Quantum Circuits
Week 2

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1 Noise in quantum circuits (Lecture 3)

Previously we have assumed that our quantum system evolves over time isolated from its surroundings. The **Schrödinger** equation captured the full system dynamics, meanwhile the **Born** rule provided an operational method to calculate the probabilities of different measurement outcomes. However, we are mostly interested in situations where our system carries an information to be *processed* and then finally *retrieved*. Furthermore, practical measurements are continuous in nature, in the sense that we gain information about our system gradually over time. In both situations a coupling to the environment is inevitable and should be incorporated somehow into our description of any quantum information processing task. Although a rigorous derivation of the dynamics of an open quantum system is beyond the scope of our course, we can still develop a solid understanding of a noisy process with the concepts we have already learned in the previous lectures.

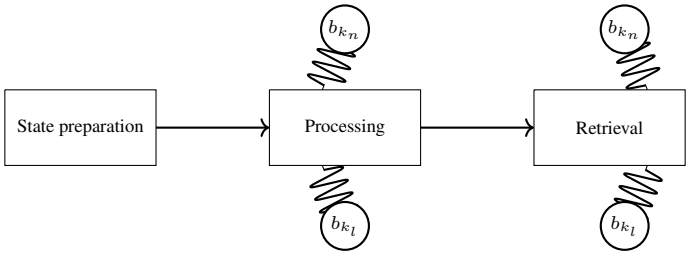

Figure 1 – *Quantum information processing*

Not surprisingly, the environment comes into play as a bath of infinitely many **harmonic oscillators**, such that each one of them is endowed with its own pair of creation and annihilation operators $\sum_k \hbar\omega_k b_k^\dagger b_k$. The net effect of theses modes is to exert a drag force on our quantum system to damp its motion.

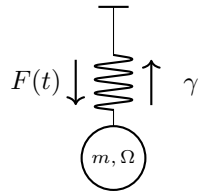

Figure 2 – *Damping of a simple harmonic oscillator*

Furthermore, we will assume that the interaction with the environment over some time interval T is **continuous**. That is, we divide T into infinitesimal increments Δt then take the limit as $\Delta t \to 0$. Moreover, during each of these small increments a new arbitrary version of the environment interacts with the system. Thus the environment always looks *unchanging* from the system's perspective. This arbitrariness, reflecting our incomplete knowledge of the vast degrees of freedom of the environment, can be modelled as a **Random process** . That is a source that picks, according to some probability distribution, an erratic function (**sample**) from its associated sample space to interact with the system. Meanwhile the "*unchanging*" property of the environment translates into two interesting characteristics of our random process, namely, **staionarity** and **ergodicity**. The former means that the signal statistics are invariant under time translations, whereas the later implies that when we observe the random process "*long enough*" , we witness a kind of "*concatenation*" of all the possible samples from the event space. In other words, we realize the entire same sample space over some long time interval instead of running the same process infinitely many times in parallel. So if we thought of the average of these Δt interaction intervals as preparing parallel environments chosen from the same random process to act on the system, ergodicity tells us that we can instead carry out an equivalent time average to measure the system. We now give a more rigorous definition of each of the previous concepts and explain their role in relating the noise present in some signal to its frequency description.

1.1 Random processes

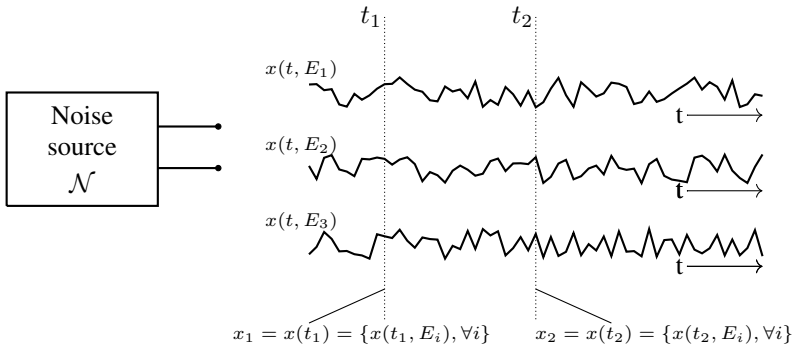

Figure 3 – *Some sample functions of a random noise source.*

In probability theory a random process is a description of the output of a system that appear to vary randomly over time [1]. We can think of the depicted noise source in Fig.3 as an event space $\mathcal{N} = \{E_i\}_n$. Then each event $E_i \in \mathcal{N}$ is mapped to a time-varying waveform $x(t, E_i)$ denoted as a **sample function**. The set of all possible sample functions $\{E_i\}_n$ is called the **ensemble** and defines the random process $x(t)$ that describes the noise source $\mathcal{N}$. Thus each observed output generated by the noise source realizes one of the sample functions. Furthermore, as shown in Fig.3 we can define a set of random variables $x_1 = x(t_1), x_2 = x(t_2), ..., x_n = x(t_n)$, such that $x(t)$ is the random process. Here the random variable $x_j = x(t_j)$ takes on values described by the set of constants $\{x(t_j, E_i), \forall i\}$ obtained by fixing a time t_j inside the observation interval. To summarize, a random process is characterized by two domains, namely, amplitude and time. When time is fixed, for some value t_i, the amplitude distribution follows some probability density function (PDF). Generally speaking different PDFs at different times are not equal, although when they are, we say the process possesses a special property as we will see shortly. On the other hand, we characterize the time behaviour of the random process when we consider evolving this PDF over time.

1.2 Stationarity

A random process $x(t)$ is called N-order stationary if the N random variables joint probability density function (PDF) is invariant under time translations

$$P_N\big[x(t_1), x(t_2), ..., x(t_N)\big] = P_N\big[x(t_1 + t_0), x(t_2 + t_0), ..., x(t_N + t_0)\big]$$

where t_0 is a constant. Thus it is straight forward to conclude that a first order stationary process is time-independent.

1.3 Ergodicity

Before giving a definition of ergodicity, we first introduce two notions of averages regarding a random process.

1.3.1 *Ensemble average*

This is the kind of average obtained when we fix time at some value t_i

$$\overline{x(t)} = \int_{-\infty}^{\infty} x\ P(x, t_i)\ dx \tag{1}$$

where $P(x)$ is the PDF of the random variable $x(t_j, E_i)\forall i$. It is clear that for a stationary process we can drop the time label.

[1] In general, the system can be parametrized by any other parameter than time.

1.3.2 *Time average*

Observing a particular realization of the random process for a sufficiently long time produces what is known as the time average

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t, E_i)\ dt \tag{2}$$

For a periodic sample function the time average can be written as

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t, E_i)\ dt$$

Thus, a random process is said to be **ergodic** if all time averages of any sample function chosen from the ensemble are equal to the corresponding ensemble averages (expectations). In other words, observing any realization of an ergodic process for a sufficiently long time produces the same statistical properties (moments) as observing simultaneously infinitely many outputs of the same random process.

1.4 Example of an ergodic process

Consider now the experiment of measuring the voltage across many identical resistors. Assume further that we are performing this experiment in finite temperature environment. This means that the electrons motion across the resistor is random which in turn implies that the ensemble average is equal to the time average. Thus the whole experiment can be simulated by observing one resistor for sufficiently long time. As we will see later in our course, this how we usually model an environment bath in a quantum circuit.

2 Autocorrelation and cross-correlation of a a random process

2.1 *Auto-correlation*

The auto-correlation function captures the time behavior of a signal. Simply put, it is the multiplication of the amplitudes at t_1 and t_2, then averaging over the ensemble. let $x(t)$ be a real random process. Its auto-correlation function is

$$R_x(t_1, t_2) = \overline{x(t_1)x(t_2)} = \int_{-\infty}^{\infty} x_1 x_2 P(x_1, x_2)\ dx_1 dx_2$$

where x_1, x_2 are random variables as depicted in Fig.3, and $P(x_1, x_2)$ is their joint PDF.

If the process is second order stationary(mean and variance are constant) then

$$P(x(t_1), x(t_2)) = P(x(t_1 + t_0), x(t_2 + t_0))$$

Setting t_1 equals to t_0 gives

$$R_x(\tau) = \overline{x(t)x(t+\tau)}$$

where $\tau = t_2 - t_1$.
Thus in this case the auto-correlation function depends only on the time difference.

3 Energy and power signals

Throughout this course we will be concerned with physically realizable waveforms. These are finite energy continuous signals. Mathematically they are objects of the infinite dimensional L_2 space. Thus by virtue of Fourier theory we can represent any physically realizable signal as a linear combination of *"basis"* signals.

3.1 Energy signal

The energy content of a random *signal* (process) $x(t)$ is just its squared norm[2]. This norm induces an inner product operation. Thus, when $x(t)$ is a complex-valued function, the L_2 space becomes our beloved **Hilbert** space. Similar to a valid wavefunction, $x(t)$ should satisfy some basic properties , such as *square integrability*, *continuity* and the *the well-definedness of its norm.* For the purpose of this discussion we only consider real-valued functions.*"Formally"* speaking, the energy content of a real-valued random process $x(t)$ is

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt, \text{ where} 0 < E < \infty$$

where the finiteness of the energy of the signal $x(t)$ is mirrored by the well-definedness property of its norm.
However, the energy of a random process, as a quantity, doesn't tell us that much about the rate of energy transfer during the measurement process. Afterall, E is just a random variable that takes some constant values. We can think of the measuring device as an analog voltmeter for example, where the energy absorbed by the meter moves its pointer. Thus we are looking for a quantity that reflects the time taken to register a pointer's movement after absorbing the energy of the random process (*gradual acquisition of information*). For this purpose we define *power signals.*

3.2 Power signal

From classical mechanics power $\mathcal{P}$ is the rate of energy transfer per unit time $\mathcal{P} = \frac{E}{T}$.
For our continuous real-valued random process $x(t)$ we can define a similar quantity

$$\mathcal{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\, dt \text{ where} 0 < \mathcal{P} < \infty$$

[2]More precisely the L_2 norm.

where T is an observation window. According to the above definitions, a finite energy signal has zero power when averaged over infinite time, whereas a finite power signal has infinite energy when $T \to \infty$.
At this point one should worry about whether the mathematically convenient description of power yields measurable experimental effects. As we mentioned earlier, our measuring device is a meter with a movable **pointer**. After absorbing the signal's energy, the pointer's movement will start deteriorating according to its **damping coefficient**. The damping coefficient determines an **observation interval**, such that outside this interval the power tends to zero. This allows us to yield finite observable values while taking the limit as $T \to \infty$. We have developed a similar concept in the first lecture when the problem of a particle trapped inside a **potential well** was considered. The assumption that the particle's wavefunction dies at the well's **boundaries** was imposed to avoid mathematical subtleties.
In the frequency domain we can similarly define the energy and power of a random process. First we define a Fourier pair as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2j\pi ft}\, dt$$
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2j\pi ft}\, df$$

Then after applying Parseval's theorem we get

$$E = \int_{-\infty}^{\infty} X^2(f)$$
$$\mathcal{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} X_F^2(f)\, df$$

where

$$X_T(t) = \begin{cases} X(t) & |T| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

is a truncated version of the random process selected over a specific period, and $X_F(f)$ is its Fourier transform.

4 Power spectral density

Power spectral density (PSD) is a function that relates the power of a signal to its frequency domain representation. In other words, it is the power per unit bandwidth.
The PSD can be written as

$$S(f) = \lim_{T\to\infty} \left[\frac{X_F^2(f)}{T} \right]$$

Thus the average power of the random process in the frequency domain can be written in terms of the PSD after exchanging the limit operation and integration.

4.1 Transmission of random processes through a linear system

When a random process $x(t)$ is applied at the input of a linear system with a transfer function $H(f)$ we can write the PSD of the output as

$$S_y(f) = |H(f)|^2 S_x(f)$$

Meanwhile the mean square voltage can be written as

$$\overline{v^2} = \int_{-\infty}^{\infty} S_y(f)\, df$$

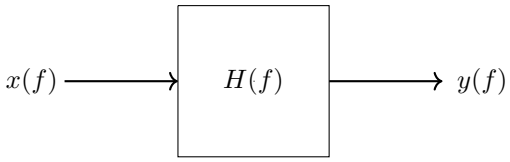

Figure 4 – Linear system transfer function

4.2 Wiener–Khintchine Theorem

The PSD is often calculated from the following property. Let's assume that $x(t)$ is a wide-sense stationary process (constant mean and variance), then the PSD can be obtained from the Fourier transform of the autocorrelation function

$$S(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau}\, d\tau$$

5 Revisiting the harmonic oscillator

Imagine now that the harmonic oscillator (HO) in Fig.2 is being acted upon with a quantum force $\hat{F}(t)$ representing an environment. Such situation arises either during a measurement process or when the HO is undergoing some transition ($|n+1\rangle \rightsquigarrow |n\rangle$ or $|n\rangle \rightsquigarrow |n+1\rangle$). In the former the environment is coupled to the HO number operator $a^\dagger a$, while in the later it is coupled to the HO position operator $x \propto (a + a^\dagger)$. Since in this situation the system's observables are not accessible, we measure the auto-correlation function of the environment force to make an inference about the state of the system. When thermal equilibrium is reached the principle of detailed balance implies that the ratio between the PSD of the two possible transition rates is

$$\frac{S_{FF}(\Omega)}{S_{FF}(-\Omega)} = 1 + \frac{1}{\bar{n}}$$

where $\bar{n}$ is the average number of black-body photons.
In the limit of large $\bar{n}$ the classical symmetry between the transition rate PSDs is restored. However, when we cool down our environment $i.e. \bar{n} \approx 0$ the symmetry no

longer holds as the ratio diverges, and the two PSDs correspond to different phenomnea heating/cooling of a harmonic oscillator. Unfortunately this is way out of the scope of our course.

6 Superconductivity (Lecture 4)

The Josephson junction (JJ) that we have studied thoroughly during the lecture is the basis for different qubit architectures. We are going to postpone the discussion of these qubit realizations till a future lecture. Meanwhile, we only focus on the analogy between the JJ and the harmonic oscillator model.
As we have seen the super current inside each side of the junction is made of pairs of electrons called *Cooper pairs*. These electrons occupy a macroscopic wavefunction with a well-defined phase. Tunneling occurs when electron pairs migrate from one side of the junction to the other, thus decreasing/increasing the number of cooper pairs at superconducting lump. Thus it is quite natural to characterize the behaviour of JJ by the number of cooper pairs, hence we define an *"observable"* corresponding to the number of cooper pairs

$$N = \sum_n |n\rangle\, n\, \langle n|$$

where $n = 0, \pm 1, \pm 2, \pm 3, ...$ and $|n\rangle$ are the eigenstates of the number operator N
By analogy with the quantum theory of electromagnetic radiation we define an operator corresponding to the condensate phase

$$e^{i\hat{\varphi}} = \frac{1}{2}\int_0^{2\pi} d\varphi'\, e^{i\varphi'}\, |\varphi'\rangle\, \langle\varphi'|$$

It is straight forward to conclude that this operator has a space of eigen-kets $|\varphi\rangle$ as defined in the exercise sheet. Pushing this analogy one step further, we impose a canonical commutation relation between the two observables similar to that between the position and momentum of a harmonic oscillator

$$[N, e^{i\hat{\varphi}}] = i$$

However our assumption that cooper pair number operator has a discrete spectrum leads us to a contradiction. To see this, consider calculating the matrix element of the previous commutator bracket for any two arbitrary number states $|n\rangle$, and $|n'\rangle$

$$\langle n'|\, [N, e^{i\hat{\varphi}}]\, |n\rangle = i\delta_{nn'}$$

Then using the definition of the number operator we get

$$\langle n'|\, \big(\sum_{n''} |n''\rangle\, n''\, \langle n''|\,\big) e^{i\hat{\varphi}}\, |n\rangle - \langle n'|\, e^{i\hat{\varphi}} \big(\sum_{n''} |n''\rangle\, n''\, \langle n''|\,\big)\, |n\rangle = i\delta_{nn'}$$

$$(n' - n)\, \langle n'|e^{i\hat{\varphi}}|n\rangle = i\delta_{nn'}$$

Thus when $n = n'$ we get a **contradiction!** This actually asserts the remark pointed out during the lecture that we shouldn't take the discrete nature of the cooper pairs

too serious and it is more convenient to think of the condensate wavefunction as a coherent macroscopic *state of knowledge* of continuous nature. Usually the problem of the quantum phase is dealt with in the context of quantum optics. We can find more concrete arguments in any standard textbook on the subject.

7 Some useful definitions of the delta function

$$\delta_{nm} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{-i(n-m)\varphi}$$

$$\delta(\varphi - \varphi') = \frac{1}{2\pi} \sum_n e^{-in(\varphi - \varphi')}$$