3 KINETICS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 11:

- ☐ Quantities and equations of classical elasticity
- ☐ Constitutive equation of linearly elastic isotropic material
- ☐ Principle of virtual work in solid mechanics
- ☐ Derivation of engineering models by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus

PREREQUISITE I: Tensor definitions and identities

- **Conjugate tensor** $\overleftrightarrow{a}_{\mathrm{c}}$: $\overleftrightarrow{a} \cdot \vec{b} = \vec{b} \cdot \overleftrightarrow{a}_{\mathrm{c}} \quad \forall \vec{b}$

- **Second order identity tensor** $\overleftrightarrow{I}$: $\overleftrightarrow{I} \cdot \vec{a} = \vec{a} \cdot \overleftrightarrow{I} = \vec{a} \quad \forall \vec{a}$

- **Fourth order identity tensor** $\overleftrightarrow{\overleftrightarrow{I}}$: $\overleftrightarrow{\overleftrightarrow{I}} : \overleftrightarrow{a} = \overleftrightarrow{a} : \overleftrightarrow{\overleftrightarrow{I}} = \overleftrightarrow{a} \quad \forall \overleftrightarrow{a}$

- **Associated vector** $\vec{a}$ **of an antisymmetric tensor** $\overleftrightarrow{a}$: $\vec{b} \cdot \overleftrightarrow{a} = \vec{a} \times \vec{b}$, when $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$

- **Scalar triple product** $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

- **Vector triple product** $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

- **Symmetric-antisymmetric double product** $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$ ja $\overleftrightarrow{b} = \overleftrightarrow{b}_{\mathrm{c}} \quad \Rightarrow \quad \overleftrightarrow{a} : \overleftrightarrow{b} = 0$

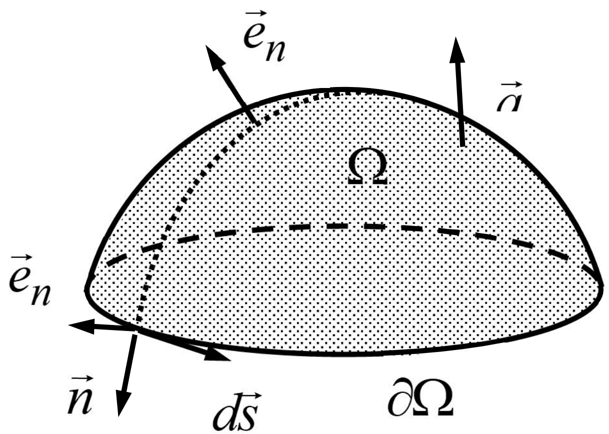
- **Symmetric-antisymmetric division** $\overleftrightarrow{a} = \overleftrightarrow{a}_s + \overleftrightarrow{a}_u = \frac{1}{2}(\overleftrightarrow{a} + \overleftrightarrow{a}_{\mathrm{c}}) + \frac{1}{2}(\overleftrightarrow{a} - \overleftrightarrow{a}_{\mathrm{c}})$

PREREQUISITE II: Integration by parts

- ☐ **Domain** $\Omega \subset \mathbb{R}^n$**, boundary** $\partial\Omega$, and a subset of the boundary $\partial\Omega_t$, $\partial\Omega_u$, etc.

- ☐ **Fundamental theorem of calculus** $a \in C^0(\Omega)$

$$\int_\Omega \nabla \cdot \vec{a} d\Omega = \int_{\partial\Omega} \vec{n} \cdot \vec{a} d\Gamma \quad \alpha \in \{x, y, z\}$$



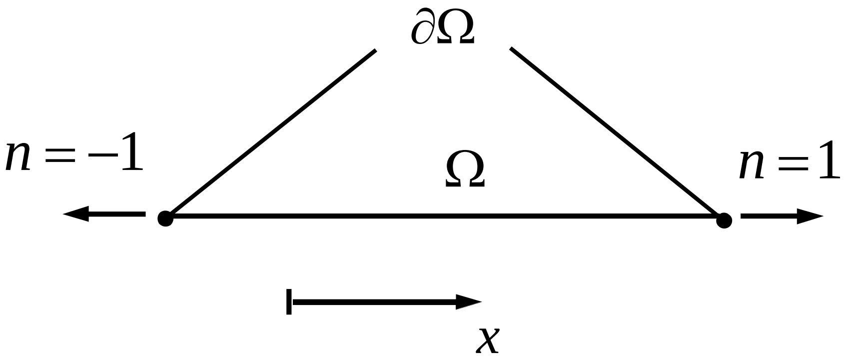
- ☐ **Kelvin-Stokes theorem** $a \in C^0(\Omega)$

$$\int_\Omega \nabla \times \vec{a} \cdot d\vec{A} = \oint_{\partial\Omega} \vec{a} \cdot d\vec{s} \quad \Rightarrow$$

$$\int_\Omega [(\nabla_0 \cdot \vec{a}) - (\nabla_0 \cdot \vec{e}_n)(\vec{a} \cdot \vec{e}_n)] dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds$$

Integration by parts is the basic tool to transform between the local and variational forms of a boundary value problem. In one-dimension and in connection with continuous functions $a, b \in C^0(\Omega)$.

$$\int_\Omega \frac{d}{dx}(ab)dx = \sum_{\partial\Omega} nab \quad \Leftrightarrow$$



$$\int_\Omega a\frac{db}{dx}dx = \sum_{\partial\Omega} (nab) - \int_\Omega b\frac{da}{dx}dx$$

Summing is over the boundary points and the unit normal to the boundary $n = \pm 1$. What kind of modification is needed if the functions are discontinuous on $I \subset \Omega$?

The generic form of the integration by parts formula is given by (Ω means domain and $\partial\Omega$ its boundary)

$$\int_{\Omega} a\frac{\partial b}{\partial \eta}d\Omega = \int_{\partial\Omega} (n_\eta ab)d\Gamma - \int_{\Omega} b\frac{\partial a}{\partial \eta}d\Omega \quad \eta \in \{x, y, z, \ldots\}.$$

In one-dimension, the first integral on the right-hand side should be interpreted as a sum. The well-known Gauss divergence theorem follows from the generic form:

$$\int_{\Omega} (\nabla \cdot \vec{a})dV = \int_{\partial\Omega} \vec{a} \cdot \vec{n}dA.$$

As a generic vector identity, Gauss theorem is valid also when a thin body has curved mid-surface geometry. However, all surfaces need to be accounted for correctly. Assuming that vector $\vec{a}$ does not depend on the transverse coordinate n, one obtains

$$\int_{V} \nabla \cdot \vec{a}dV = \int_{\partial V} \vec{n} \cdot \vec{a}dA \quad \Leftrightarrow$$

$$\int_{\Omega} [(\nabla_0 \cdot \vec{a}) - (\nabla_0 \cdot \vec{e}_n)(\vec{a} \cdot \vec{e}_n)]dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a})ds \, .$$

In the latter form, the area integral is over the mid-surface and the boundary integral over the boundary of the mid-surface and $\nabla = \nabla_0 + \vec{e}_n \partial / \partial n$. Term $\nabla_0 \cdot \vec{e}_n$ is twice the mean curvature of the mid-surface or the trace of curvature tensor $\overset{\leftrightarrow}{\kappa} : \overset{\leftrightarrow}{I}$ (one may consider the outcome also as a version of the Kelvin-Stokes theorem).

PREREQUISITE III: Fundamental lemma of variation calculus

- $a, b \in \mathbb{R}$: $ab = 0 \;\; \forall b \quad \Leftrightarrow \quad a = 0$

- $\{a\}, \{b\} \in \mathbb{R}^n$: $\{a\}^{\mathrm{T}} \{b\} = 0 \;\; \forall \{b\} \quad \Leftrightarrow \quad \vec{a} = 0$

- $\vec{a}, \vec{b} \in \mathbb{R}^3$: $\vec{a} \cdot \vec{b} = 0 \;\; \forall \vec{b} \quad \Leftrightarrow \quad \vec{a} = 0$

- $a, b \in C^0(\Omega)$: $\int_\Omega ab d\Omega = 0 \;\; \forall b \quad \Leftrightarrow \quad a = 0 \;$ in Ω

- $a, b \in C^2(\Omega)$: $\int_\Omega \nabla a \cdot \nabla b d\Omega = 0 \;\; \forall b \quad \Leftrightarrow \quad \nabla^2 a = 0$ in Ω, $a = \underline{a}$ or $\vec{n} \cdot \nabla a = 0$ on $\partial\Omega$

In connection with principle of virtual work, b is taken to be kinematically admissible variation $\delta\vec{u}$ of displacement $\vec{u}$ (vanishes whenever $\vec{u}$ is known).

3.1 CLASSICAL LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant. ←

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

BALANCE LAWS

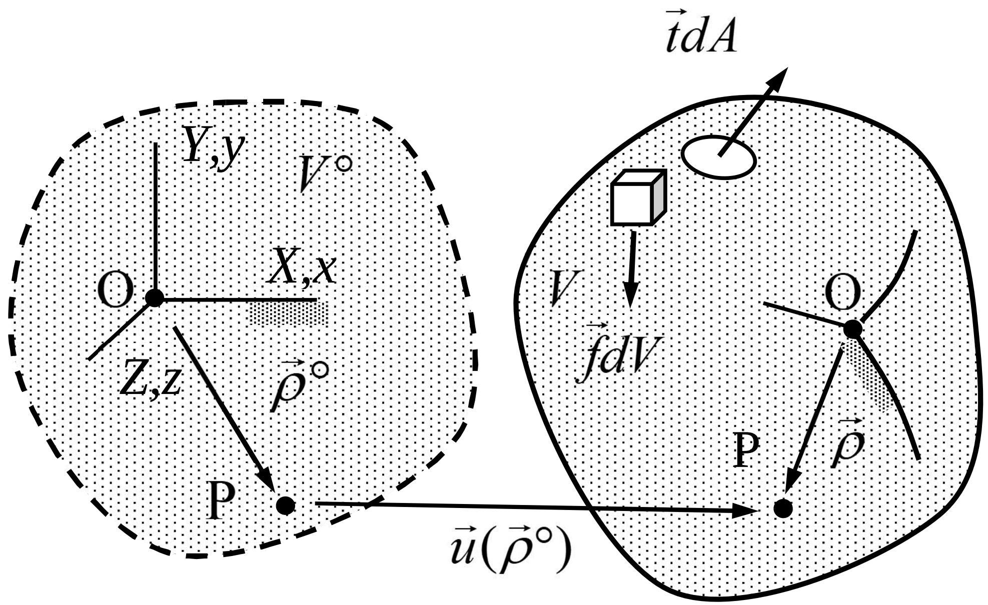
Given the solution $\vec{\vec{\sigma}}^\circ$, $\vec{u}^\circ$ (usually $\vec{u}^\circ = 0$) on the initial geometry V°, the goal is to find a new solution $\vec{\vec{\sigma}}$, $\vec{u}$ on V, when, e.g., external given forces are changed.

$\dot{m} = 0 \quad : \quad \rho^\circ = \rho J \quad \text{in } V$

$\dot{\vec{p}} = \vec{F} \quad : \quad \nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{in } V$

$\dot{\vec{p}} = \vec{F} \quad : \quad \vec{\sigma} \equiv \vec{n} \cdot \vec{\vec{\sigma}} = \vec{t} \quad \text{on } \partial V_t$

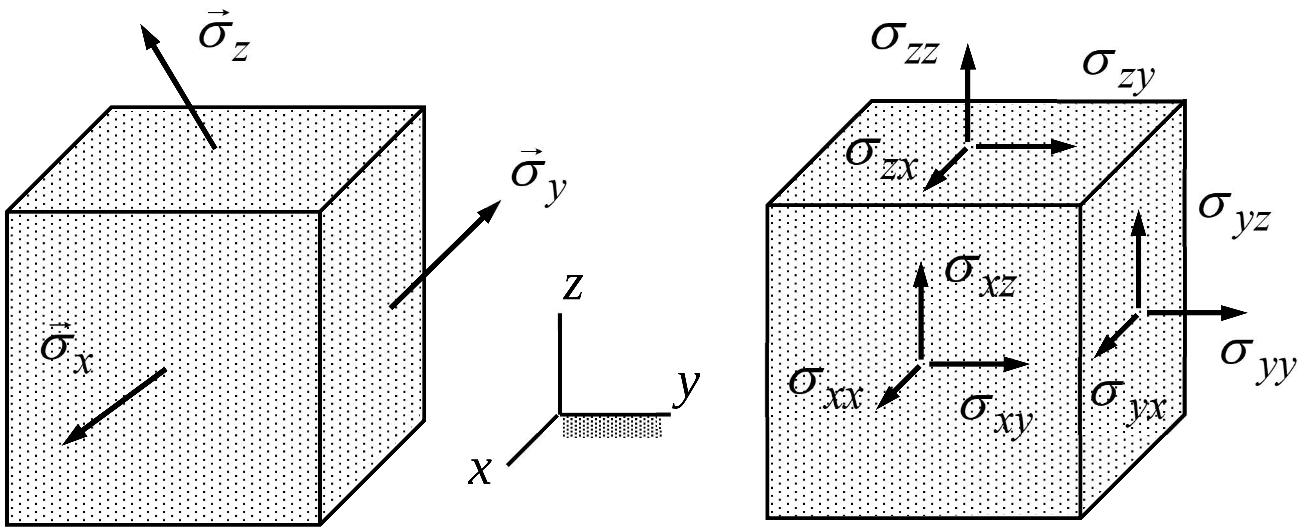
$\dot{\vec{L}} = \vec{M} \quad : \quad \vec{\vec{\sigma}} = \vec{\vec{\sigma}}_\mathrm{c} \quad \text{in } V$



A constitutive equation $f(\vec{\vec{\sigma}}, \vec{u}) = 0$ brings the material details into the model. For an unique solution, a displacement boundary condition is needed somewhere on ∂V.

TRACTION AND STRESS

In continuum mechanics, traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Cauchy stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component.

LINEAR STRAIN

Linear strain $\vec{\vec{\varepsilon}} = [\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]/2$ is a measure of material element shape deformation. The components of the (invariant) tensor quantity depend on the selection of the coordinate system. In a Cartesian $(x, y, z)-$coordinate system

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}.$$

Normal strains: $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}$, $\varepsilon_{zz} = \dfrac{\partial u_z}{\partial z}$

Shear strains: $\varepsilon_{xy} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y})$, $\varepsilon_{yz} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z})$, $\varepsilon_{zx} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z})$

LINEARLY ELASTIC MATERIAL

Assuming that the material coordinates coincide with symmetry (orthotropy) coordinates, the generalized Hooke's laws for the isotropic and orthotropic materials can be expressed in forms:

Component: $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}$, $\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$, and $\begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix}$

Tensor: $\vec{\sigma} = \overset{\leftrightarrow}{E} : \nabla \vec{u}$ where $\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$

in which the symmetric elasticity matrices $[E]$ and $[G]$ depend on material type.

The tensor form can be obtained from the component form by writing

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} : \overleftrightarrow{\varepsilon}, \quad \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix} : \overleftrightarrow{\varepsilon} \text{ and } \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix} : \overleftrightarrow{\varepsilon}$$

$$\begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = \overleftrightarrow{\sigma}.$$

Therefore, using the constitutive equations in their component forms

$$\overleftrightarrow{\sigma} = \left(\begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix} \right) : \overleftrightarrow{\varepsilon} \equiv \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \overleftrightarrow{\varepsilon} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla \vec{u}\,.$$ ⬅

CONSTITUTIVE EQUATION VARIANTS

Stress-displacement relationship of linearly elastic material model can be expressed in various equivalent forms depending on the symmetry conditions imposed on the fourth order elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$:

(a) $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$ and $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\mathrm{c}}$ $\Leftrightarrow$

(b) $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$ and $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\mathrm{c}}$ and $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\cdot\mathrm{c}}$ $\Leftrightarrow$ **Last index pair conjugate!**

(c) $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$ and $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\cdot\mathrm{c}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\mathrm{c}\cdot} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\mathrm{cc}}$ ←

Also, other kinetic conditions like $\sigma_{zz} = 0$ can be satisfied ‘a priori’ by the selection of elasticity tensor. The conditions of (c) are called as the minor and major symmetries.

ISOTROPIC MATERIAL

The generalized Hooke's law for an isotropic material follows with the elasticity matrices

$$[E] = E\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix},$$

$$[G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} = \frac{E}{2(1+\nu)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in which the material parameters E and ν are the Young's modulus and the Poisson's ratio, respectively, and $G = E/(2+2\nu)$ the shear modulus. Using these, one may deduce the elasticity matrices for the engineering models.

In the coordinate system invariant form $\overleftrightarrow{\sigma} = \overleftrightarrow{\overleftrightarrow{E}} : \overleftrightarrow{\varepsilon} = \overleftrightarrow{\overleftrightarrow{E}} : \nabla \vec{u}$, the elasticity tensor (satisfying the major and minor symmetries) is given by

$$\overleftrightarrow{\overleftrightarrow{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Elasticity tensor of plate model ($\sigma_{zz} = 0$)

$$\overleftrightarrow{\overleftrightarrow{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

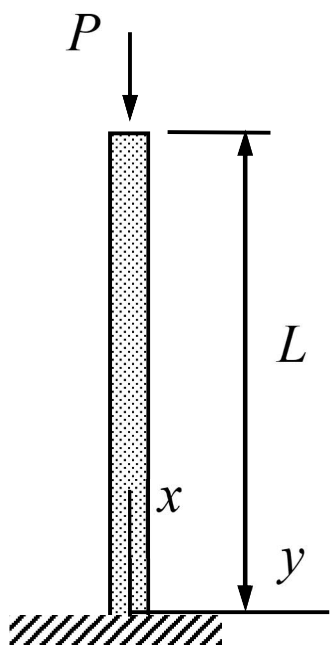
Elasticity tensor of the beam model ($\sigma_{yy} = \sigma_{zz} = 0$)

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Representation in some other system can be obtained from the Cartesian $(x, y, z)-$ system representation by using the relationships between the basis vectors. For example, in the cylindrical $(r, \phi, z)-$coordinate system

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix} + \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}.$$

EXAMPLE The cross section of the column is square of side length h. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. The column is loaded by a constant traction of magnitude P/h^2 at its free end. Determine stress $\vec{\vec{\sigma}}$ and displacement $\vec{u}$ starting from the generic equations for linear elasticity. Assume that the transverse (to the axis) displacement is not constrained by the support.



Answer $\vec{u} = \frac{P}{Eh^2}(-x\vec{i} + \nu y\vec{j} + \nu z\vec{k})$, $\quad \vec{\vec{\sigma}} = -\frac{P}{h^2}\vec{i}\vec{i}$

The component forms of the equilibrium equations and constitutive equations of a linearly elastic isotropic material in a Cartesian (x, y, z)−coordinate system

$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + \partial\sigma_{zx}/\partial z + f_x \\ \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{zy}/\partial z + f_y \\ \partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + f_z \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = G\begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}.$$

Let us assume that the only non-zero stress component $\sigma_{xx}(x)$ and displacement components $u_x = u(x)$, $u_y = v(y)$ and $u_z = w(z)$. The axial stress follows from the equilibrium equation and traction is known at the free end $x = L$. Therefore

$$\frac{d\sigma_{xx}}{dx} = 0 \quad 0 < x < L \quad \text{and} \quad \sigma_{xx}(L) = -\frac{P}{h^2} \quad \Rightarrow \quad \sigma_{xx}(x) = -\frac{P}{h^2}.$$

Generalized Hooke's law written for the uniaxial stress implies that

$$\frac{du}{dx} = \frac{\sigma_{xx}}{E} = -\frac{P}{Eh^2}, \quad \frac{dv}{dy} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}, \quad \frac{dw}{dz} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}.$$

Axial displacement vanishes at the support and the transverse displacement at the axis:

$$\frac{du}{dx} = -\frac{P}{Eh^2} \quad 0 < x < L \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = -\frac{P}{Eh^2}x, \quad \leftarrow$$

$$\frac{dv}{dy} = \nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < y < \frac{1}{2}h \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad v(y) = \nu\frac{P}{Eh^2}y, \quad \leftarrow$$

$$\frac{dw}{dz} = -\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < z < \frac{1}{2}h \quad \text{and} \quad w(0) = 0 \quad \Rightarrow \quad w(z) = \nu\frac{P}{Eh^2}z. \quad \leftarrow$$

3.2 PRINCIPLE OF VIRTUAL WORK

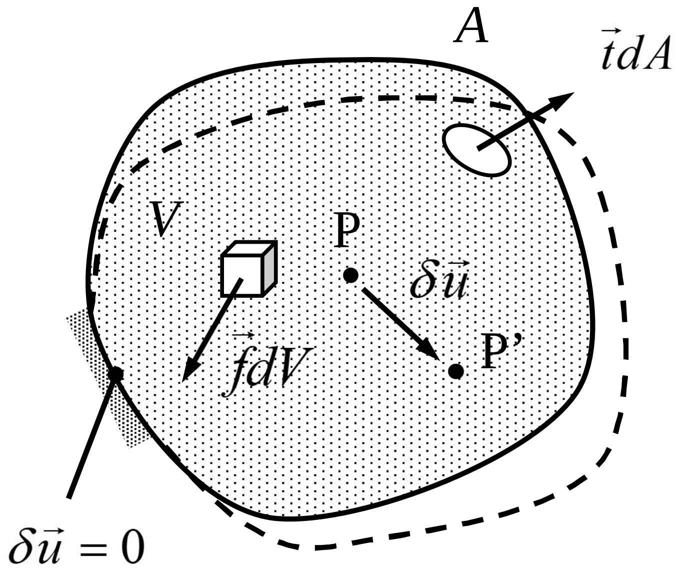
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\vec{u} \in U$ is just one representation of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

virtual work density

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_c) dV$$

$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta\vec{u}) dA$$



The details of the expressions vary case by case, but the principle itself does not!

In what follows, we skip some of the technical details and assume that displacement boundary conditions are satisfied ‘a priori’. The local and variational forms of elasticity problem are equivalent, i.e., the local form implies the variational form and the other way around. Let us consider first the derivation of the variational form:

$$\left.\begin{aligned} &\nabla\cdot\overleftrightarrow{\sigma}+\vec{f}=0 \quad\text{and}\quad \overleftrightarrow{\sigma}=\overleftrightarrow{\sigma}_{\mathrm{c}} \quad\text{in } V, \\ &\vec{u}-\underline{\vec{u}}=0 \quad\text{or}\quad \vec{n}\cdot\overleftrightarrow{\sigma}-\vec{t}=0 \quad\text{on } \partial V. \end{aligned}\right\} \quad \text{local form}$$

Multiplication of the momentum equation by virtual displacement $\delta\vec{u}$, integration over the solution domain, and integration by parts with $(\nabla\cdot\overleftrightarrow{a})\cdot\vec{b}=\nabla\cdot(\overleftrightarrow{a}\cdot\vec{b})-\overleftrightarrow{a}:(\nabla\vec{b})_{\mathrm{c}}$ (selections $\overleftrightarrow{a}=\overleftrightarrow{\sigma}$ and $\vec{b}=\delta\vec{u}$), and division of the displacement gradient into its symmetric and anti-symmetric parts according to $\nabla\vec{u}=\overleftrightarrow{\varepsilon}+\overleftrightarrow{\phi}$ give

$$\int_V (\nabla\cdot\overleftrightarrow{\sigma}+\vec{f})\cdot\delta\vec{u}dV=0 \quad \forall\delta\vec{u}\in U \quad \Rightarrow$$

$$\int_V (-\overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_{\partial V} (\vec{n}\cdot\overleftrightarrow{\sigma}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

The boundary conditions of the local form imply that either $\delta\vec{u}=0$ or $\vec{n}\cdot\overleftrightarrow{\sigma}=\vec{t}$ at all points of ∂V. Therefore, one ends up with

$$\delta W=\int_V (-\overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_{\partial V} (\vec{t}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

variational form

The derivation assumes that $\overleftrightarrow{\sigma}=\overleftrightarrow{\sigma}_{\mathrm{c}}$ (where exactly?). In practice, symmetry of stress is satisfied ‘a priori’ by the form of the constitutive equation.

In derivation to the reverse direction (with the assumption $\overleftrightarrow{\sigma}=\overleftrightarrow{\sigma}_{\mathrm{c}}$ for consistency), the starting point is the variational form. One substitutes first division $\overleftrightarrow{\varepsilon}=\nabla\vec{u}-\overleftrightarrow{\phi}$ to get

$$\delta W = \int_V [-\vec{\vec{\sigma}} : (\nabla \delta \vec{u})_c] dV + \int_V (\vec{f} \cdot \delta \vec{u}) dV + \int_{\partial V} (\vec{t} \cdot \delta \vec{u}) dA = 0 \quad \forall \delta \vec{u} \ .$$

Integration by parts with $(\nabla \cdot \vec{\vec{a}}) \cdot \vec{b} = \nabla \cdot (\vec{\vec{a}} \cdot \vec{b}) - \vec{\vec{a}} : (\nabla \vec{b})_c$ (selections $\vec{\vec{a}} = \vec{\vec{\sigma}}$ and $\vec{b} = \delta \vec{u}$) gives an equivalent but more convenient form

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \vec{f}) \cdot \delta \vec{u} dV + \int_{\partial V} (-\vec{n} \cdot \vec{\vec{\sigma}} + \vec{t}) \cdot \delta \vec{u} dA = 0 \quad \forall \delta \vec{u} \ .$$

The variational form, together with the assumed symmetry of stress and the conditions for the function set U, implies equations

$$\left.\begin{aligned} &\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{and} \quad \vec{\vec{\sigma}} - \vec{\vec{\sigma}}_c = 0 \quad \text{in } V, \\ &\vec{n} \cdot \vec{\vec{\sigma}} - \vec{t} = 0 \quad \text{or} \quad \vec{u} - \underline{\vec{u}} = 0 \quad \text{on } \partial V. \end{aligned}\right\} \text{The starting point}$$

EXAMPLE Principle of virtual work for a Bernoulli beam problem is given by: find $w \in U$ such that $\forall \delta w \in U$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_{\Omega} \left(-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b\right) dx = 0$$

in which $\Omega = (0, L)$, $U = \{w \in C^4(\Omega) : w = dw/dx = 0 \text{ at } x = 0\}$ and the bending stiffness $EI(x)$ and $b(x)$ are given. Deduce the underlying boundary value problem by using integration by parts and the fundamental lemma of variation calculus.

Answer $-\dfrac{d^2}{dx^2}\left(EI \dfrac{d^2 w}{dx^2}\right) + b = 0$ in $(0, L)$, $\dfrac{d}{dx}\left(EI \dfrac{d^2 w}{dx^2}\right) = 0$ at $x = L$,

$-EI \dfrac{d^2 w}{dx^2} = 0$ at $x = L$, $\dfrac{dw}{dx} = 0$ at $x = 0$, and $w = 0$ at $x = 0$

Integration by parts twice in the first term gives an equivalent form (notice that $\delta w \in U$ and therefore $\delta w = d\delta w / dx = 0$ at $x = 0$)

$$\delta W = \int_{\Omega} (-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} [\frac{d\delta w}{dx}\frac{d}{dx}(EI \frac{d^2 w}{dx^2}) + \delta w b] dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L} \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} [-\frac{d^2}{dx^2}(EI \frac{d^2 w}{dx^2}) + b] \delta w dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2}) - \delta w \frac{d}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L} .$$

According to principle of virtual work $\delta W = 0 \; \forall \delta w \in U$. Let us first consider a subset $U_0 \subset U$ for which $\delta w = d\delta w / dx = 0$ at $x = L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W = \int_\Omega [-\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) + b]\delta w dx = 0 \quad \Rightarrow \quad -\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) + b = 0 \ \text{ in } (0, L).$$

After that, let us consider U with restriction $d\delta w / dx = 0$ first and then with $\delta w = 0$ at $x = L$ and simplify the virtual work expression by using the equilibrium equation already obtained. The natural boundary conditions follow from the fundamental lemma of variation calculus

$$\delta W = [\delta w \frac{d}{dx}(EI\frac{d^2w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad \frac{d}{dx}(EI\frac{d^2w}{dx^2}) = 0 \ \text{ at } \ x = L,$$

$$\delta W = -[\frac{d\delta w}{dx}(EI\frac{d^2w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad -EI\frac{d^2w}{dx^2} = 0 \ \text{ at } \ x = L\ .$$

Boundary conditions $w = dw/dx = 0$ at $x = 0$ follow from assumption $w \in U$.

3.3 DERIVATION OF ENGINEERING MODELS

First, write the virtual work expression by using the virtual work densities of an engineering model. If not available, start with the generic virtual work expression, kinematical and kinetic assumptions of the model, and integrate over the small dimensions.

Second, use the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus to deduce the field equation(s) and (natural) boundary conditions in terms of stress resultants. Consider suitable subset of function space U to deduce first the equilibrium equation and thereafter the conditions at the boundaries.

Third, use the definitions of the stress resultants to derive the constitutive equations corresponding to the material model required.

THIN BODY ASSUMPTIONS

Bar: $\vec{u}(x,y,z)=\vec{u}_0(x)$ and $\sigma_{yy}=\sigma_{zz}=\sigma_{xy}=\sigma_{yz}=\sigma_{zx}=0$

String: $\vec{u}(s,n,b)=\vec{u}_0(s)$ and $\sigma_{nn}=\sigma_{bb}=\sigma_{sn}=\sigma_{nb}=\sigma_{bs}=0$

Straight beam: $\vec{u}(x,y,z)=\vec{u}_0(x)+\vec{\theta}(x)\times\vec{\rho}(y,z)$ and $\sigma_{yy}=\sigma_{zz}=0$

Curved beam: $\vec{u}(s,n,b)=\vec{u}_0(s)+\vec{\theta}(s)\times\vec{\rho}(n,b)$ and $\sigma_{nn}=\sigma_{bb}=0$

Thin slab: $\vec{u}(x,y,z)=\vec{u}_0(x,y)$ and $\sigma_{zz}=\sigma_{yz}=\sigma_{zx}=0$

Membrane: $\vec{u}(\alpha,\beta,n)=\vec{u}_0(\alpha,\beta)$ and $\sigma_{nn}=\sigma_{\beta n}=\sigma_{n\alpha}=0$

Plate: $\vec{u}(x,y,z)=\vec{u}_0(x,y)+\vec{\theta}(x,y)\times\vec{\rho}(z)$ and $\sigma_{zz}=0$

Shell: $\vec{u}(z,s,n)=\vec{u}_0(z,s)+\vec{\theta}(z,s)\times\vec{\rho}(n)$ and $\sigma_{nn}=0$

BAR EQUATIONS

Bar is one of the loading modes of the beam model and it can be considered also as the elasticity problem in one dimension. The model assumes that displacement and stress have just axial components depending on the axial coordinate only. The bar boundary value problem

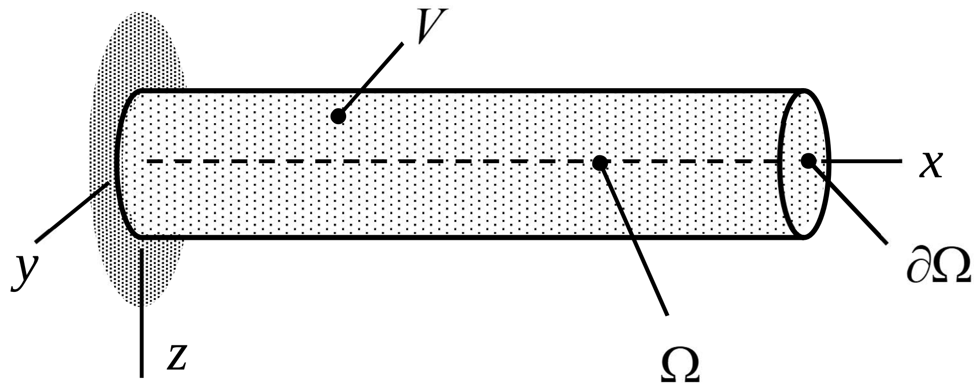
$$\frac{d\vec{N}}{dx}+\vec{b}=0 \text{ in } \Omega \quad \text{and} \quad \vec{n}\cdot\vec{N}-\underline{F}=0 \text{ or } \vec{n}\cdot\vec{u}-\underline{u}=0 \text{ on } \partial\Omega,$$

where

$$\vec{N}=\int \vec{\vec{\sigma}}\cdot d\vec{A}=\int \vec{\sigma}dA,\ \vec{b}=\int \vec{f}dA,\ \text{and } F=\int \vec{t}dA.$$

For a closed equation system (number of equations and unknown functions should match) a material model is also needed.

The physical domain of the bar model is V occupied by a body althought the solution domain of the equations is the mid-line Ω. The starting point is the virtual work expression written for the physical domain.



Let us consider the steps in the Cartesian $(x, y, z)-$coordinate system for clarity. The bar model assumes that displacement and stress have just axial components depending on the axial coordinate only. Representations of stress, displacement and gradient operator are $\vec{\vec{\sigma}} = \sigma_{xx}\vec{i}\vec{i}$ and $\vec{u}(x) = u(x)\vec{i}$, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$

$$\delta W^{\text{int}} = -\int_V (\nabla \delta \vec{u})_{\text{c}} : \vec{\sigma} dV = -\int_\Omega \frac{d\delta u}{dx} (\int \sigma_{xx} dA) dx = -\int_\Omega \frac{d\delta u}{dx} N dx$$

$$\delta W^{\text{ext}} = \int_V \delta \vec{u} \cdot \vec{f} dV + \int_{\partial V} \delta \vec{u} \cdot \vec{t} dA = \int_\Omega \delta u b dx + \sum_{\partial \Omega} \delta u F$$

in which

$$N = \int \sigma_{xx} dA, \;\; b = \int f_x dA, \;\; \text{and} \;\; F = \int t_x dA.$$

According to the principle of virtual work $\delta W = 0 \;\; \forall \delta u \in U$. Integration by parts is used first to obtain a more convenient form for deducing the bar equations.

$$\delta W = -\int_\Omega (N \frac{d\delta u}{dx}) dx + \int_\Omega (b \delta u) dx + \sum_{\partial \Omega} (F \delta u) = 0 \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\frac{dN}{dx} + b) \delta u dx + \sum_{\partial \Omega} (-nN + F) \delta u = 0 \;\; \text{in which} \;\; n = \pm 1.$$

Let us consider first a subset of variations $\delta u \in U$ with restriction $\delta u = 0$ on $\partial\Omega$ and use the fudamental lemma of variational calculus to deduce that

$$\delta W = \int_{\Omega} (\frac{dN}{dx} + b)\delta u dx = 0 \quad \forall \delta u \in U \quad \Leftrightarrow \quad \frac{dN}{dx} + b = 0 \text{ in } \Omega. \tag{1}$$

After that, we consider $\delta u \in U$ without restrictions on the boundary (and use the equilibrium equation to get rid of the first term of the virtual work expression) to deduce

$$\delta W = \sum_{\partial\Omega} (-nN + F)\delta u = 0 \quad \forall \delta u \in U \quad \Leftrightarrow \quad nN - F = 0 \text{ on } \partial\Omega.$$

The boundary term vanishes also if $\delta u = 0$ on $\partial\Omega$ which implies that u is given on $\partial\Omega$. Therefore, on the boundary either $u - \underline{u} = 0$ or $nN - F = 0$ but not both.

The bar model boundary value problem combines the equations

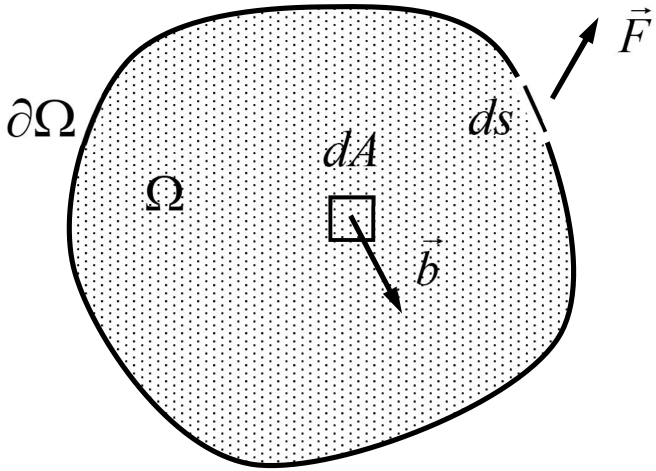
$$\left.\begin{aligned}&\frac{dN}{dx}+b=0 \quad \text{in} \quad \Omega,\\ &nN-F=0 \quad \text{or} \quad u-\underline{u}=0 \quad \text{on} \quad \partial\Omega.\end{aligned}\right\} \text{ Local form}$$

For a unique solution, the displacement boundary condition should be given at least on one boundary point. The constitutive equation for an elastic material follows from the generalized Hooke's law for the bar model $\sigma_{xx} = Edu/dx$ and the definition of stress resultant

$$N = \int \sigma_{xx} dA = EA\frac{du}{dx}.$$

THIN SLAB EQUATIONS

Thin slab model assumes that the transverse displacement (perpendicular to the mid-plane) and stress components vanish and that the quantitities do not depend on the transverse coordinate. Principle of virtual work gives



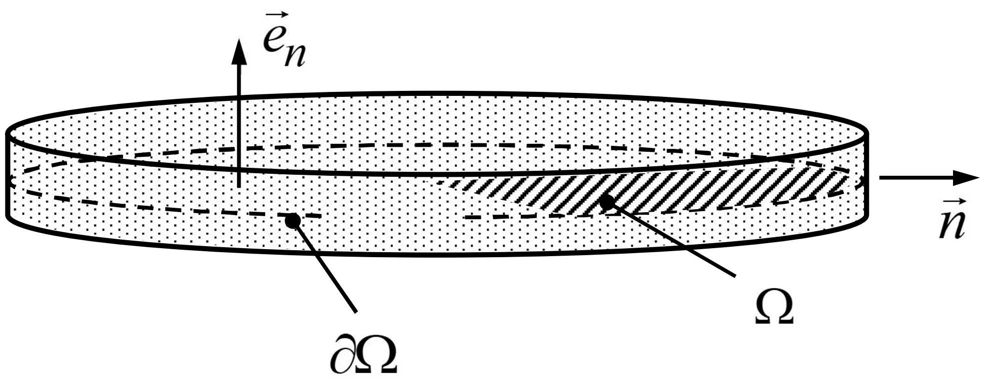
$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = 0 \quad \text{in} \quad \Omega,$$

$$\vec{n} \cdot \overleftrightarrow{N} - \vec{F} = 0 \quad \text{or} \quad \vec{u} - \underline{\vec{u}} = 0 \quad \text{on} \quad \partial\Omega,$$

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn, \; \vec{b} = \int \vec{f} dn, \; \text{and} \; \vec{F} = \int \vec{t} dn.$$

Constitutive equation $f(\overleftrightarrow{N}, \vec{u}) = 0$, which is needed for a closed system of equations, follows form a material model and the stress resultant definition. Further calculations require specification of the coordinate system.

The physical domain of the thin-slab model is a prismatic body althought the solution domain of the equations is the mid plane. The starting point is virtual work expression written for the physical domain.



If the external forces on the top and bottom surfaces vanish and stress is symmetric 'a priori', virtual work expressions of the internal and external forces simplify to (volume element $dV = dndA$ and area element on the boundary $dA = dnds$)

$$\delta W^{\text{int}} = -\int \overleftrightarrow{\sigma} : \delta(\nabla\vec{u})_{\mathrm{c}} dV = -\int_{\Omega} (\int \overleftrightarrow{\sigma} dn) : \delta(\nabla\vec{u})_{\mathrm{c}} dA = -\int_{\Omega} \overleftrightarrow{N} : \delta(\nabla\vec{u})_{\mathrm{c}} dA,$$

$$\delta W_V^{\text{ext}} = \int \vec{f} \cdot \delta\vec{u} dV = \int_\Omega (\int \vec{f} dn) \cdot \delta\vec{u} dA = \int_\Omega \vec{b} \cdot \delta\vec{u} dA,$$

$$\delta W_A^{\text{ext}} = \int \vec{t} \cdot \delta\vec{u} dA = \int_{\partial\Omega} (\int \vec{t} dn) \cdot \delta\vec{u} ds = \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds$$

in which the stress resultants

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn, \quad \vec{b} = \int \vec{f} dn, \quad \text{and} \quad \vec{F} = \int \vec{t} dn.$$ **integrals over the thickness!**

Integration by parts with the vector identity $\vec{a} : (\nabla\vec{b})_c = \nabla \cdot (\vec{a} \cdot \vec{b}) - (\nabla \cdot \vec{a}) \cdot \vec{b}$ in the virtual work expression gives an equivalent but more convenient form for the next step

$$\delta W = -\int_\Omega \overleftrightarrow{N} : \delta(\nabla\vec{u})_c dA + \int_\Omega \vec{b} \cdot \delta\vec{u} dA + \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\nabla \cdot \overleftrightarrow{N} + \vec{b}) \cdot \delta\vec{u} dA + \int_{\partial\Omega} (-\vec{n} \cdot \overleftrightarrow{N} + \vec{F}) \cdot \delta\vec{u} ds.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the local forms. Consider first a subset of variations $\delta\vec{u} \in U$ with restriction $\delta\vec{u} = 0$ on $\partial\Omega$ to get

$$\delta W = \int_{\Omega} (\nabla \cdot \vec{\vec{N}} + \vec{b}) \cdot \delta\vec{u} dA = 0 \quad \forall \delta\vec{u} \in U \quad \Leftrightarrow \quad \nabla \cdot \vec{\vec{N}} + \vec{b} = 0 \;\text{ in }\; \Omega.$$

Now using the equilibrium equation, the first term of the virtual work expression vanishes. Consider then $\delta\vec{u} \in U$ without restrictions on the boundary to get

$$\delta W = \int_{\partial\Omega} (-\vec{n} \cdot \vec{\vec{N}} + \vec{F}) \cdot \delta\vec{u} ds = 0 \quad \Rightarrow \; -\vec{n} \cdot \vec{\vec{N}} + \vec{F} = 0 \;\text{ or }\; \delta\vec{u} = 0 \;\text{ on }\; \partial\Omega.$$

Vanishing of variation $\delta\vec{u} = 0$ on $\partial\Omega$ implies that displacement is given, i.e., $\vec{u} = \underline{\vec{u}}$. The boundary value problem implied by the principle of virtual work

$$\left.\begin{aligned} &\nabla\cdot\overset{\leftrightarrow}{N}+\vec{b}=0 \quad \text{in} \quad \Omega, \\ &\vec{n}\cdot\overset{\leftrightarrow}{N}-\vec{F}=0 \quad \text{or} \quad \vec{u}-\underline{\vec{u}}=0 \quad \partial\Omega. \end{aligned}\right\}$$

NOTICE (1) Integration by parts of the derivation uses the Gauss theorem for a flat geometry which may exclude domains of non-vanishing curvature (it turns out later that the form is valid also in curved geometry). (2) A constitutive equation is needed for a closed system of equations (here the number of unknown stress components is 3, whereas the number of equilibrium equations is 2 in flat geometry). The additional equations are given by the stress resultant definition when the stress expression of the material model is substituted there.

THIN SLAB EQUATIONS IN (x, y)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the Cartesian $(\vec{i}, \vec{j})-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + b_x \\ \dfrac{\partial N_{yy}}{\partial y} + \dfrac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0, \text{ where } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

Representations in the Cartesian system (notice that the second form of the gradient is valid only when basis vectors are constants)

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \ \overset{\leftrightarrow}{N} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \ \vec{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

$$\nabla \cdot \overset{\leftrightarrow}{N} + \vec{b} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \cdot \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0$$

$$\nabla \cdot \overset{\leftrightarrow}{N} + \vec{b} = \left(\begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0. \quad \Leftarrow$$

A constitutive equation is needed for a closed system of equations (here the number of unknown stress components is 3, whereas the number of equations is 2. Assuming that the thin slab is made of isotropic homogeneous and linearly elastic material of thickness t (steel, aluminum etc.), stress-displacement relationship, kinematic assumption of the model, and elasticity tensor of the plane-stress case give

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \quad \text{so} \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}. \quad \Leftarrow$$

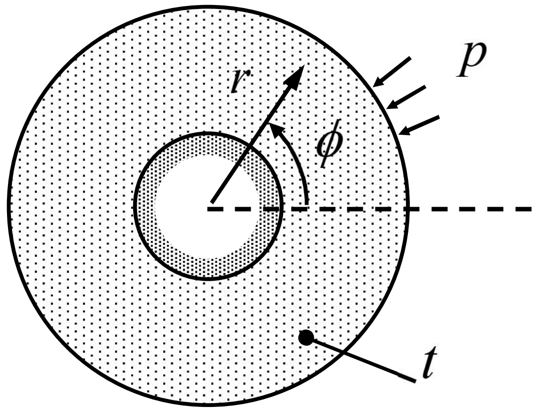
THIN SLAB EQUATIONS IN (r,ϕ)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the polar $(\vec{e}_r,\vec{e}_\phi)-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \dfrac{1}{r}[\dfrac{\partial(rN_{rr})}{\partial r}+\dfrac{\partial N_{r\phi}}{\partial\phi}-N_{\phi\phi}]+b_r \\ \dfrac{1}{r}[\dfrac{1}{r}\dfrac{\partial(r^2N_{r\phi})}{\partial r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi}]+b_\phi \end{Bmatrix}=0, \text{ where } \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \dfrac{\partial u_r}{\partial r} \\ \dfrac{1}{r}(u_r+\dfrac{\partial u_\phi}{\partial\phi}) \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial\phi}+r\dfrac{\partial}{\partial r}(\dfrac{u_\phi}{r}) \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

EXAMPLE Consider a disk $r \in [\varepsilon R, R]$ which is loaded by traction $\vec{t} = -p\vec{e}_r$ on the outer edge $r = R$ (p is constant). Assuming rotation symmetry i.e. that all quantities depend only on the distance r from the center point, find the displacement components $u_r = u(r)$ and $u_\phi = v(r)$ for a linearly elastic material when Young's modulus E and Poisson's ratio ν are constants.



Answer $u = \dfrac{(\varepsilon R)^2 - r^2}{r} \dfrac{p}{E} \dfrac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)}$

If the displacement and stress resultant components depend only on the radial coordinate, the equilibrium equations and the constitutive equations of the polar coordinate system simplify to (here $b_r = b_\phi = 0$)

$$\frac{dN_{rr}}{dr} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) = \frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \quad \frac{\partial N_{r\phi}}{\partial r} + \frac{2}{r}N_{r\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 N_{r\phi}) = 0$$

and

$$N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}), \quad N_{r\phi} = tG(\frac{dv}{dr} - \frac{v}{r}) = tGr\frac{d}{dr}(\frac{v}{r}).$$

On the inner edge $r = \varepsilon R$ displacement vanishes, i.e., $u_r \equiv u = 0$. On the outer edge $r = R$, $\vec{n} = \vec{e}_r$, $\vec{n} \cdot \overleftrightarrow{N} - \vec{F} = 0$, and $\vec{F} = -pt\vec{e}_r$. These conditions give the boundary value problem,

$$\frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \ N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}) \ \text{ in } \ (\varepsilon R, R),$$

$$u = 0 \ \text{ at } \ r = \varepsilon R \quad \text{and} \quad N_{rr} = -pt \ \text{ at } \ r = R.$$

Elimination of the stress resultants from the equilibrium equation and boundary conditions gives the boundary value problem for the radial displacement component

$$\frac{d}{dr}[\frac{1}{r}\frac{d(ru)}{dr}] = 0 \quad \text{in} \ \ (\varepsilon R, R),$$

$$u = 0 \ \text{ at } \ r = \varepsilon R \quad \text{and} \quad \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}) = -pt \ \text{ at } \ r = R.$$

The generic solution to the differential equation is $u = a/r + br$. Thereafter, the boundary conditions give the values of the integration constants and solution,

$$u = \frac{(\varepsilon R)^2 - r^2}{r} \frac{p}{E} \frac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)} .$$ 🡐

The boundary value problem for the displacement component in the angular direction (in terms of displacement component and stress resultant) is given by,

$$\frac{1}{r^2}\frac{d}{dr}(r^2 N_{r\phi}) = 0 \text{ and } N_{r\phi} = tGr\frac{d}{dr}(\frac{v}{r}) \text{ in } (\varepsilon R, R),$$

$$v = 0 \text{ at } r = \varepsilon R \text{ and } N_{r\phi} = 0 \text{ at } r = R.$$

Equilibrium equation and the condition on the outer edge imply first $N_{r\phi}(r) = 0$. After that, the constitutive equation, and the displacement boundary condition result into

$$v = 0.$$ 🡐