Problem set 2

Due March 15, 10:15 AM

Noise in quantum circuits (Lecture 3)

(1) Let a random process be given by

$$x(t) = A\sin(\omega_0 t + \phi)$$

where A and ω_0 are constants and ϕ is a random variable that is uniformaly distribute over $[0, 2\pi]$. Discuss whether $x(t)$ is a stationary process.

(2) Suppose that a random process is described as $x(t) = At$, such that A is a random variable uniformly distributed over $[-2, 2]$.

a Sketch a few sample function from the ensemble.

b Calculate the auto-correlation function $R_x(t_1, t_2)$ of the random variable $x(t)$.

c Is the process stationary? Check whether it is ergodic.

(3) Determine the power and the rms value for each of the following signals:

(a)$5\cos(300\pi t + \frac{\pi}{6})$ (b)$5\sin 55t \sin \pi t$ (c)$e^{j\alpha t}\sin\omega_0 t$ (d) $10\sin 5t\cos 10t \cdot u(t)$
where $u(t)$ is the unit step function.

(4) Show that $y(t) = e^{-\alpha t}$ starting at $-\infty$ is neither an energy nor a power signal for any real α. Nonetheless, when α is purely imaginary, it is a power signal with power equal to 1 regardless of the value of α

(5) Let $x(t)$ be a superposition of two sinusoidal signals oscillating at different frequencies

$$x(t) = A_1\cos(\Omega_1 t + \phi_1) + A_2\cos(\Omega_2 t + \phi_2)$$

(a) Calculate the power of $x(t)$.

(b) Assume now that $\Omega_1 = \Omega_2$, calculate again the power of $x(t)$ and discuss the difference between the two cases.

(6) An RC circuit has two parallel resistors R_1, and R_2. Calculate the rms value of the thermal noise voltage V_0 across the capacitor in the following two cases:

(a) We treat each resistor independently with respective thermal noise voltages of PSD $2KTR_1$, $2KTR_2$.

(b) We consider the equivalent resistor of their parallel combination with its thermal noise voltage of PSD $2KTR_{\text{eq}}$.

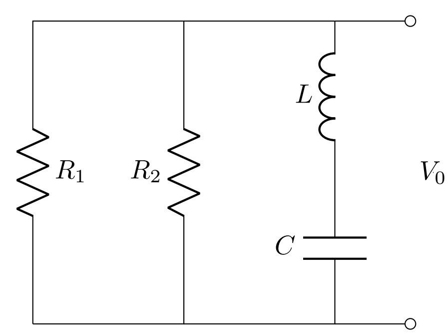

Figure 1

Superconductivity (Lecture 4)

In a Josephson Junction (JJ) the super current inside each superconducting lump is characterized by the number of cooper pairs n and a macroscopic phase. Thus the super-current wave-function in each region is written as $\Psi_i = \sqrt{n_j} e^{i\theta_i}$.

(1) Define now an operator $N = \sum_n n |n\rangle \langle n|$ representing the number of cooper pairs on one side of the junction, such that

$$N |N\rangle = n |N\rangle$$

where $n = 0, \pm 1, \pm 2, ...$ Similarly, define a set of eigen-states representing the phase difference between the super current wavefunctions. Both sets are related via the following pair of transformations

$$|N\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{-in\varphi} |\varphi\rangle$$

$$|\varphi\rangle = \sum_{n=-\infty}^{\infty} e^{in\varphi} |N\rangle$$

(a) Verify the previous relation by inserting the definition of $|\varphi\rangle$ into $|N\rangle$

(b) Calculate the inner product between two different phase states $\langle \varphi | \varphi' \rangle$, where $\{|N\rangle\}$ constitute an orthogonal set of states.

(2) Define now the following operator

$$e^{i\hat{\varphi}} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \, e^{i\varphi'} |\varphi'\rangle \langle \varphi'|$$

(a) Show how $e^{i\hat{\varphi}}$ acts on on a phase state $|\varphi\rangle$

(b) What happens when we act with $e^{i\hat{\varphi}}$ on a number state N? Does the relation between $|N\rangle$ and $|\varphi\rangle$ resembles something we have encountered before? Explain.

(c) Derive a number state representation for both $e^{i\hat{\varphi}}$ and its Hermitian conjugate.

When Cooper pairs start to tunnel between the two superconducting regions, the Hamiltonian describing the tunneling process can be written as

$$H_{\mathrm{T}} = \frac{-E_{\mathrm{J}}}{2} \sum_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|)$$

where E_{J} is the Josephson energy we derived in lecture 4.

(3) Show that $|\varphi\rangle$ is an eigenstate of the H_{T} with eigen value $-E_{\mathrm{J}} \cos \varphi$.

(4) Now define the current operator as $I = 2e\frac{dN}{dt}$. Derive a number state representation of the operator I. **Hint:** Recall the Heisenberg equation of motion for operator evolution.

(5) Show that $I |\varphi\rangle = I_{\mathrm{c}} \sin \varphi |\varphi\rangle$, where $I_{\mathrm{c}} = \frac{2e}{\hbar} E_{\mathrm{J}}$.