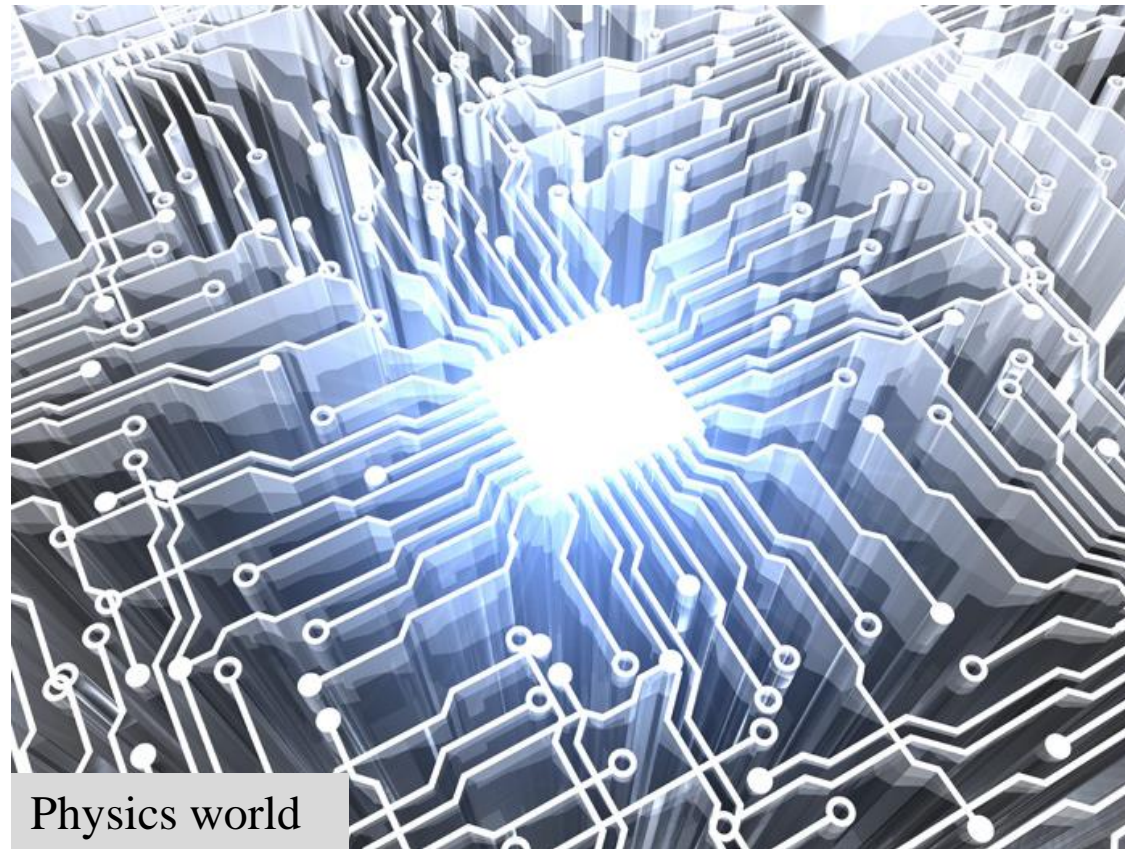


COMSOL

Circuit simulation

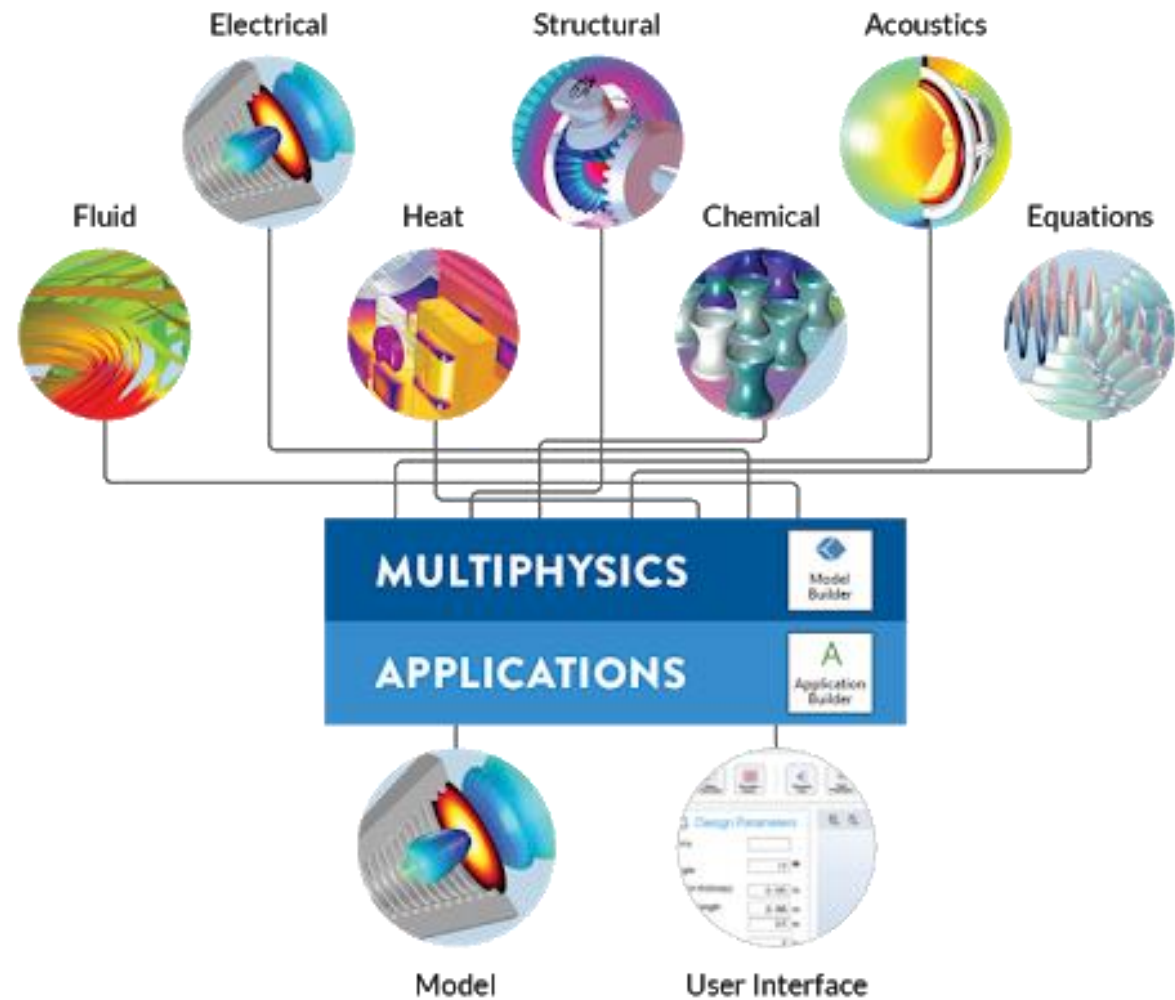


Physics world

Manohar Kumar
Aalto University

COMSOL: Multiphysics

Multiphysics: Systems involving more than one simultaneously occurring physical field and the studies of and knowledge about these processes and systems (*def: Wikipedia*).



COMSOL: Multiphysics

Earlier approach: Different domains and their aspects of a structure are studied individually.

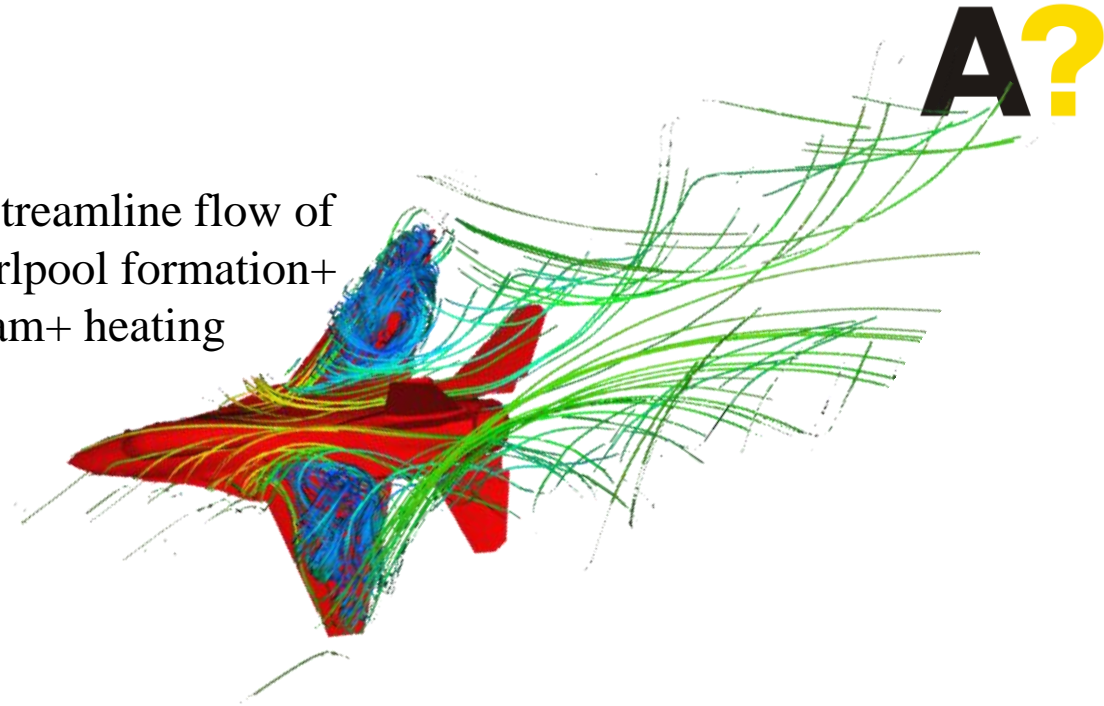
Multiphysics simulation: It could be done simultaneously

System requirement

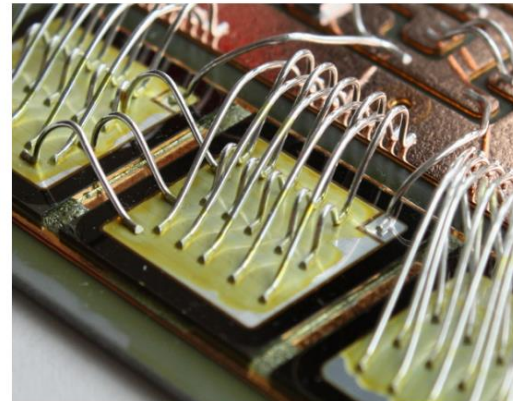
- *At least 1 GB memory, but 4 GB or more per processor core is recommended.*
- *1-5 GB of disk space, depending on your licensed products and installation options.*

Aerodynamics

Structural integrity + streamline flow of the air stream and whirlpool formation+ cooling via air jet stream+ heating

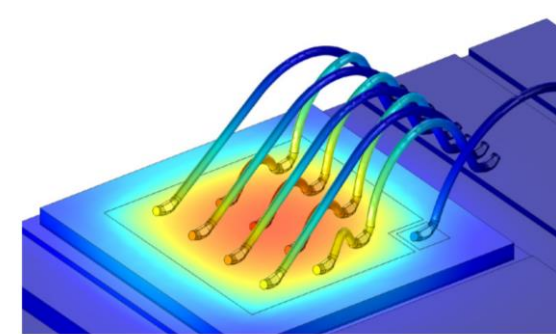


A?

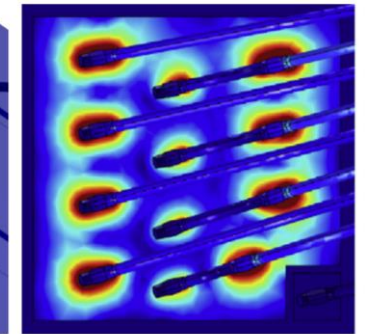


Wire bonding of a chip

Wire bonding of chips



Temperature distribution



Current distribution

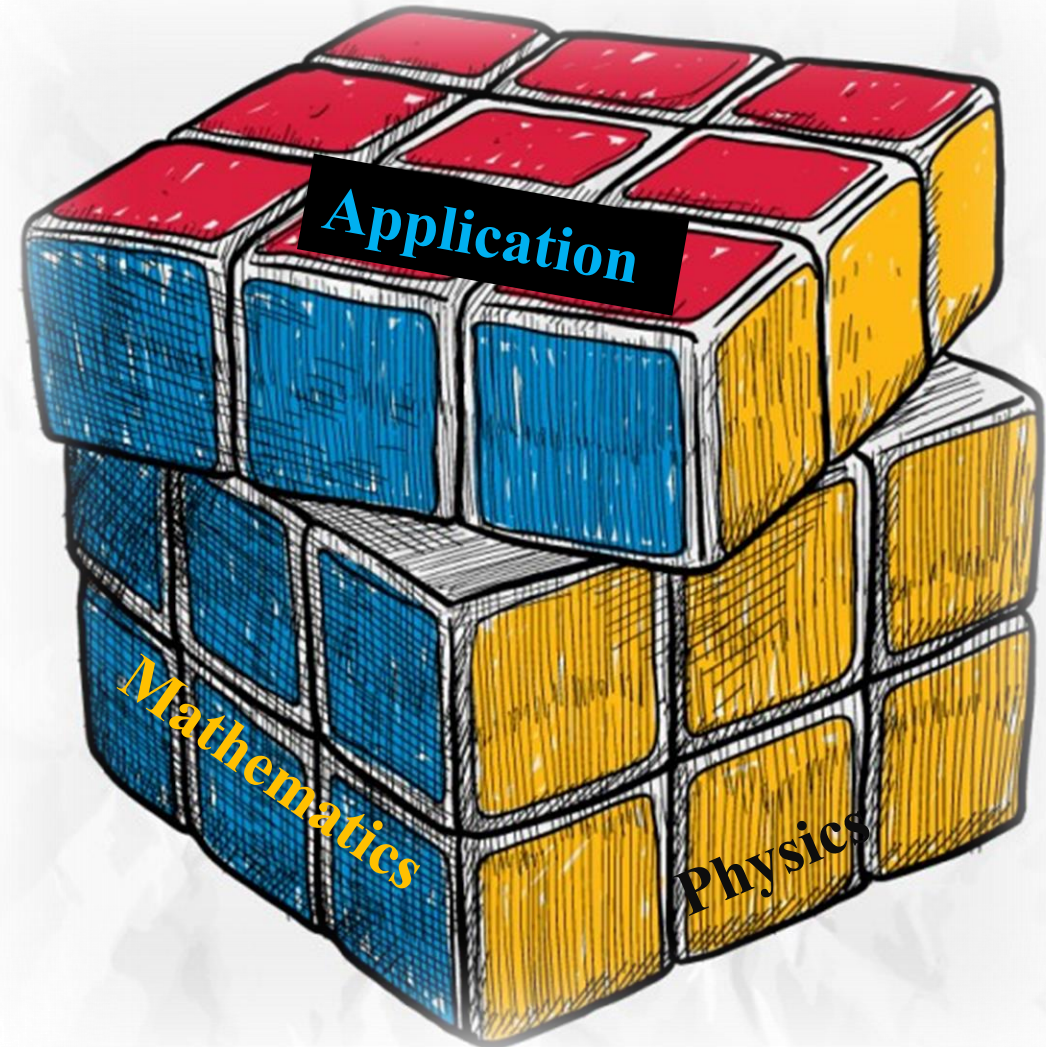
COMSOL: Multiphysics

Three facets of Multiphysics problems

- Mathematics
- Physics
- Applications

Top-down approach

Define the problem first, then physics and then use mathematics to solve it



Multiphysics Rubik

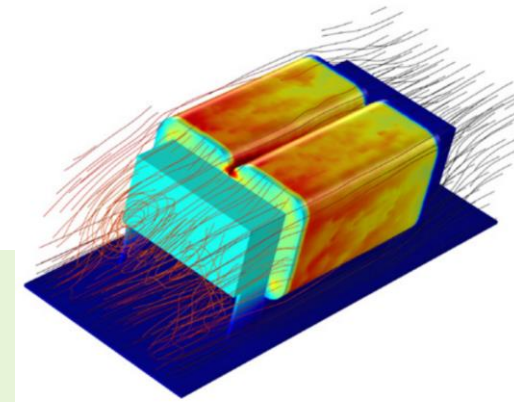
COMSOL: Multiphysics

Applications in physics:

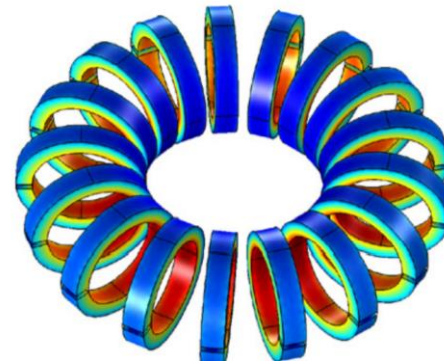
- Electrical
- Mechanical
- Fluid
- Chemical

Electrical Applications

- Joule heating
- Induction heating
- Microwave heating
- Electromagnetic waves
- Piezoelectric heating
- Piezoresistive effect
- Electrochemical effect



Air cooled DC choke



*Magnetic flux density
in toroidal choke*

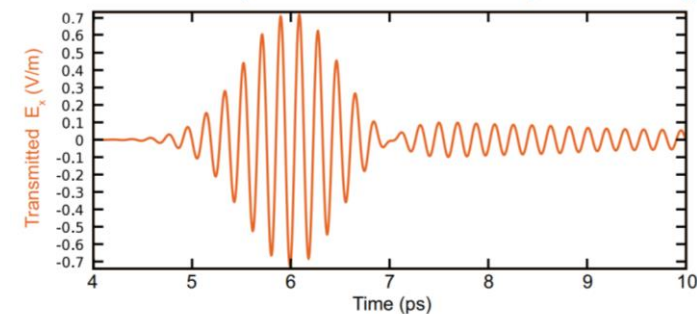
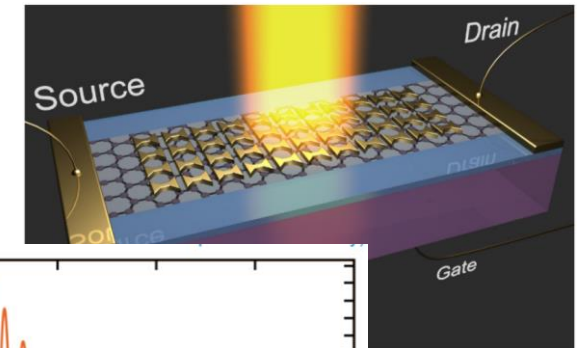


FIGURE 4: The time-dependent electric field of a Gaussian pulse transmitted through an array of graphene nanoribbons.

See Physics sections

COMSOL: Multiphysics

Mathematics module:

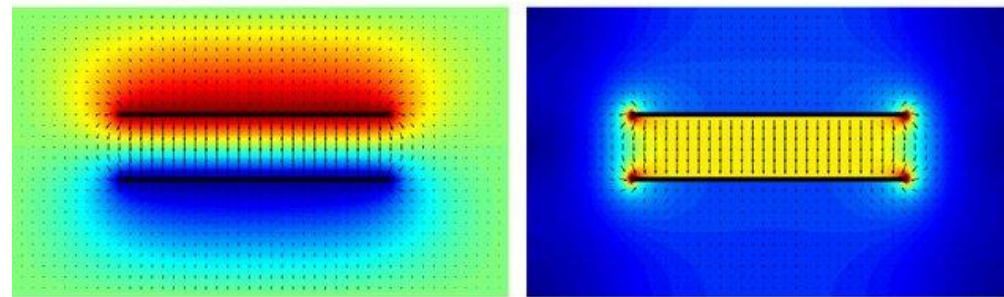
- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Partial differential equations

- Laplace Equation
- Poisson's Equation
- Wave Equation
- Stabilized Convection-Diffusion Equation
- Helmholtz equation
- Heat Equation
- Convection-Diffusion Equation

(depends upon more than one variable, different than ordinary differential equations)

Solution of 2D Poisson's Equation



$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$$

*The voltage potential and Electric field distribution
in parallel plate capacitor*

COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Partial differential equations

For second-order PDE for the function $u(x_1, x_2, \dots, x_n)$

$$F\left(\frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_n \partial x_n}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, x_1, x_2, \dots, x_n\right) = 0$$

x_i 's general co-ordinates: Spatial coordinates (Physics)

In spatial 2-D space co-ordinates

$$F\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial x}, \dots, \frac{\partial u}{\partial y}, x, y\right) = 0$$

For symmetric conditions $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

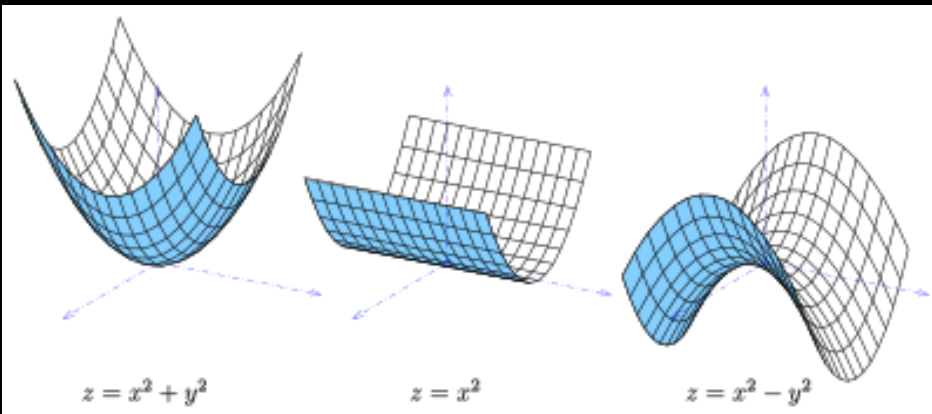
$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \dots (\text{lower order terms}) = 0$$

Quadratic equation: Discriminant: $B^2 - AC$

Elliptic PDE $B^2 - 4AC < 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Parabolic PDE $B^2 - 4AC = 0 \Rightarrow \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = 0$

Hyperbolic PDE $B^2 - 4AC > 0 \Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = 0$



COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Partial differential equations

$$Au_{xx} + 2Bu_{xy} + CU_{yy} + \dots (\text{lower order terms}) = 0$$

Quadratic equation: Discriminant: $B^2 - AC$

Elliptic PDE

$$B^2 - 4AC < 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

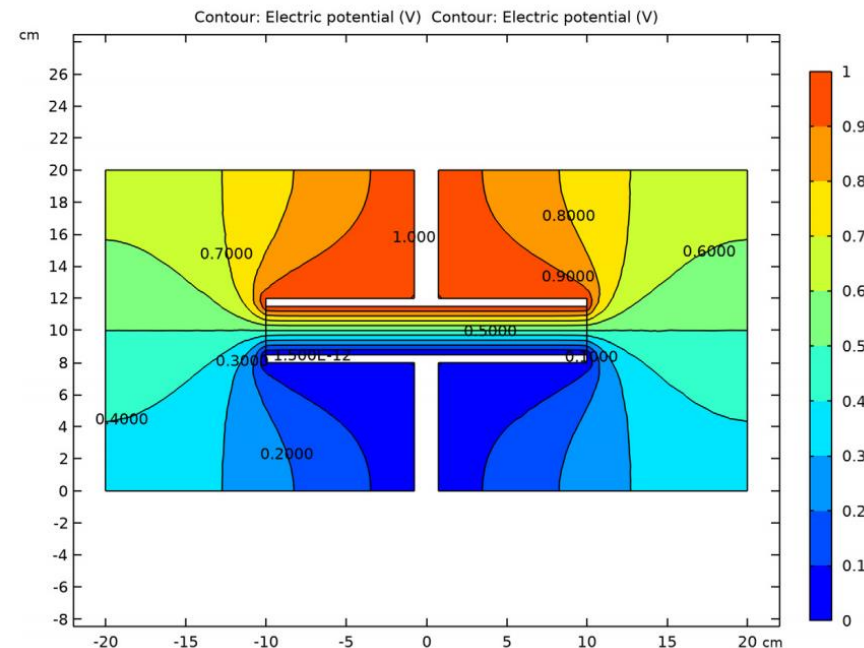
$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

$$x \in \mathbb{R}^n$$

$\Delta u = 0$ Laplace equations

$\Delta u = -\kappa\rho$ Poisson's equations

→ { Electric field, gravitational field, fluid potentials
Equilibrium state
(*No temporal/time dependent*)



COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Partial differential equations

$$Au_{xx} + 2Bu_{xy} + CU_{yy} + \dots (\text{lower order terms}) = 0$$

Quadratic equation: Discriminant: $B^2 - AC$

Parabolic PDE
 $B^2 - 4AC = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = 0$

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

$$x \in \mathbb{R}^n$$

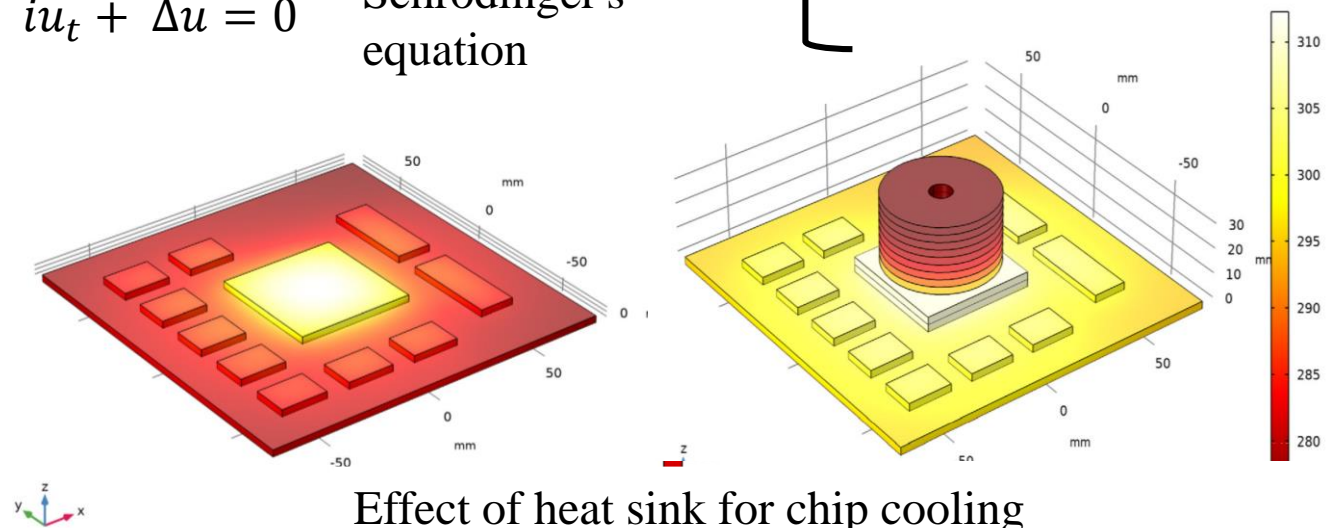
$$-\Delta u = \lambda u \quad \text{Helmholtz's equations}$$

$$u_t - \Delta u = 0 \quad \text{Heat and diffusion equation}$$

$$iu_t + \Delta u = 0 \quad \text{Schrodinger's equation}$$

Electromagnetic radiation,
acoustic, thermal/heat
transport, fluid transport

Time dependent equations



COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Partial differential equations

$$Au_{xx} + 2Bu_{xy} + CU_{yy} + \dots (\text{lower order terms}) = 0$$

Quadratic equation: Discriminant: $B^2 - AC$

Hyperbolic PDE
 $B^2 - 4AC > 0 \quad \longrightarrow \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = 0$

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

$$x \in \mathbb{R}^n$$

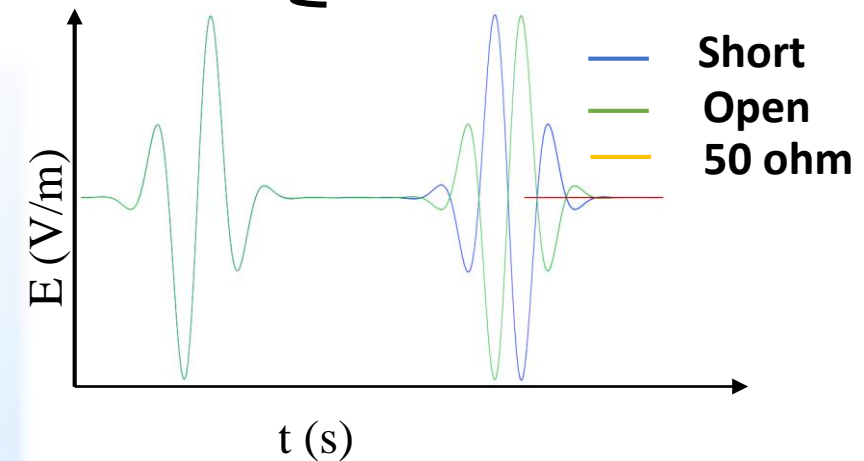
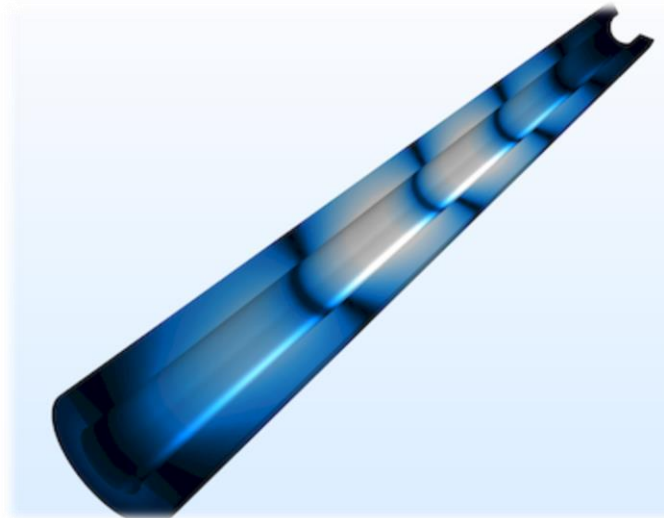
$$u_{tt} - c^2 \Delta u = 0$$

Wave equations

$$u_{tt} + u_t - u_{xx} = 0$$

Current/voltage
distribution over
transmission line

Electro-mechanical wave
Transmission line
Time dependent equations



Transient model of a coaxial cable

COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Interface and boundary value

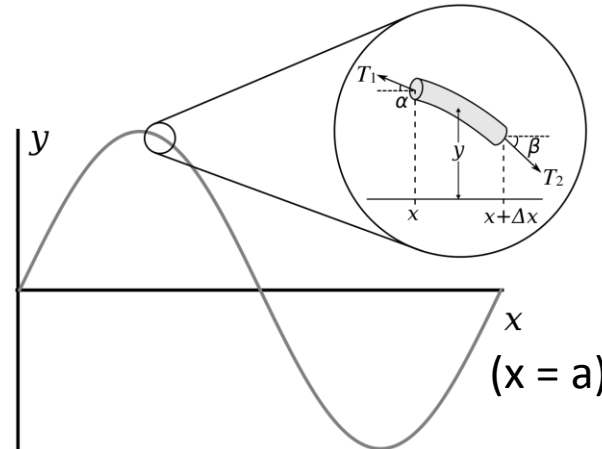
Ordinary differential equation



Initial value problem



Value of function and
its derivative at $t = 0$



Vibrating string

Any suggestion for the solution?

Partial differential equation



Boundary value problem



The value of function defined at the
boundary of the domain, where
solution is supposed to be defined

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Domains $t \in [0, \infty[$, $x \in [0, a]$

$$u(0, t) = u(a, t) = 0 \quad \forall t \in [0, \infty[$$

$$u(x, 0) = u(a, 0) = 0$$

$$\frac{\partial u(x, 0)}{\partial t} = \frac{\partial u(a, 0)}{\partial t} = 0$$

COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

Interface and boundary value (RF Module)

Terminating impedance	$\frac{\mathbf{n} \cdot \nabla V}{R + j\omega L} + \frac{V}{Z_L} = 0$	Only at external boundary
Open circuit	$\mathbf{n} \cdot \nabla V = 0$	Infinite impedance and zero current
Short circuit	$V = 0$	Zero impedance and zero voltage
Matched port	$Z_0 = Z_L$	Default value of $Z_L = 50 \Omega$
Scattering boundary condition	$\mathbf{n} \times \mathbf{E} = Z_0 \mathbf{H}$	
Perfect Magnetic conductor	$\mathbf{n} \times \mathbf{H} = 0$	High impedance at the boundary, meaning current density = 0
Zero charge	$\mathbf{n} \cdot \mathbf{D} = 0$	Exterior boundary and edges with charge conservation

COMSOL: Multiphysics

Mathematics module:

- Partial differential equations
- Ordinary differential equations
- Optimization and sensitivity
- Interface and boundary value
- Coordinates

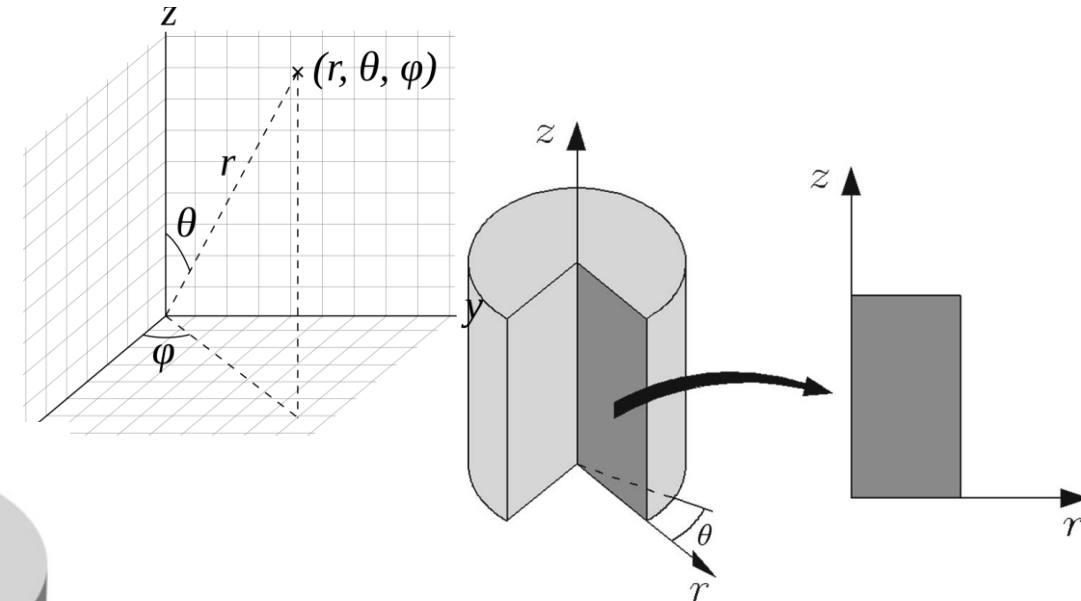
Coordinate systems

Geometry	Default coordinates
2D	x y
3D	x y z
Axial symmetry 2D	r φ z

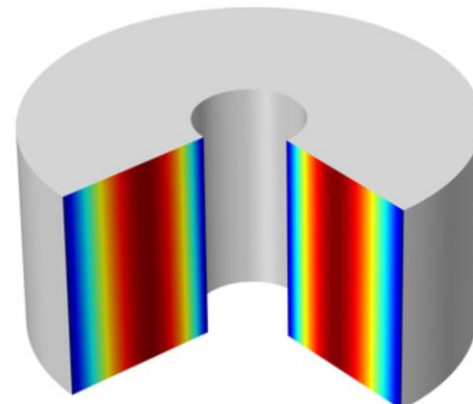
*Additional geometry
1D, axial symmetry
1D and 0 D*

3D Geometry

- Spherical
- Cylindrical
- Cartesian



Example: Axial symmetry 2D



Norm of the electric
field for the m=1
mode at 2.122 GHz

COMSOL: Multiphysics

Application (Physics) module:

- AC/DC
- RF
- Semiconductor
- Mathematics
- Acoustic
- Chemical species transport
- Electrochemistry
- Fluid flow
- Heat Transport
- Optics
- Plasma
- Structural Mechanics

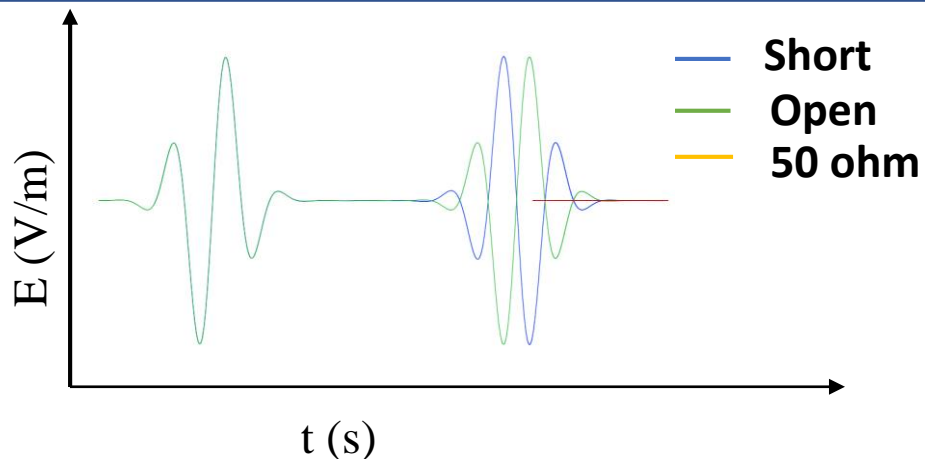
Physics module (Electrical mainly)

- Electrical Fields and currents
 - Magnetic Fields, No currents
 - Electromagnetic fields
 - Electromagnetic heating
 - Electromagnetic and mechanics
 - Particle tracing
 - Electrical circuits
 - Electromagnetic waves and scattering
 - Electromagnetic waves, frequency domains
 - Electromagnetic waves, Time explicit
 - Electromagnetic waves, Transient
 - Transmission line
 - Semiconductor
 - Schrodinger equation
 - Schrodinger-Poisson equation
 - Semiconductor Optoelectronics Beam envelopes
 - Semiconductor Optoelectronics Frequency domains
 - .
 - .
 - Mathematics (PDE, ODE, etc.)
- Diagram illustrating the categorization of Physics module features:
- AC/DC** (Grouped by a blue bracket):
 - Electrical Fields and currents
 - Magnetic Fields, No currents
 - Electromagnetic fields
 - Electromagnetic heating
 - Electromagnetic and mechanics
 - Particle tracing
 - Electrical circuits
 - RF** (Grouped by a blue bracket):
 - Electromagnetic waves and scattering
 - Electromagnetic waves, frequency domains
 - Electromagnetic waves, Time explicit
 - Electromagnetic waves, Transient
 - Transmission line
 - Semiconductor** (Grouped by a blue bracket):
 - Semiconductor
 - Schrodinger equation
 - Schrodinger-Poisson equation
 - Semiconductor Optoelectronics Beam envelopes
 - Semiconductor Optoelectronics Frequency domains

COMSOL: Multiphysics

Study module:

- Frequency Domain
- Stationary
- Time domain
- Eigen Frequency
- Custom studies



Physics studies module (Electrical mainly)

Frequency domain

$$E = E_0 \sin(\omega t + \varphi)$$

$$B = B_0 \sin(\omega t + \varphi)$$

It compute the response of a linear (linearized) model for harmonic excitations via one or several frequencies. Its output is typically displayed as a transfer function, for example, magnitude or phase of deformation, sound pressure, impedance, or scattering parameters versus frequency.

Stationary studies

$$\frac{\partial E}{\partial t} = 0, \frac{\partial B}{\partial t} = 0$$

It is used for field variables which doesn't change over time. Examples: In electromagnetics, it is used to compute static electric or magnetic fields, as well as direct currents. In heat transfer, it is used to compute the temperature field at thermal equilibrium.

Time domain

$$E(t), B(t)$$

This study is used when field variables change over time. Examples: In electromagnetics, it is used to compute transient electromagnetic fields, including electromagnetic wave propagation in the time domain. In heat transfer, it is used to compute temperature changes over time

Eigen Frequency





It is used for computing eigenmodes and eigenfrequencies of a linear (linearized) model. Examples: In electromagnetics, the eigenfrequencies correspond to the resonant frequencies and the eigenmodes correspond to the normalized electromagnetic field at the eigenfrequencies.

Physics studies module (Summary)

COMSOL: Multiphysics

Study module: **AC/DC**

- Frequency Domain
- Stationary
- Time domain
- Eigen Frequency
- Custom studies





PHYSICS INTERFACE	ICON	TAG	SPACE DIMENSION	AVAILABLE STUDY TYPE
 AC/DC				
Electric Currents ¹		ec	all dimensions	stationary; stationary source sweep; frequency domain; time dependent; small signal analysis, frequency domain; eigenfrequency
Electric Currents in Shells		ecis	3D	stationary; frequency domain; time dependent; eigenfrequency
Electric Currents in Layered Shells		ecis	3D	stationary; frequency domain; time dependent; eigenfrequency

Physics studies module (Summary)

COMSOL: Multiphysics






Study module: **AC/DC**

- Frequency Domain
- Stationary
- Time domain
- Eigen Frequency
- Custom studies

PHYSICS INTERFACE	ICON	TAG	SPACE DIMENSION	AVAILABLE STUDY TYPE
Electrical Circuit		cir	Not space dependent	stationary; frequency domain; time dependent; small signal analysis, frequency domain; eigenfrequency
Electrostatics ¹		es	all dimensions	stationary; time dependent; stationary source sweep; eigenfrequency; frequency domain; small signal analysis, frequency domain; eigenfrequency
Electrostatics, Boundary Elements		esbe	3D, 2D	stationary; stationary source sweep; frequency domain; small signal analysis, frequency domain
Magnetic Fields ¹		mf	3D, 2D, 2D axisymmetric	stationary; frequency domain; time dependent; small signal analysis, frequency domain; coil geometry analysis (3D only); time to frequency

Physics studies module (Summary)COMSOL:
MultiphysicsStudy module: **AC/DC**

- Frequency Domain
- Stationary
- Time domain
- Eigen Frequency
- Custom studies







PHYSICS INTERFACE	ICON	TAG	SPACE DIMENSION	AVAILABLE STUDY TYPE
Magnetic and Electric Fields		mef	3D, 2D, 2D axisymmetric	stationary; frequency domain; small signal analysis, frequency domain; coil geometry analysis (3D only)
Magnetic Field Formulation		mfh	3D, 2D, 2D axisymmetric	stationary; frequency domain; time dependent; small signal analysis, frequency domain; time to frequency losses
Magnetic Fields, No Currents		mfnc	3D, 2D, 2D axisymmetric	stationary; frequency domain; time dependent; time to frequency losses
Magnetic Fields, No Currents, Boundary Elements		mfncbe	3D, 2D	stationary
Magnetic Fields, Currents Only		mfco	3D	stationary; stationary source sweep with initialization

COMSOL: Multiphysics

Study module: **RF**

- Frequency Domain
- Stationary
- Time domain
- Eigen Frequency
- Custom studies

Physics studies module (Summary)

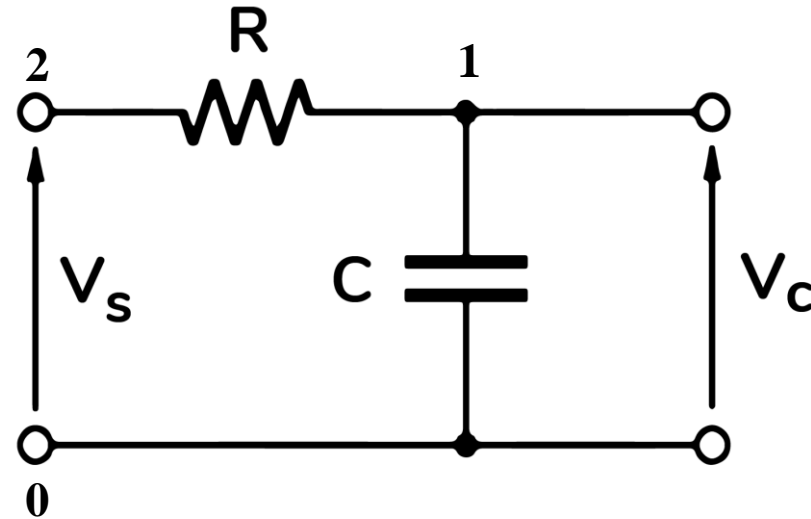
Microwave Heating ¹		—	3D, 2D, 2D axisymmetric	frequency-stationary; frequency-transient; sequential frequency-stationary; sequential frequency-transient
 Radio Frequency				
Electromagnetic Waves, Frequency Domain		emw	3D, 2D, 2D axisymmetric	adaptive frequency sweep; boundary mode analysis; eigenfrequency; frequency domain; frequency domain, modal; mode analysis (2D and 2D axisymmetric models only)
Electromagnetic Waves, Time Explicit		ewte	3D, 2D, 2D axisymmetric	time dependent; time dependent with FFT
Electromagnetic Waves, Transient		temw	3D, 2D, 2D axisymmetric	eigenfrequency; time dependent; time dependent, modal; time dependent with FFT
Transmission Line		tl	3D, 2D, 1D	eigenfrequency; frequency domain
¹ This physics interface is a predefined multiphysics coupling that automatically adds all the physics interfaces and coupling features required.				

COMSOL: Multiphysics

Modelling: RF/AC-DC

- Understand the problem
- Space dimension
- Use boundary condition to reduced the space dimension
- Source for excitations
- Study module: time domain, frequency domain, etc.
- Variables to study: Eigen function, voltage, electric field, etc.
- Materials: Conductive, insulating etc.
- Meshing
- Solve

Modelling: Low pass filter



Circuit simulation

Surprisingly, you could do this here in COMSOL but waste of time, unless until you would like to couple it to the some 3D model

Dimension: Anyone could be used but 1 D will be most appropriate

Boundary condition: $V_s = V$ at 2 and ground node at 1

Source of excitations: $V_s = V \sin(\omega t + \varphi) + A$

Study module: time domain then $V = 0$ and if frequency domain $A = 0$

Output: Current or Voltage

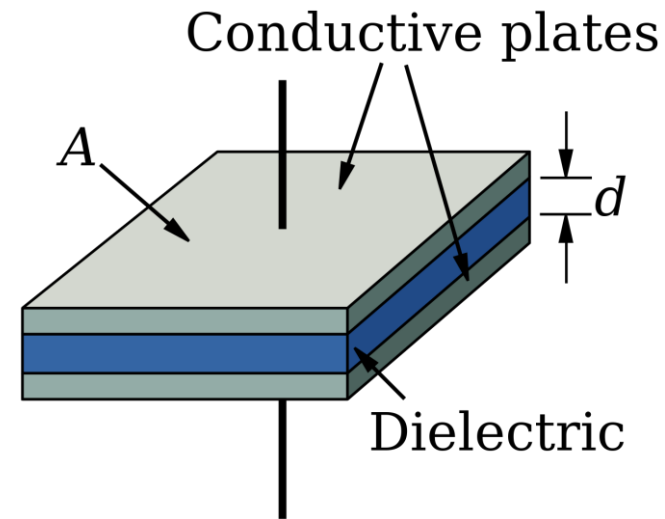
Components: Resistors and capacitors

COMSOL: Multiphysics

Modelling: **RF/AC-DC**

- Understand the problem
- Space dimension
- Use boundary condition to reduced the space dimension
- Source for excitations
- Study module: time domain, frequency domain, etc.
- Variables to study: Eigen function, voltage, electric field, etc.
- Materials: Conductive, insulating etc.
- Meshing
- Solve

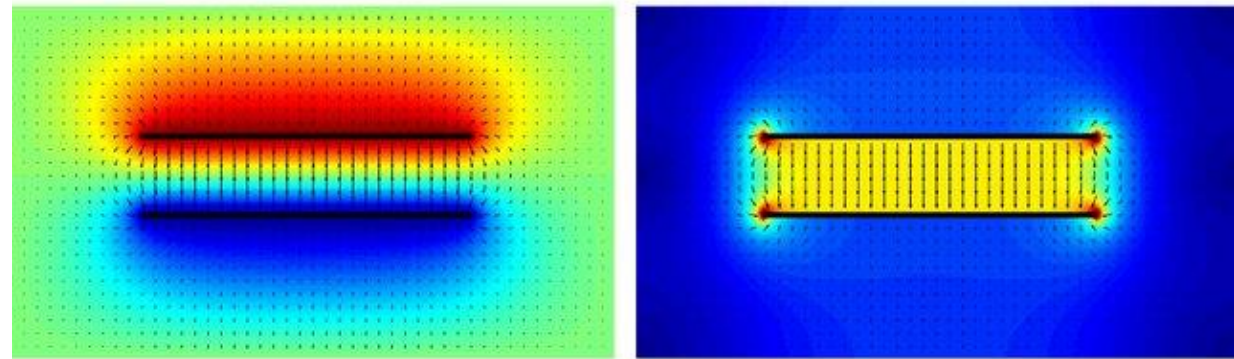
Modeling: Parallel plate capacitors



Geometry?

PDE?

Boundary condition?



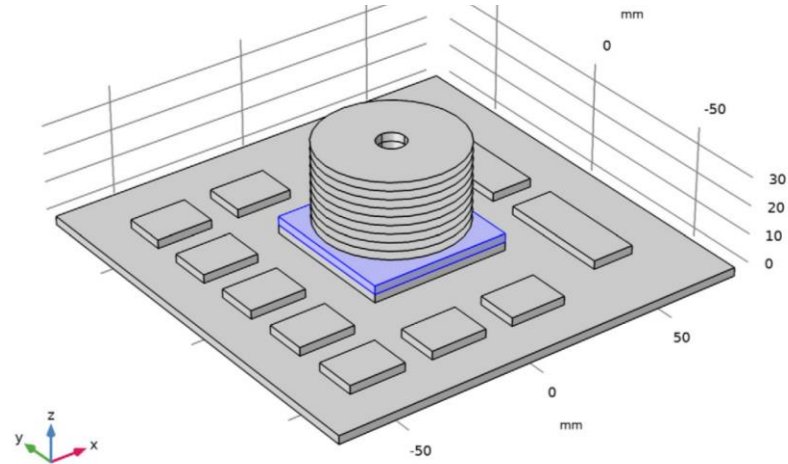
The voltage potential and Electric field distribution

Modeling: Heat Sink

COMSOL: Multiphysics

Modelling: RF/AC-DC

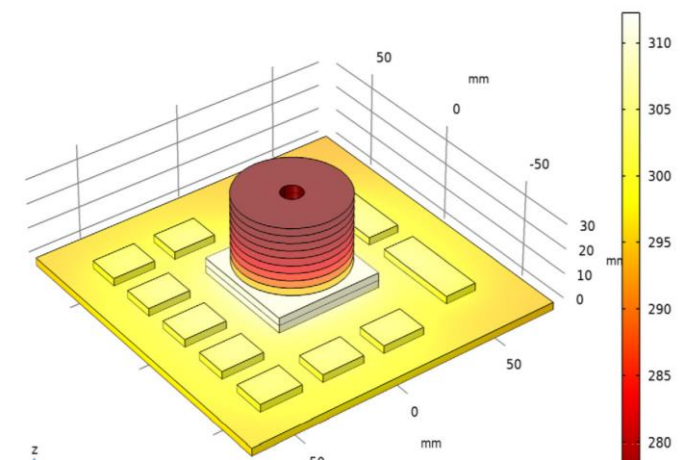
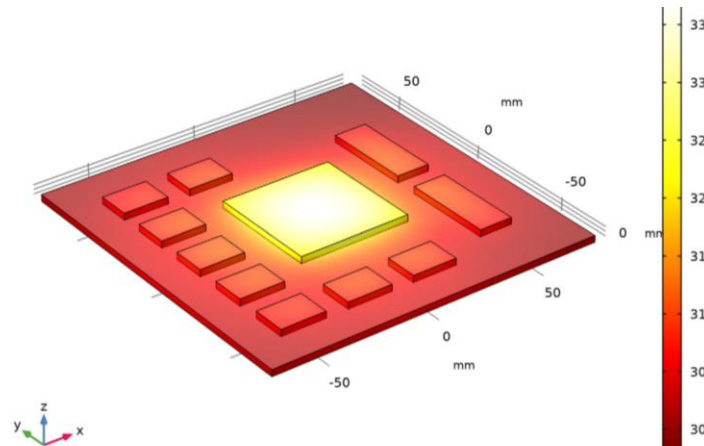
- Understand the problem
- Space dimension
- Use boundary condition to reduced the space dimension
- Source for excitations
- Study module: time domain, frequency domain, etc.
- Variables to study: Eigen function, voltage, electric field, etc.
- Materials: Conductive, insulating etc.
- Meshing
- Solve



Geometry?

PDE?

Boundary condition?



Temperature of the PCB in absence and presence of heat sink

COMSOL: Multiphysics

Modelling: **RF/AC-DC**

- Meshing

Two option for meshing in COMSOL

1. Physics controlled mesh: Fully automatic
2. User controlled mesh

Modeling: Mesh structure (3 D/2-D solid construction)

A?

Finite element method

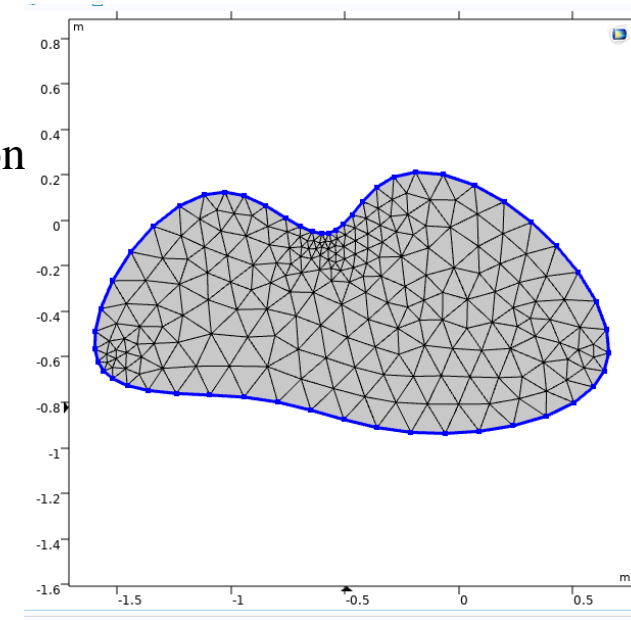
- Geometry divided in smaller pieces
- Solution piece wise continuous function

Discretization of the geometry

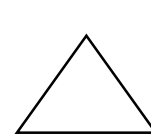
- Sequence of the division
- Geometry use for the division
- Element order: Interpolation between the nodes

Geometry

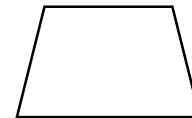
Controlled by physics



Common meshing element available



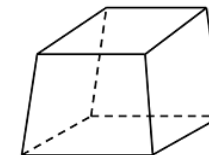
Triangle



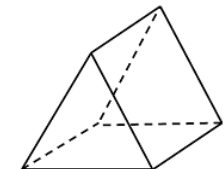
Tetrahedron



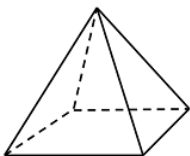
Tetrahedron



Hexahedron



Triangular
Prism



Prism

Default element is triangle for 2D and tetrahedron for 3 D

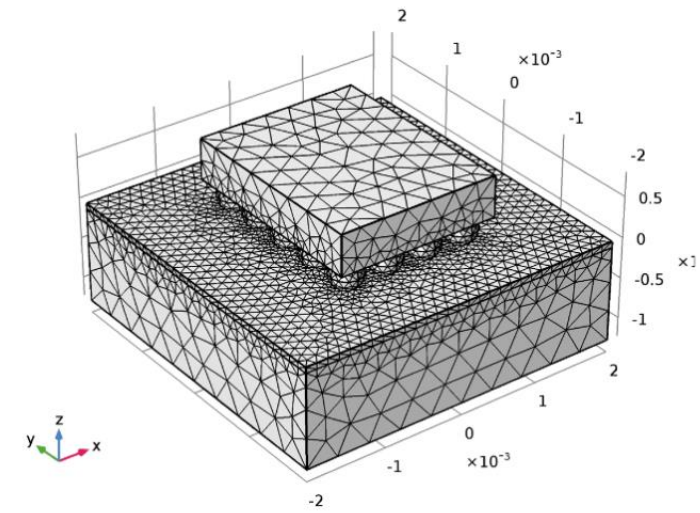
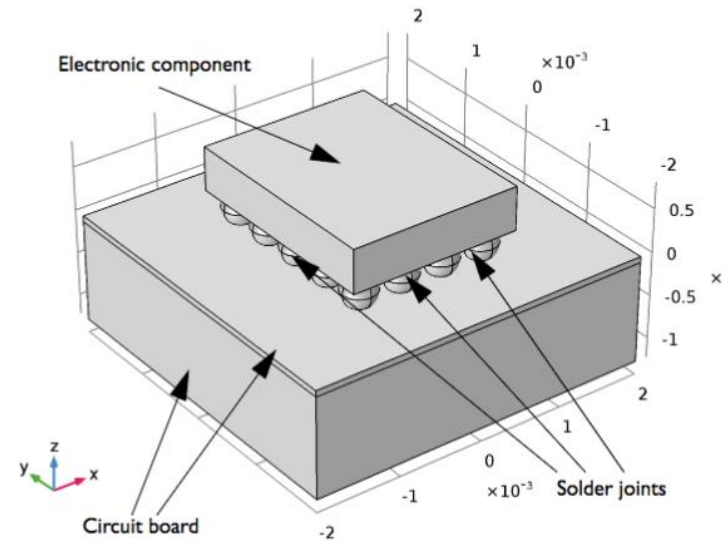
COMSOL: Multiphysics

Modelling: **RF/AC-DC**

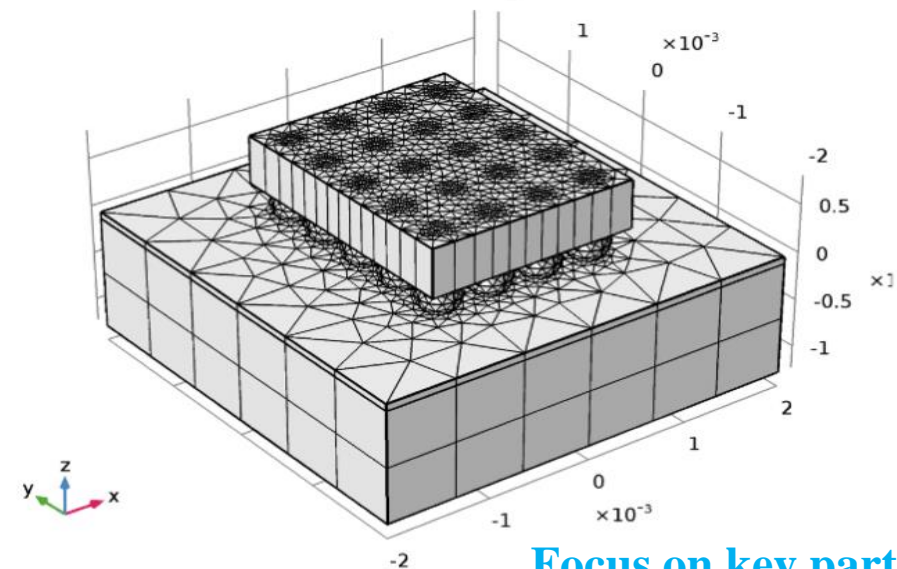
- Meshing

Modeling: Mesh structure (3 D/2-D solid construction)

A?



Is this optimized mesh structure?



Focus on key part of structure

COMSOL Multiphysics Desktop



Coaxial_line_1.mph - COMSOL Multiphysics

File Home Definitions Geometry Materials Physics Mesh Study Results Developer Electric Field (emw)

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Model Builder

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Settings

3D Plot Group

Plot

Label: Electric Field (emw)

Data

Dataset: Study 1/Solution 1 (sol1)

Parameter value (freq (GHz)): 0.1

Selection

Title

Plot Settings

View: Automatic

☒ Show hidden entities

☒ Propagate hiding to lower dimensions

☒ Plot dataset edges

Color: From theme

Frame: Spatial (x, y, z)

Color Legend

☒ Show legends

☒ Show maximum and minimum values

☐ Show units

Position: Right

Text color: Blue

Number Format

Window Settings

Graphics Plot 1

freq(1)=0.1 GHz

Multislice: Electric field norm (V/m) Arrow Volume: Electric field Arrow Volume: Magnetic field Arrow Volume: Electric field + Power flow, time average

1.96 x 10⁻⁴ x 10⁻⁵

18 16 14 12 10 8 6 4 2 0

Messages Progress Log Table

COMSOL Multiphysics 5.6.0.280

[Mar 14, 2021 11:35 PM] Opened file: C:\Users\kumarm1\OneDrive\Documents\Coaxial_line.mph

[Mar 14, 2021 11:35 PM] Finalized geometry is empty.

[Mar 14, 2021 11:36 PM] Opened file: C:\Users\kumarm1\OneDrive\Documents\Coaxial_line_1.mph

1.84 GB | 1.8 GB

Impedance of a coaxial cable

$$V = V_i - V_o = - \int_{r_o}^{r_i} \mathbf{E} \cdot d\mathbf{r} \quad (1)$$

Similarly, the current is obtained as a line integral of the magnetic field along the boundary of either conductor or any closed contour, C , bisecting the space between the conductors:

$$I = \oint_C \mathbf{H} \cdot d\mathbf{r}$$

The voltage and current in the direction out of the plane are positive for integration paths oriented as in [Figure 2](#).

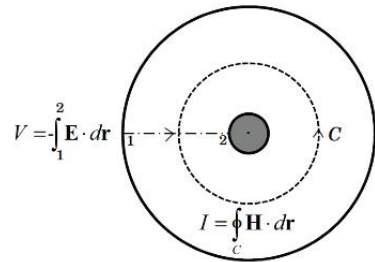


Figure 2: The impedance of a coaxial cable can be found from the voltage, V , and current, I , which are computed via line integrals as shown.

The value of Z_0 obtained in this way, should be compared with the analytic result

$$Z_{0,\text{analytic}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} \log\left(\frac{r_o}{r_i}\right) \approx 74.5 \, \Omega$$



Electromagnetic wave model to an electrical circuit

Introduction

An application built with the RF Module can be connected to an electrical circuit equivalent, if there is some structure outside of the model space that you want to approximate as a circuit equivalent. An example is shown in Figure 1, the 3D model of a coaxial cable is connected to a voltage source, in series with a matched impedance, and sees a load, also of matched impedance.

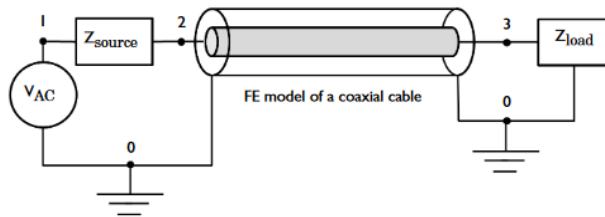


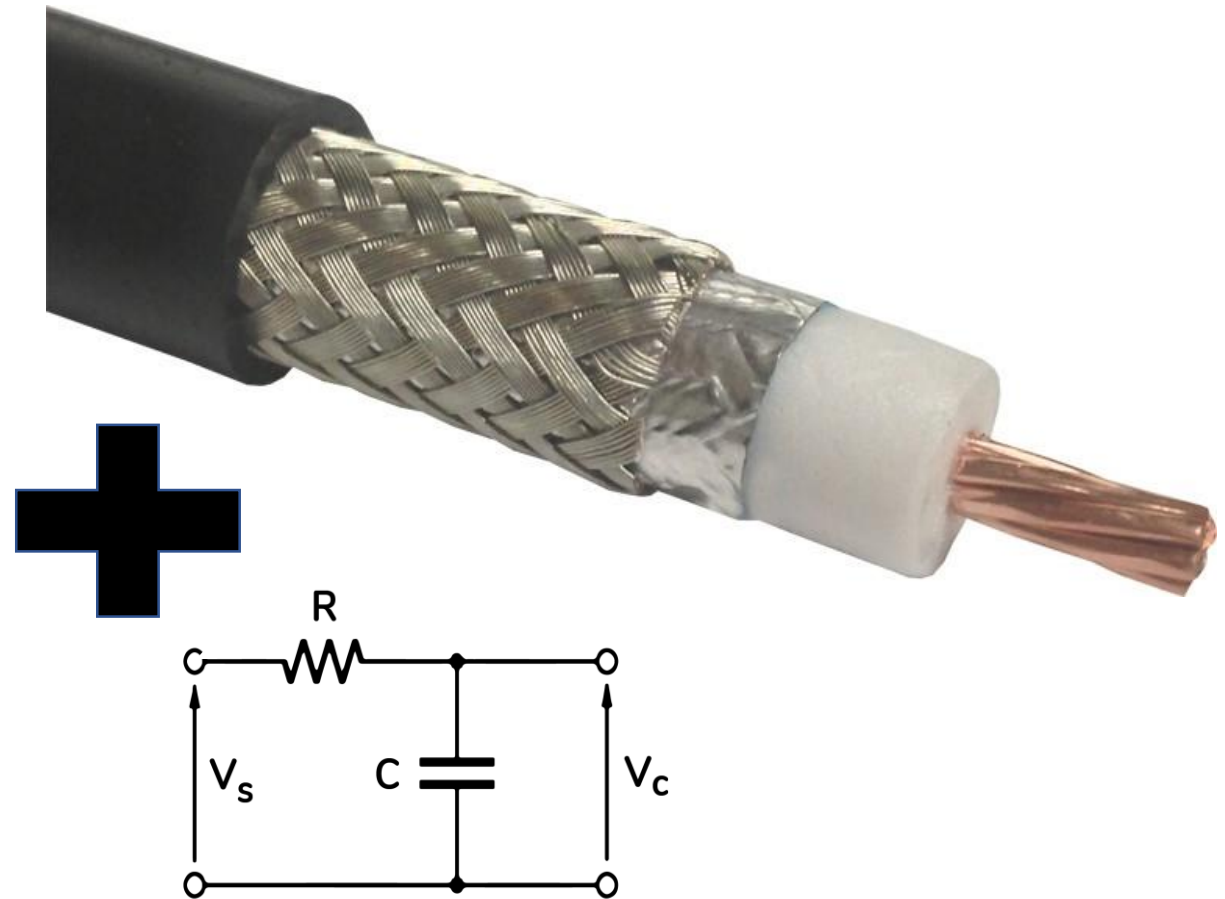
Figure 1: Schematic of a section of a coaxial transmission line connected to a voltage source, source impedance, and load.

Model Definition

The geometry in this example is a short section of a air-filled coaxial transmission line, shown schematically in Figure 1. A 3D modeling space is used to model the coaxial cable. The walls of the coax are treated as perfect electric conductors. This is appropriate when the skin depth, and the losses in the conductors, are insignificant.

At one end of the coaxial cable, Lumped Port boundary condition is used to connect the model to nodes 0 and 2 of the Electrical Circuit. A Voltage Source between circuit nodes 0 and 1 excites the system, and a Resistor representing the source impedance is added between nodes 1 and 2. Node 0 is specified as the Ground Node by default, which fixes the absolute voltage. The connection from the Electrical Circuit model to the Electromagnetic Waves interface is via the External I Vs. U features.

At the other end of the coaxial cable, another Lumped Port boundary condition is used to connect the model to nodes 3 and 0 of the Electrical Circuit. A Resistor which works as a matched load is added between nodes 3 and 0. At any nonzero frequency, the absolute voltage has no well-defined meaning, voltage only has a meaning as the path integral of



COMSOL: Multiphysics

COMSOL simulation

Assignment and tutorial on
Wednesday

Things to do for the next class

Please download the software: download.aalto.fi

Please go through COMSOL desktop

Enough text and videos available online [COMSOL Documentation](#)

Brush up partial differential equation and Electromagnetism