Solutions, Exam Feb. 24, 2021

1. A gas obeys the following equation of state,

$$p(V-Nb) = Nk_B T \exp\left(-\frac{a}{V-Nb}\right)$$

(a) Presented in one of the exercise sessions.

(b) We have

$$dU = TdS - pdV = T\left[\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV\right] - pdV$$

$$= T\left[\frac{\partial S}{\partial T}\right]_V dT + \left[T\left(\frac{\partial S}{\partial V}\right)_T - p\right] dV$$

from Maxwell relation: $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$dU = T\left(\frac{\partial S}{\partial T}\right)_V dT + \left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right] dV$$

from equation of state $\left(\frac{\partial P}{\partial T}\right)_V = \frac{Nk_B}{V-Nb}\exp\left(-\frac{a}{V-Nb}\right) = \frac{P}{T}$

$$\Rightarrow dU = T\left(\frac{\partial S}{\partial T}\right)_V dT + \overset{\to 0}{\left[T\cdot\frac{\cancel{P}}{\cancel{T}} - P\right]} dV$$

$$\Rightarrow dU = T\left(\frac{\partial S}{\partial T}\right)_V dT$$

U is just function of T.

2.(b) $\Omega_f = 3^N \longrightarrow S_f = k_B \ln \Omega_f = N k_B \ln 3$

$\Omega_i = 1 \longrightarrow S_i = k_B \ln 1 = 0$

$$\Delta S_{system} = S_f - S_i = N k_B \ln 3$$

$$Q_{system} = T \Delta S_{system} = N k_B T \ln 3$$

ideal gas

$$\Delta W = \int_{V_i}^{3V_i} p\,dV = \int_{V_i}^{3V_i} \frac{N k_B T}{V}\,dV = N k_B T \ln 3$$

T is not changing $\Rightarrow U_i = U_f$

$$0 \equiv \Delta U = \Delta W + \Delta Q$$

heat to the system

$$Q = -\Delta Q = -(-\Delta W) = \Delta W = N k_B T \ln 3$$

3.(a) Presented in details in one of the lectures.

(b) $I = 0.48 \frac{\Delta}{eR_T} \sqrt{\frac{k_B T}{\Delta}}$ $\dot{Q}_{NIS} = 0.59 \frac{\Delta^2}{e^2 R_T} \left(\frac{k_B T}{\Delta}\right)^{3/2}$

The efficiency in a refrigerator in given by

$$\eta = \frac{\text{Cooling power}}{\text{Power by the source}}$$

$$\text{Power} = VI = \frac{\Delta}{e} \cdot 0.48 \frac{\Delta}{eR_T} \sqrt{\frac{k_B T}{\Delta}} = 0.48 \frac{\Delta^2}{e^2 R_T} \sqrt{\frac{k_B T}{\Delta}}$$

$$\Rightarrow \eta = \frac{0.59 \cancel{\frac{\Delta^2}{e^2 R_T}} \left(\frac{k_B T}{\Delta}\right)^{3/2}}{0.48 \cancel{\frac{\Delta^2}{e^2 R_T}} \sqrt{\frac{k_B T}{\Delta}}} = 1.23 \frac{k_B T}{\Delta}$$

4.(a) $Z = e^{-\beta E/2} + e^{-\beta(-E/2)} = 2\cosh\left(\frac{\beta E}{2}\right)$

population: $P_i = \frac{e^{-\beta E_i}}{Z}$

population of the ground state: $P_g = \frac{e^{\beta E/2}}{e^{\beta E/2} + e^{-\beta E/2}} = \frac{1}{1+e^{-\beta E/2}}$ ②

" " excited " : $P_e = \frac{e^{-\beta E/2}}{e^{\beta E/2} + e^{-\beta E/2}} = \frac{1}{1+e^{\beta E/2}}$

4.(b) $\dot{P}_g = -\Gamma_\uparrow P_g + \Gamma_\downarrow P_e$

$\dot{P}_g = -\Gamma_\Sigma P_g + \Gamma_\downarrow$ (1) , where $\Gamma_\Sigma = \Gamma_\uparrow + \Gamma_\downarrow$

from (1): $\int_{P_g(0)=1}^{P_g(t)} \frac{dP_g}{P_g - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}} = \int_0^t -\Gamma_\Sigma \, dt$ ③

$\Rightarrow \ln \frac{P_g(t) - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}}{1 - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}} = -\Gamma_\Sigma t$

$P_g(t) = \frac{\Gamma_\downarrow}{\Gamma_\Sigma} + \left(1 - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}\right) e^{-\Gamma_\Sigma t}$

②

$P_g(\infty) = \frac{\Gamma_\downarrow}{\Gamma_\Sigma} = \frac{\Gamma_\downarrow}{\Gamma_\downarrow + \Gamma_\uparrow} = \frac{1}{1+\frac{\Gamma_\uparrow}{\Gamma_\downarrow}} \equiv \frac{1}{1+e^{-\beta E/2}}$

we have $\frac{\Gamma_\uparrow}{\Gamma_\downarrow} = e^{-\beta E}$ (Detailed Balance condition)

4. (c) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)|$$

$$= \frac{1}{2}|g\rangle\langle g| + \frac{1}{2}|e\rangle\langle e|$$

$$+ \frac{1}{2}|g\rangle\langle e| + \frac{1}{2}|e\rangle\langle g|$$

$$\rho_{gg}(0) \equiv P_g(0) = \langle g|\rho(0)|g\rangle = \frac{1}{2}$$

Now we solve Eq. ③ with the new condition $P_g(0) = \frac{1}{2}$

$$\int_{1/2}^{P_g(t)} \frac{dP_g}{P_g - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}} = -\Gamma_\Sigma t$$

$$\Rightarrow \ln \frac{P_g(t) - \Gamma_\downarrow/\Gamma_\Sigma}{\frac{1}{2} - \Gamma_\downarrow/\Gamma_\Sigma} = -\Gamma_\Sigma t$$

$$P_g(t) = \frac{\Gamma_\downarrow}{\Gamma_\Sigma} + \left(\frac{1}{2} - \frac{\Gamma_\downarrow}{\Gamma_\Sigma}\right) e^{-\Gamma_\Sigma t}$$

$$\dot{\rho}_{ge} = -\frac{1}{2}\Gamma_\Sigma \rho_{ge}$$

$$\int_{1/2}^{\rho_{ge}(t)} \frac{d\rho_{ge}}{\rho_{ge}} = -\frac{1}{2}\int \Gamma_\Sigma dt \Rightarrow \rho_{ge} = \frac{1}{2} e^{-\frac{1}{2}\Gamma_\Sigma t}$$