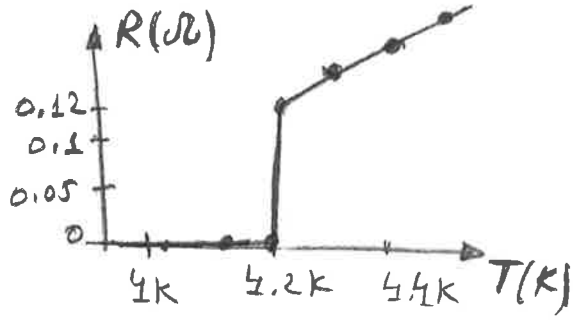
Superconductivity

1911 - Heike Kamerlingh Onnes

Electrical resistance of Hg (metal!) dropped to $< 10^{-5}\Omega$ at $T_c = 4.2K$



Other typical metals become superconductors:

$$T_c = 1.2K \text{ for Al}$$
$$T_c = 7.2K \text{ for Pb}$$
$$T_c = 9.2K \text{ for Nb}$$

1986 – discovery of high-T_c compounds by J.G. Bednorz and K.A. Müller

$$T_c = 95K \text{ for } YBa_2Cu_3O_{7-\delta}$$
$$T_c = 125K \text{ for } Tl_2Ba_2Ca_2Cu_3O_{10}$$
$$T_c = 133K \text{ for } HgBa_2Ca_2Cu_3O_{8+\delta}$$

These are not metals, they are ceramic materials at room temperature!

Meissner effect

- In the beginnings of superconductivity research it was hoped that the electromagnetic properties could be derived from the property of infinite conductivity.

$$\sigma = \infty \qquad \left.\begin{array}{l}\vec{J} = \sigma \cdot \vec{E} \\ \vec{J} = \text{finite}\end{array}\right| \Rightarrow \vec{E} = 0 \Rightarrow \vec{\nabla} \times \vec{E} = 0$$

$$\text{Maxwell: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Bigg\} \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0$$

So $\vec{B}$ = constant inside a superconductor and also we expect it to be dependent on the way it was cooled down (e.g. either in the presence or absence of the magnetic field)

BUT in 1933 Meissner and Ochsenfeld discovered that $\vec{B} = 0$.
The magnetic field inside the superconductor is not just constant, but it is exactly <u>zero</u>. Magnetic field lines are <u>expelled</u>.
A superconductor is a perfect diamagnet.

Theory development

- 1935 – phenomenological theory developed by F. & H. London (two brothers!)
- 1957 – BCS (Bardeen–Cooper–Schrieffer) theory
- high-T_c superconductivity – maybe YOU?

Elements of London theory

Consider a particle of mass m^* and charge e^*. It will turn out that $m^* = 2m_e$ and $e^* = -2e$; these particles are Cooper pairs, and a complete understanding of what they are is provided by the BCS theory.

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad$ $\vec{A}$ = vector potential

Schrödinger equation: $$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})\right)^2 \psi(\vec{r},t)$$

Note: the Hamiltonian of a free particle in a magnetic field is

$$H = \frac{\Pi^2}{2m^*} \quad \text{where } \underbrace{\Pi}_{\text{canonical momentum}}(\vec{r}) = -i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})$$

Probability density

$$P(\vec{r},t) = |\psi(\vec{r},t)|^2$$

$$\frac{\partial P(\vec{r},t)}{\partial t} = \frac{\partial \psi^*(\vec{r},t)}{\partial t}\psi(\vec{r},t) + \psi^*(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t}$$

$$= \frac{i}{\hbar}\Big\{\Big[\frac{1}{2m^*}\left(i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})\right)^2\psi^*(\vec{r},t)\Big]\psi(\vec{r},t) - \psi^*(\vec{r},t)\Big[\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})\right)^2\psi(\vec{r},t)\Big]\Big\}$$

$$= -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

So $$\frac{\partial P(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

where $$\vec{J}(\vec{r},t) = \frac{1}{2m^*}\Big[\left(-i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\Big]^*\psi(\vec{r},t) + \frac{1}{2m^*}\psi^*(\vec{r},t)\cdot\Big[\left(-i\hbar\vec{\nabla} - e^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\Big]$$

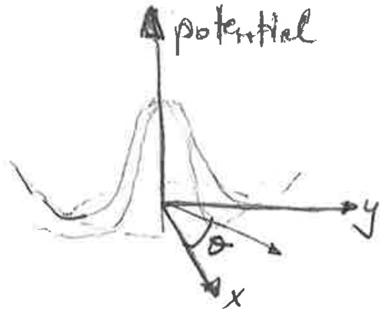
Key point: The wavefunction $\psi(\vec{r},t)$ for a superconductor can be regarded as an <u>order</u> <u>parameter</u> (a macroscopic wavefunction!)

Ginzburg-Landau order parameter $\quad \psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\, e^{i\theta(\vec{r},t)}$

$\rho(\vec{r},t)$ = charge density

$\theta(\vec{r},t)$ = superconducting phase

Appears as a result of a <u>broken</u> <u>symmetry</u>.



So $\vec{J}(\vec{r},t) = \frac{\hbar}{m^*}\left[\vec{\nabla}\theta(\vec{r},t) - \frac{e^*}{\hbar}\vec{A}(\vec{r},t)\right]\rho(\vec{r},t)$ $\quad$ Electrical current: $\vec{J} = e^*\vec{J}$

$$\vec{J}_s(\vec{r},t) = \frac{\hbar e^*}{m^*}\left[\vec{\nabla}\theta(\vec{r},t) - \frac{e^*}{\hbar}\vec{A}(\vec{r},t)\right]\rho(\vec{r},t)$$

supercurrent

Gauge-invariant phase: $\begin{cases}\theta \to \theta + \frac{e^*}{\hbar}\chi \\ \vec{A} \to \vec{A} + \vec{\nabla}\chi\end{cases}$

CONSEQUENCES:

Let us consider $\rho(\vec{r},t) \equiv \rho_s \equiv$ constant = density of "superelectrons" or Cooper pairs

and θ = constant in $\vec{r}$, $\vec{\nabla}\theta = 0$

- PERFECT CONDUCTIVITY

$$\vec{J}_s = -\frac{e^{*2}}{m^*}\rho_s\vec{A} \Rightarrow$$

$$\frac{d\vec{J}_s(\vec{r},t)}{dt} = -\frac{e^{*2}}{m^*}\rho_s\frac{d\vec{A}(\vec{r},t)}{dt} \quad \text{or} \quad \boxed{\frac{d\vec{J}_s(\vec{r},t)}{dt} = +\frac{e^{*2}\rho_s}{m^*}\vec{E}(\vec{r},t)}$$

Recall Maxwell: $\begin{cases}\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A}\end{cases}$

What does it mean?

Take a ballistic superelectron (no collisions with atoms, impurities etc.)

$$\left.\begin{aligned} m^*\frac{d\vec{v}_s}{dt} &= e^*\vec{E} \\ \vec{J}_s &= \rho_s \cdot e^* \cdot \vec{v}_s \end{aligned}\right\} \Rightarrow \boxed{\frac{d\vec{J}_s}{dt} = \frac{e^{*2}\rho_s}{m^*}\vec{E}}$$

Note the difference w.r.t $\vec{J} = \sigma\vec{E}$ (Ohm's law)!

• MEISSNER EFFECT

Let us look at Maxwell's equations: $\begin{cases} \vec{\nabla}\cdot\vec{B}=0 \\ \vec{\nabla}\times\vec{B}=\mu_0\vec{J}_s \end{cases}$

Now $\vec{B}=\vec{\nabla}\times\vec{A}$ so

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{\text{we can use Coulomb gauge } \vec{\nabla}\cdot\vec{A}=0}-\vec{\nabla}^2\vec{A}=-\vec{\nabla}^2\vec{A}$$

So $\left.\begin{aligned} \vec{\nabla}^2\vec{A}&=-\mu_0\vec{J}_s \\ \vec{J}_s&=-\frac{e^{*2}}{m^*}\rho_s\vec{A} \end{aligned}\right\} \Rightarrow \boxed{\vec{\nabla}^2\vec{A}=\frac{\mu_0 e^{*2}\rho_s}{m^*}\vec{A}}$

Notation: $\lambda_L=\sqrt{\frac{m^*}{\mu_0\rho_s e^{*2}}}$ = London penetration length.

Since $\vec{J}_s=-\frac{e^{*2}\rho_s}{m^*}\vec{A}$ we have $\boxed{\vec{J}_s=-\frac{1}{\mu_0\lambda_L^2}\vec{A}}$ and $\boxed{\vec{\nabla}^2\vec{A}=\frac{1}{\lambda_L^2}\vec{A}}$

So $\mu_0\lambda_L^2\vec{J}_s=-\vec{A} \Rightarrow \mu_0\lambda_L^2\,\vec{\nabla}\times\vec{J}_s=-\vec{\nabla}\times\vec{A}=-\vec{B}$

$\Downarrow$

$$\mu_0\lambda_L^2\frac{\partial}{\partial t}\underbrace{(\nabla\times\vec{J}_s)}_{\text{but } \vec{J}_s=\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}}=-\vec{\nabla}\times\underbrace{\frac{\partial\vec{A}}{\partial t}}_{=-\vec{E}}$$

$\Downarrow$

$$\lambda_L^2\frac{\partial}{\partial t}\cdot\underbrace{\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{B})\right)}_{=\vec{\nabla}\cdot(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0})-\vec{\nabla}^2B}=\underbrace{\vec{\nabla}\times\vec{E}}_{=-\frac{\partial\vec{B}}{\partial t}}$$

$\Rightarrow \lambda_L^2\vec{\nabla}^2\vec{B}=+\vec{B}$ or $\left[\frac{1}{\lambda_L^2}-\vec{\nabla}^2\right]\vec{B}(\vec{r})=0$

Take $\vec{B}(\vec{r})=(0,0,B(z)) \Rightarrow B(z)=B_0\exp(-z/\lambda_L)$

This is the Meissner effect. The field decays exponentially in the superconductor.

To review: we found $\boxed{\vec{J}_s = -\frac{1}{\mu_0 \lambda_L^2} \vec{A}}$

and $\boxed{\nabla^2 \vec{A} = \frac{1}{\lambda_L^2} \vec{A}}$

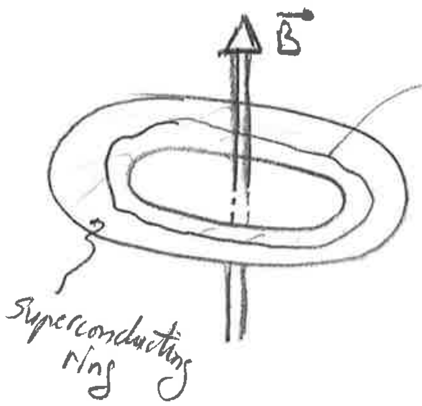
or $\boxed{\frac{d\vec{J}_s}{dt} = \frac{1}{\mu_0 \lambda_L^2} \vec{E}}$ – called 1^{st} London equation

$\boxed{\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{J}_s}$ – called 2^{nd} London equation

So the magnetic field can penetrate at most to depths $\simeq \lambda_L$.
Currents can flow in this region, but deep in the bulk they will be zero.

QUANTIZATION OF FLUX

So far we did not discuss the phase θ from the general expression of the current. Now it's the time.... with a spectacular example!



→ contour of integration deep in the bulk, where $\vec{J}(\vec{r}, t) = 0$

$$\Rightarrow \hbar \vec{\nabla} \theta(\vec{r}) = e^* \vec{A}(\vec{r})$$

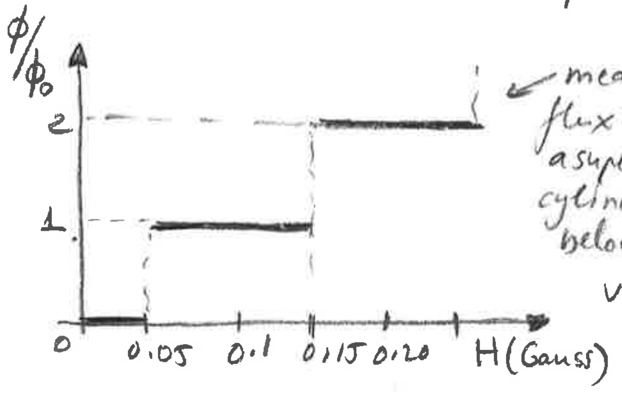
$$\Rightarrow \hbar \underbrace{\oint \vec{\nabla} \cdot \theta(\vec{r}) \, d\vec{\ell}}_{= 2\pi n} = e^* \Phi$$

Φ = magnetic flux

$\phi = \iint \vec{B} \, d\vec{S}$

$$\Rightarrow \quad \phi = \frac{2\pi n \hbar}{e^*} = \frac{h}{e^*} \cdot n$$

n = integer no.



← measured flux in a superconducting cylinder cooled below T_c in various magnetic fields

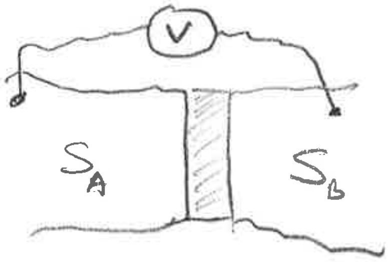
$$\Phi_0 = \frac{h}{2e} = \text{flux quantum}$$

$$= 2{,}067 \times 10^{-15} \, \text{Wb}$$

JOSEPHSON EFFECT

• What happens when we put a voltage across a <u>weak link</u> between two superconductors?

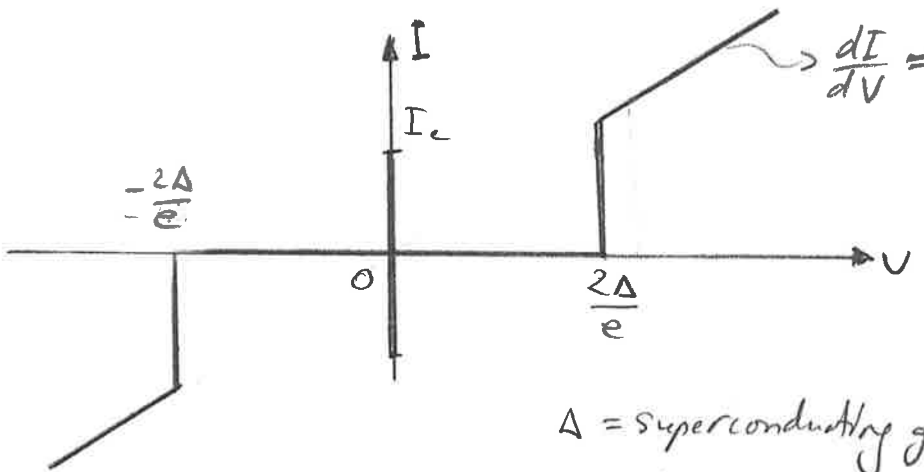
Weak link = can be a S-I-S (insulator between two superconductors)



S-N-S (a metal in-between)

S-s-S (a constriction)

• What we measure:



Δ = superconducting gap

$\Delta = 1.764\, k_B T_c$

T_c = critical temperature

(from BCS theory)

R_N = normal-state resistance

– Currents flowing for $|V| \geq \frac{2\Delta}{e}$ are no surprise – they are associated to breaking the Cooper pairs by the voltage.

But, at $V=0$ there is a current flowing, with max. value = I_c (critical current of the junction).

This is the Josephson effect.

– Circuit symbol

• A simple model based on the Ginzburg - Landau order parameter

The Cooper pairs experience a voltage difference V, Therefore there is a potential energy e^*V between superconductor A and superconductor B. Also the Cooper pairs can tunnel from $A \Leftrightarrow B$ with a certain probability, meaning that A and B are coupled by an interaction (hopping) term of strength K,

$$H_{eff} = \underbrace{\frac{1}{2}\begin{pmatrix} -e^*V & 0 \\ 0 & +e^*V \end{pmatrix}}_{\text{potential energy}} + \underbrace{\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}}_{\text{hopping}}$$

Order parameter : can be seen as a spinor: $\psi = \begin{pmatrix} \sqrt{\rho_A}\, e^{i\theta_A} \\ \sqrt{\rho_B}\, e^{i\theta_B} \end{pmatrix}$

Schrödinger equation: $i\hbar \frac{\partial}{\partial t}\psi = H_{eff}\,\psi$

$$\Rightarrow \frac{\partial \sqrt{\rho_A}}{\partial t} + i\sqrt{\rho_A}\frac{\partial \theta_A}{\partial t} = \frac{-e^*V}{2i\hbar}\sqrt{\rho_A} + \frac{K}{i\hbar}\sqrt{\rho_B}\, e^{i(\theta_B-\theta_A)}$$

$\downarrow$

equate the real and imaginary parts

$$\Rightarrow \begin{cases} \dfrac{\partial\sqrt{\rho_A}}{\partial t} = \dfrac{K}{\hbar}\sqrt{\rho_B}\sin(\theta_B-\theta_A) \\[2ex] \dfrac{\partial \theta_A}{\partial t} = +\dfrac{e^*V}{2\hbar} - \dfrac{K}{\hbar}\sqrt{\dfrac{\rho_B}{\rho_A}}\cos(\theta_B-\theta_A) \end{cases}$$

or $\frac{\partial \rho_A}{\partial t} = \frac{2K\sqrt{\rho_A\rho_B}}{\hbar}\sin\varphi$

where

$\varphi \equiv \theta_B - \theta_A$ = relative phase between the superconductors

Similarly for B:

$$\begin{cases} \dfrac{\partial\sqrt{\rho_B}}{\partial t} = -\dfrac{K\sqrt{\rho_A}}{\hbar}\sin(\theta_B-\theta_A) \\[2ex] \dfrac{\partial \theta_B}{\partial t} = \dfrac{-e^*V}{2\hbar} - \dfrac{K}{\hbar}\sqrt{\dfrac{\rho_B}{\rho_A}}\cos(\theta_B-\theta_A) \end{cases}$$

or $\frac{\partial \rho_B}{\partial t} = -\frac{2K\sqrt{\rho_A\rho_B}}{\hbar}\sin\varphi$

$e^* = -2e$

Let $I_c = \frac{2K\sqrt{\rho_A\rho_B}}{\hbar}$ = a constant that depends on the junction

$$\Rightarrow \boxed{I = I_c \sin\varphi}$$ current-phase relation

$$\boxed{\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V}$$ phase-voltage relation

flux

or: $V = \frac{\partial}{\partial t}\left(\frac{\phi_0}{2\pi} \cdot \varphi\right)$

very similar to Faraday's law

- Consequences:

DC JOSEPHSON EFFECT

$V = 0 \Rightarrow \frac{\partial \varphi}{\partial t} = 0 \Rightarrow \varphi = \text{const.}$

$I = I_c \sin\varphi$ — The current can reach a max. value of I_c

AC JOSEPHSON EFFECT

$V = \text{const} \neq 0 \Rightarrow \varphi = \frac{2e}{\hbar} V \cdot t$

$\rightarrow I = I_c \sin\left(\frac{2e}{\hbar} V \cdot t\right) = I_c \sin\left(2\pi \frac{V}{\phi_0} t\right)$

$f_J = \frac{V}{\phi_0}$ = Josephson frequency $= 483 \times 10^{12} V_0$ (Hz)

JOSEPHSON INDUCTANCE

$$\frac{\partial I}{\partial t} = \frac{\partial I}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial t} = I_c \cos\varphi \cdot \frac{2\pi}{\phi_0} V$$

or $V = L_J(\varphi) \frac{\partial I}{\partial t}$ $\quad L_J(\varphi)$ = Josephson inductance

$$L_J(\varphi) = \frac{\phi_0}{2\pi I_c \cos\varphi}$$

— depends on phase!
- can be ∞ if $\varphi = \frac{\pi}{2} + n\pi$

JOSEPHSON ENERGY

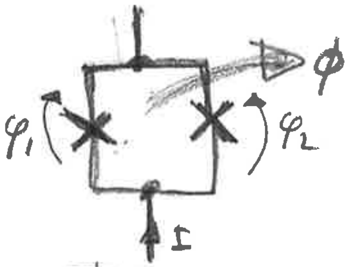
$$E_J = \int dt\, I \cdot V = \int d\varphi \cdot I_c \sin\varphi \cdot \frac{\phi_0}{2\pi}$$

$$E_J(\varphi) = -\frac{I_c \phi_0}{2\pi} \cos\varphi = -E_J \cos\varphi$$

$E_J = \frac{\phi_0 I_c}{2\pi}$ = Josephson energy

THE SQUID

- superconducting quantum interference device



- assume for simplicity identical junctions and negligible geometric inductance of the loop.

$$\varphi_1 - \varphi_2 = 2\pi \frac{\phi}{\phi_0} + 2\pi n$$

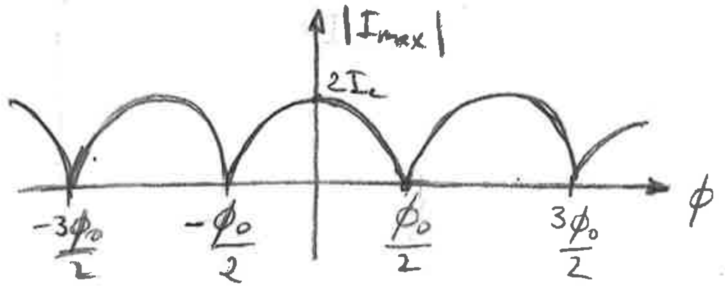
$$I = I_1 + I_2 = I_c \sin\varphi_1 + I_c \sin\varphi_2 = 2 I_c \sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}$$

Let $\varphi = \frac{\varphi_1+\varphi_2}{2} \Rightarrow$

$$I = 2 I_c \sin\varphi \cos\left(\frac{\pi\phi}{\phi_0} + \pi n\right) = I_{max}(\phi)\sin\varphi$$

The SQUID behaves as a single Josephson junction with critical current controlled by the magnetic flux.

The maximum current will be



$|I_{max}|$ never exceeds $2 I_c$,
and can be also zero (destructive interference in the two branches of the SQUID)

References

- Terry P. Orlando and Kevin A. Delin – Foundations of Applied Superconductivity
- D.R. Tilley and J. Tilley – Superfluidity and Superconductivity
- Antonio Barone and Gianfranco Paternò – Physics and Applications of the Josephson Effect