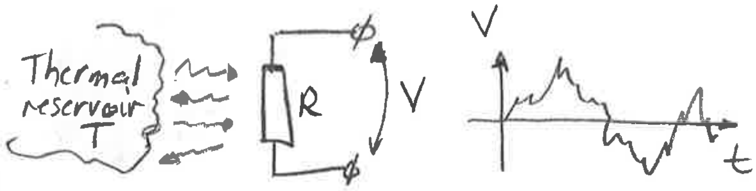
$$\mathcal{S}_x(f) = \left.\frac{\Delta\overline{X^2}}{\Delta f}\right|_{\Delta f\to 0}$$

But it works!

Units of $\mathcal{S}_x(f)$: if $x \equiv$ voltage, then V^2/Hz.

Thermal noise in electrical circuits

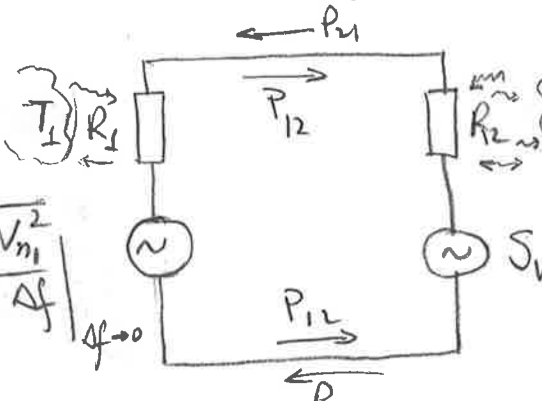
Resistor at a finite temperature:



- Discovered by John Johnson (Bell Labs) in 1926
- Theory by Harry Nyquist (Bell Labs)

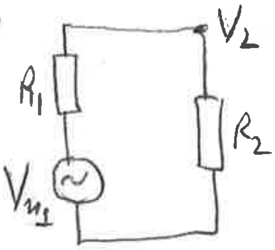
Derivation of the Nyquist formula (H. Nyquist, Thermal agitation of electric charge in conductors, Phys. Rev. 32, 110 (1928))

1. Estimates and general considerations



$$S_{V_1} = \frac{\Delta \overline{V_{n_1}^2}}{\Delta f}\bigg|_{\Delta f \to 0} \qquad S_{V_2} = \frac{\Delta \overline{V_{n_2}^2}}{\Delta f}\bigg|_{\Delta f \to 0}$$

Calculate P_{12}:



$$V_2 = V_{n_1} \frac{R_2}{R_1 + R_2}$$

$$P_{12} = \frac{V_2^2}{R_2} = V_{n_1}^2 \cdot \frac{R_2}{(R_1+R_2)^2}$$

Similarly for P_{21}:

$$P_{21} = V_{n_2}^2 \cdot \frac{R_1}{(R_1+R_2)^2}$$

1.a. Case $T_1 = T_2 = T$

2nd law of thermodynamics: $\overline{P_{12}} = \overline{P_{21}}$.

Otherwise by collecting the difference between them you would be able to extract work from a single reservoir at temperature T!

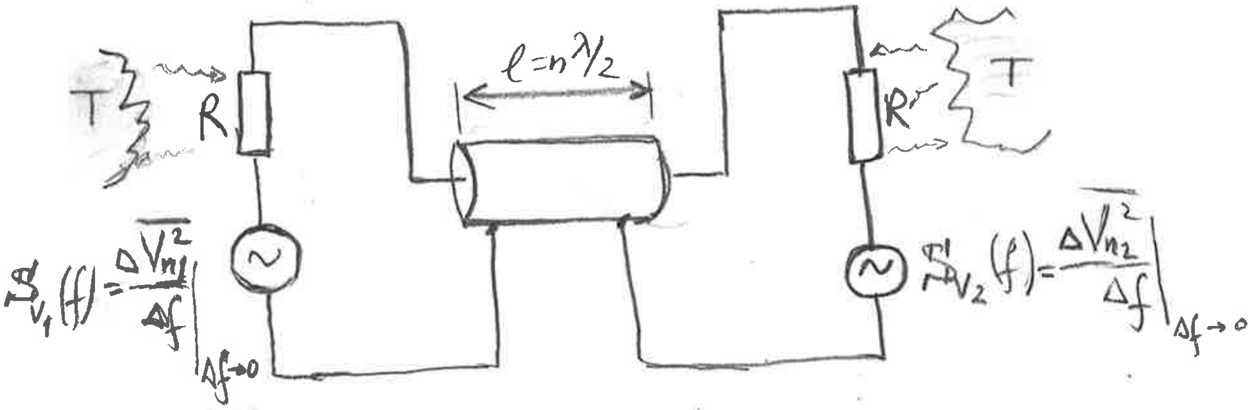
$$\overline{P_{12}} = \overline{P_{21}} \Rightarrow \frac{\overline{V_{n_1}^2}}{R_1} = \frac{\overline{V_{n_2}^2}}{R_2} = \text{const.} \Rightarrow \text{we expect } \overline{V_n^2} \sim R$$

2.b. Case $T_1 \neq T_2$

We expect $(\overline{P_{12}} - \overline{P_{21}}) \sim (T_1 - T_2)$. This is just because usually energy $\sim K_B T$ in Thermodynamics

$$\left.\begin{array}{l}\text{Make } T_2 = 0\,K \\ P_{21} = 0\end{array}\right\} \Rightarrow \overline{P_{12}} \sim T_1 \Rightarrow \overline{V_n^2} \sim T$$

2. A more rigurous derivation



This time $R_1 = R_2 = R$ but the connection is done through a TL (transmission line).

Consider a frequency interval df.

We want to find $S_V(f)$:
for a bandwidth df
around f.

How many modes dn of the TL are in df?

v = speed of light in the TL

$$l = n\frac{\lambda}{2} = n\frac{v}{2f} \Rightarrow f = n\frac{v}{2l} \Rightarrow \underline{dn = \frac{2l}{v}df}$$

(a) $$dE = dn \cdot \underbrace{\frac{hf}{e^{\frac{hf}{K_B T}} - 1}}_{\text{Planck's energy per mode}} = \frac{2l}{v} \cdot \frac{hf}{e^{hf/K_B T} - 1}\, df$$ = energy contained in the TL in the frequency interval df

On the other hand, from $\overline{P_{12}} = \overline{V_{n_1}^2} \frac{R}{(R+R)^2} = \frac{\overline{V_{n_1}^2}}{4R} = \overline{P_{21}}$, $\overline{P} = \frac{\overline{V_n^2}}{4R}$

we have $$d\overline{P} = \frac{1}{4R} S_V(f) \cdot df$$

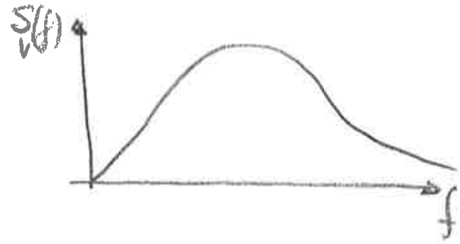
(b) Therefore $$dE = 2 \cdot \frac{l}{v} \cdot d\overline{P} = 2\frac{l}{v} \cdot \frac{S_V(f)}{4R} \cdot df$$

2 sources inject this power in TL

time of propagation in TL

Now combine (a) and (b) to find $S_V(f)$:

$$\Rightarrow \boxed{S_V(f) = \frac{4hfR}{e^{hf/k_BT} - 1}}$$



• <u>Classical limit</u>

If the temperature is high, $k_BT \gg hf \Rightarrow$

$$\boxed{S_V(f) \simeq 4k_BTR}$$

This could have been obtained also by applying the <u>equipartition theorem</u>:

$$dE = \frac{2\ell}{v} \cdot \underbrace{2}_{\substack{\text{two degrees}\\\text{of freedom}\\\text{in the TL;}\\ \vec{E} \text{ and } \vec{B}}} \cdot \underbrace{\frac{k_BT}{2}}_{\substack{\frac{k_BT}{2} \text{ per}\\\text{degree of}\\\text{freedom.}}} \cdot df \qquad \text{and} \qquad dE = \frac{2\ell}{v}\frac{S_V(f)}{4R}df$$

$$\Rightarrow \underline{S_V(f) \simeq 4k_BTR}$$

• <u>Cross-over quantum-classical</u>

From the exponent we can define $f_{cr} = \frac{k_BT}{h}$

$$S_V(f) = \frac{4hfR}{e^{hf/k_BT} - 1} \simeq 4hfR \cdot \frac{1}{\cancel{1} + \frac{hf}{k_BT} + \frac{1}{2}\left(\frac{hf}{k_BT}\right)^2 - \cancel{1}}$$

$$\simeq 4k_BTR\left(1 - \frac{hf}{2k_BT}\right)$$

$$f_{cr}\Big|_{T=300K} = \frac{1.38 \cdot 10^{-23}\, J/K \cdot 300K}{6.62 \cdot 10^{-34}\, J \cdot s} = 6.2\, THz$$

$$f_{cr}\Big|_{T=10mK} = \frac{1.38 \cdot 10^{-23}\, J/K \cdot 10^{-2}K}{6.62 \cdot 10^{-34}\, J \cdot s} = 0.2\, GHz = 200\, MHz$$

– Thermal noise appears from the motion of electrons in the resistor. But where is the electron charge e in our final equations ?!? It is in fact "hidden" in the microscopic models that we might use for R (for example the Drude model, $\sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m}$)

Discussion:

- How much power it emits? $\overline{P}_{tot} = \int_0^\infty df \, \frac{1}{R} \cdot \frac{\overline{V_n^2}}{\Delta f}\Big|_{\Delta f \to 0} = \int_0^\infty df \, \frac{S_V(f)}{R}$

$$= \frac{4(k_B T)^2}{h} \int_0^\infty \frac{u \, du}{e^u - 1} = \frac{2\pi^2}{3} \frac{(k_B T)^2}{h}$$

$$u = \frac{hf}{k_B T}$$

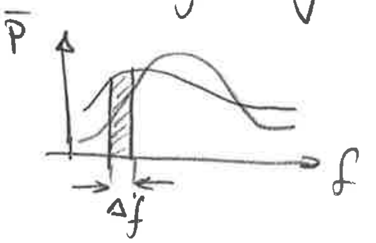
- Does not depend on R!
- Let's calculate it, $T = 300K$

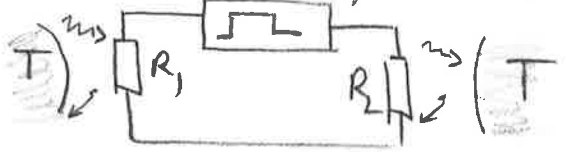
$$\overline{P}_{tot} = \frac{2\pi^2}{3} \cdot \frac{(1.38 \cdot 10^{-23} \cdot 300)^2}{6.62 \cdot 10^{-34}} \frac{J}{s} = 172 nW$$

very small!

- Note that $\Delta \overline{P} = \frac{\Delta \overline{V_n^2}}{R} = \frac{S_V(f)}{R} \cdot \Delta f$ = independent of R!

Why it is so? It is a consequence of argument 1a in the beginning. Had it been otherwise, it would be possible to create work from a single reservoir by using 2 resistors and a bandpass filter Δf.





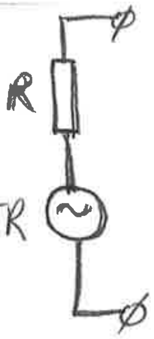
This would contradict the 2nd law of thermodynamics (perpetuum mobile of 2nd kind)

- A more general statement: the *fluctuation–dissipation theorem*.
  - in order for an object to emit radiation (due to fluctuations) it must also dissipate! Not restricted to electric circuits! E.g. in nanomechanics $S_{FF}(\omega) = 2 m k_B T \cdot \gamma$

Example: A non-dissipative circuit element (ideal capacitor, inductance) will not emit radiation. There is no fluctuating voltage across an ideal capacitor! The ideal capacitor only stores energy. Also if you have an I-V characteristic and define dV/dI it does not mean that there will be accompanying fluctuations!

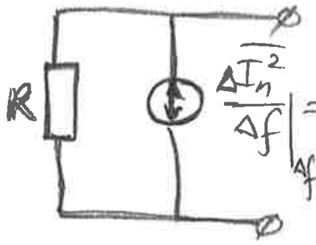
- **Circuit equivalent representation**

Thévenin



$$\left.\frac{\overline{\Delta V_n^2}}{\Delta f}\right|_{\Delta f \to 0} = 4 k_B T R$$

Norton equivalent



$$\left.\frac{\overline{\Delta I_n^2}}{\Delta f}\right|_{\Delta f \to 0} = \frac{4 k_B T}{R}$$

(~) = ideal voltage source (zero internal resistance) with fluctuating voltage

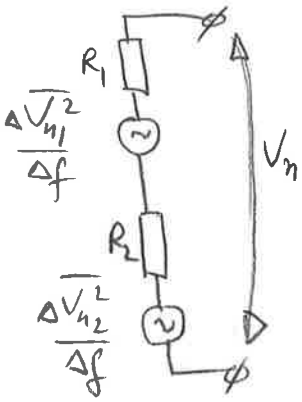
(↕) = ideal current source (∞ internal resistance) with fluctuating current

Does it work?

Example: Resistors in series:

$S_{V_{n1}} = 4 k_B T$

$S_{V_{n2}} = 4 k_B T$



$$\frac{\overline{\Delta V_n^2}}{\Delta f} = \frac{\overline{\Delta V_{n1}^2}}{\Delta f} + \frac{\overline{\Delta V_{n2}^2}}{\Delta f}$$ ← these fluctuations are uncorrelated!

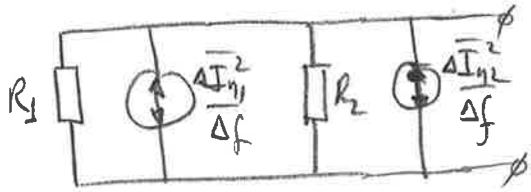
so $S_{V_n} = 4 k_B T (R_1 + R_2)$

$R = R_1 + R_2$ = series

Resistors in parallel:

$$\left.\frac{\overline{\Delta I_{n1}^2}}{\Delta f}\right|_{\Delta f \to 0} = \frac{4 k_B T}{R_1} \equiv S_{I_{n1}}$$

$$\left.\frac{\overline{\Delta I_{n2}^2}}{\Delta f}\right|_{\Delta f \to 0} = \frac{4 k_B T}{R_2} \equiv S_{I_{n2}}$$

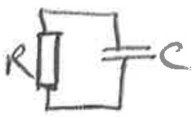


$$\frac{\overline{\Delta I_n^2}}{\Delta f} = \frac{\overline{\Delta I_{n1}^2}}{\Delta f} + \frac{\overline{\Delta I_{n2}^2}}{\Delta f}$$ ← again uncorrelated fluctuations of current

$$R_{\parallel} = \frac{R_1 R_2}{R_1 + R_2}$$

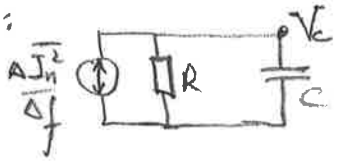
So $$S_{I_n} = \frac{4 k_B T}{R_{\parallel}}$$

∘ Coupling of a dissipative element to a non-dissipative one.
What happens for example when the fluctuations of the resistor charge and discharge a capacitor?



$$Z_c = \frac{1}{iC\cdot 2\pi f} = \frac{1}{iC\omega} \qquad \omega = 2\pi f$$

Norton:



$$\frac{\Delta \overline{I_n^2}}{\Delta f}\bigg|_{\Delta f\to 0} = \frac{4k_BT}{R}$$

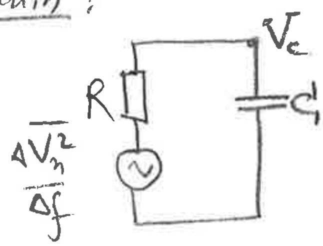
$$\frac{V_c}{Z_c} + \frac{V_c}{R} = I_n \quad \text{therefore}$$

$$V_c = \frac{I_n}{\frac{1}{R} + iC\cdot 2\pi f} = \frac{I_n R}{1+2\pi i f RC}$$

So we obtain

$$\frac{\Delta \overline{V_c^2}}{\Delta f}\bigg|_{\Delta f\to 0} = \frac{4k_BTR}{1+(2\pi RC)^2 f^2}$$

Thevenin:



$$V_c = V_n\cdot \frac{Z_c}{R+Z_c} = V_n \cdot \frac{1}{1+i2\pi RCf}$$

$$\Rightarrow \frac{\Delta \overline{V_c^2}}{\Delta f}\bigg|_{\Delta f\to 0} = \frac{\Delta \overline{V_n^2}}{\Delta f}\bigg|_{\Delta f\to 0}\cdot \frac{1}{1+(2\pi RC)^2 f^2} = \frac{4k_BTR}{1+(2\pi RC)^2 f^2}$$

Same result! So per bandwith the capacitors experiences $S_{V_c}(f) = \frac{\Delta \overline{V_c^2}}{\Delta f}\bigg|_{\Delta f\to 0} = \frac{4k_BTR}{1+(2\pi RC)^2 f^2}$

What is the total $\overline{V_c^2}$ (over all frequencies)?

$$\overline{V_c^2}_{(tot)} = \int_0^\infty S_{Vc}(f)\cdot df = \frac{4k_BT}{2\pi C}\cdot \int_0^\infty df\, \frac{2\pi RC}{1+(2\pi RCf)^2} = \frac{4k_BT}{2\pi C}\int_0^\infty dy\cdot \frac{1}{1+y^2}$$

$$y = 2\pi RCf$$

$$= \frac{4k_BT}{2\pi C}\cdot \arctan y\Big|_0^\infty$$

$$= \frac{4k_BT}{2\pi C}\cdot\frac{\pi}{2} = \frac{k_BT}{C}$$

So $\boxed{\overline{V_c^2}_{(tot)} = \frac{k_BT}{C}}$

This is called K-T-over C-noise.

- Note that it does not depend on R! Why? - Noise increases with R but bandwidth decreases $\sim \frac{1}{R}$. So they compensate exactly.
- Suppose we want to increase the speed of a circuit (say a switch, AD convertor, sampler, etc.) by making C smaller (smaller RC constant). Then $\overline{V_c^2}_{(tot)}$ will increase! The circuit will be more noisy!
- We can derive the formula above from the equipartition theorem

$$\frac{1}{2}C\overline{V_c^2}_{(tot)} = \frac{1}{2}k_BT \longleftarrow \text{There is only 1 degree of freedom in a capacitor.}$$

• Based on this we can make the following heuristic argument:
for a resistor we can define an intrinsic capacitance C_R by

$$\underbrace{\tfrac{1}{2}K_B T}_{\text{equipartition Theorem}} = \underbrace{\int df \cdot \tfrac{1}{2} C_R \cdot \overline{V_n^2}}_{\text{= total energy available per all frequencies}}$$

or

$$\frac{K_B T}{C_R} = \int_0^\infty S_{V_n}(f)\,df$$

where $S_{V_n}(f) = \dfrac{\overline{\Delta V_n^2}}{\Delta f}$

$$\frac{K_B T}{C_R} = \frac{2\pi^2}{3} \cdot \frac{(K_B T)^2}{h}$$

$$\Rightarrow \boxed{C_R = \frac{3}{2\pi^2} \frac{h}{K_B T R}}$$

$R = 1\,K\Omega$, $T = 300\,K$ $\longrightarrow$ $C_R \sim 24\,aF$

$R = 1\,\Omega$, $T = 300\,K$ $\longrightarrow$ $C_R \sim 24\,fF$

$R = 1\,\Omega$, $T = 3\,K$ $\longrightarrow$ $C_R \sim 24\,pF$

– C_R establishes a cutoff in the circuit. Having a capacitance bellow C_R in the circuit is meaningless – the circuit will be dominated by the resistive C_R.

References

- Hermann A. Haus – Electromagnetic Noise and Quantum Optics
- Ali Hajimiri – Analog Circuit Design (lectures at Caltech)
- A.M. Zagoskin – Quantum Engineering

Highly advanced:

- C. Gardiner – Quantum Noise
- M. Cattaneo, GSP – arXiv:2103.16946