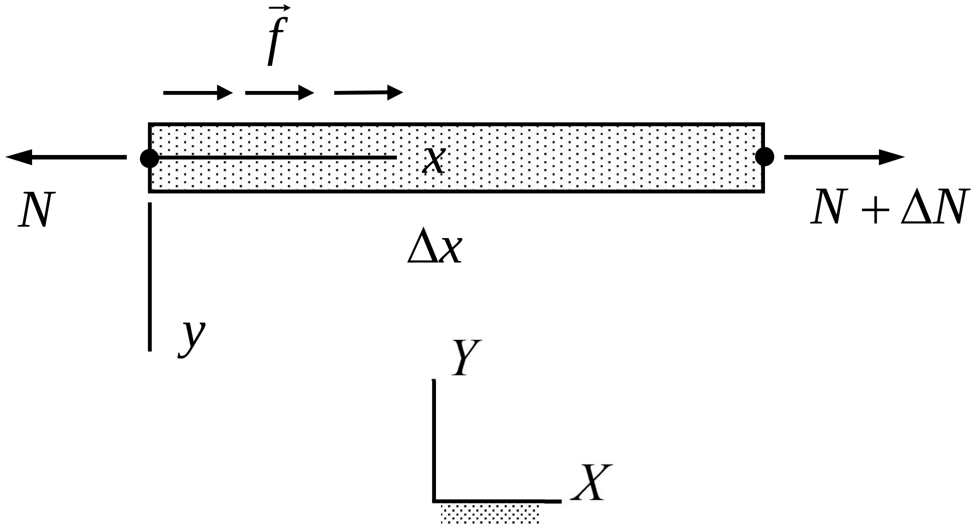
4 BEAM

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 12:

- □ Timoshenko and Bernoulli beam models.
- □ Derivation of the beam equations by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus. Beam equilibrium and constitutive equations in their tensor forms.
- □ Component representation of the beam equations in $(x, y, z)-$ and $(s, n, b)-$coordinate systems.
- □ Kinematics, virtual work density, and constitutive equation in $(s, n, b)-$coordinates

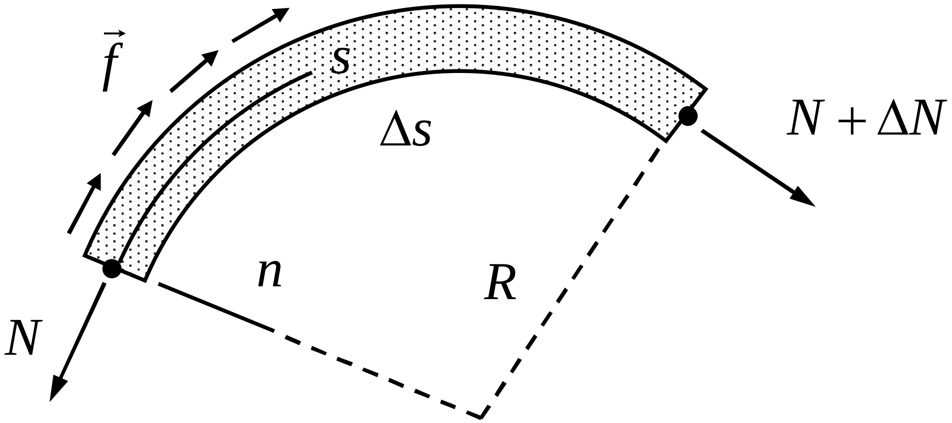
THE CURVATURE EFFECT



$$\vec{N} = N\vec{i}$$

$$\vec{f} = f_x\vec{i} + f_y\vec{j}$$

$$\frac{d\vec{N}}{dx} + \vec{f} = 0$$



$$\vec{N} = N\vec{e}_s$$

$$\vec{f} = f_s\vec{e}_s + f_n\vec{e}_n$$

$$\frac{d\vec{N}}{ds} + \vec{f} = 0$$

The basis vectors of the material $(x, y, z)-$coordinate system are constants

$$\frac{d\vec{N}}{dx} + \vec{f} = \frac{d(N\vec{i})}{dx} + f_x\vec{i} + f_y\vec{j} = (\frac{dN}{dx} + f_x)\vec{i} + f_y\vec{j} = 0 \quad \Leftrightarrow$$

$$\frac{dN}{dx} + f_x = 0 \quad \text{and} \quad f_y = 0. \quad \leftarrow$$

The basis vectors of the material (s, n, b) coordinate system are *not* constants

$$\frac{d\vec{N}}{ds} + \vec{f} = \frac{d(N\vec{e}_s)}{ds} + f_s\vec{e}_s + f_n\vec{e}_n = (\frac{dN}{ds} + f_s)\vec{e}_s + (\frac{N}{R} + f_n)\vec{e}_n = 0 \quad \Leftrightarrow$$

$$\frac{dN}{ds} + f_s = 0 \quad \text{and} \quad \frac{N}{R} + f_n = 0 \ . \quad \leftarrow$$

EXAMPLE Consider an inextensible string having constant mass per unit length (m) under its own weight. Write the equilibrium equations in the structural (x, y, z) system with the selection x as the curve parameter and show that $y - c = a\cosh[(x-b)/a]$, in which a, b, c are constants, is a solution (Catenary curve)

www.math.udel.edu/.../Chain/Demo%20015.jpg

teachers.sduhsd.k12.ca.us/.../GatewayArch.jpg

Let us write the equilibrium equations $dN/ds + f_s = 0$ and $N/R + f_n = 0$ in terms of x and y as we would like to get the solution in form $y = y(x)$. Using $y' = dy/dx$,

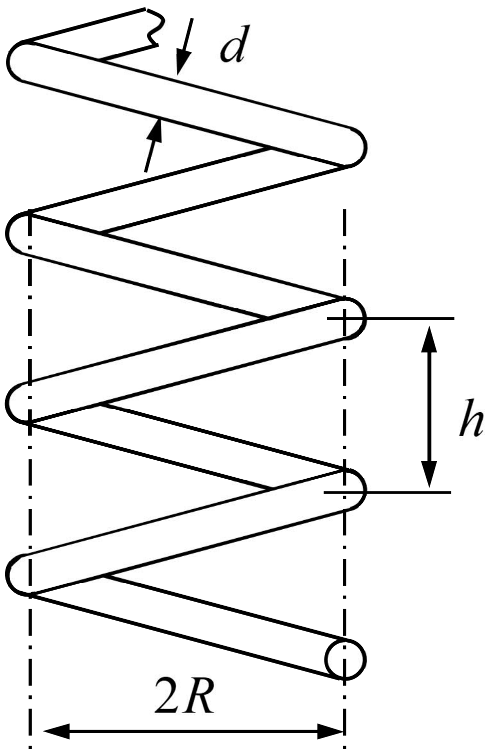
$$\frac{d}{ds} = \frac{1}{(1+y'^2)^{1/2}}\frac{d}{dx}, \quad \frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}}, \quad f_s = -\frac{mgy'}{(1+y'^2)^{1/2}}, \text{ and } f_n = -\frac{mg}{(1+y'^2)^{1/2}}$$

elimination of the forces gives the equation for the geometry

$$\left(\frac{1+y'^2}{y''} - y\right)' = 0 \quad \Rightarrow \quad y - c = a\cosh\left(\frac{x-b}{a}\right).$$ ←

Hence, the shape is a Catenary curve. The solution to the non-linear differential equation can be obtained by using, e.g., Mathematica.

EXAMPLE Equilibrium equations in (s,n,b) system can be used, e.g., to derive the well-known formula for the elastic spring coefficient ($\Delta F = k\Delta L$). The geometrical parameters are coil radius R, pitch h, number of coils n, and diameter d of wire. Material parameters are Young's modulus E and shear modulus G.

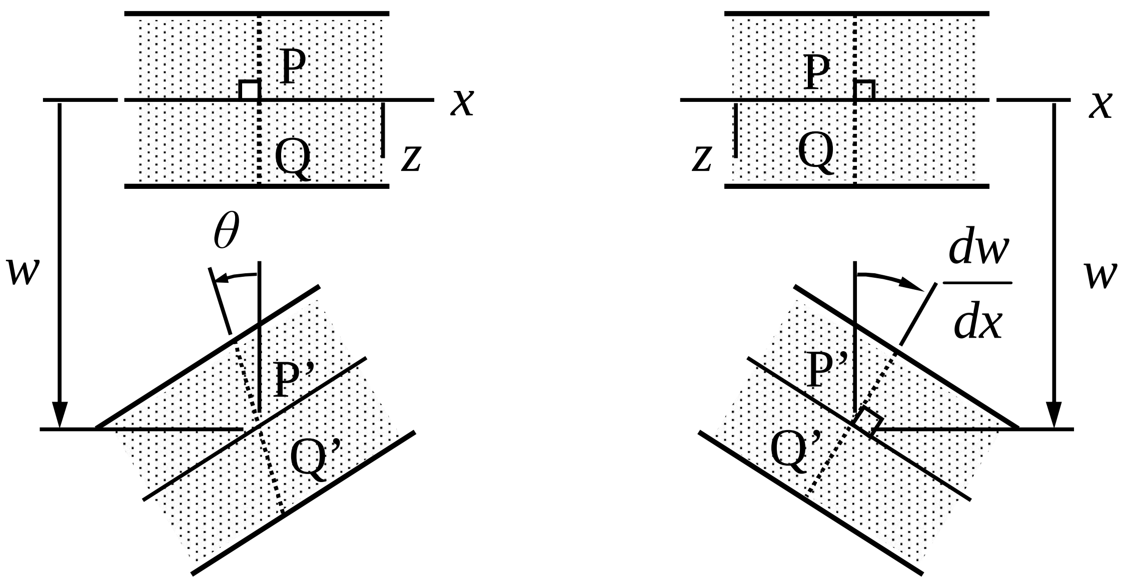


$$k = \frac{EG\pi d^4\sqrt{h^2+4\pi^2R^2}}{4n(Gd^2h^2+16Gh^2R^2+4E\pi^2d^2R^2+32E\pi^2R^4)} \Rightarrow$$

$$k \approx \frac{Gd^4}{64nR^3} \text{ when } (\frac{d}{R})^2 \ll 1 \text{ and } (\frac{h}{R})^2 \ll 1$$

the formula of literature

BEAM MODEL



Timoshenko $(\vec{u}_{\mathrm{P}} = 0)$ **Bernoulli** $(\vec{u}_{\mathrm{P}} = 0)$

Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta} \times \vec{\rho}_{\mathrm{PQ}}$ (see any textbook on statics and/or dynamics).

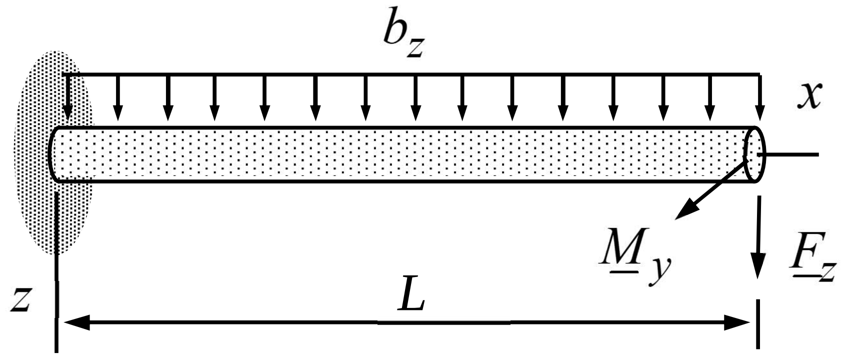
The kinematic assumption means that the normal planes to the mid-curve move as rigid bodies in deformation. In terms of displacement of the translation point $y = z = 0$ and small rotation of the cross-section, displacement of a particle identified by (x, y, z) is given by $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j} + \psi\vec{k}) \times (y\vec{j} + z\vec{k})$. According to the kinetic assumption of the beam model $\sigma_{zz} = \sigma_{yy} = 0$.

In the Bernoulli model, the cross-sections are assumed to remain normal planes to the mid-curve in deformation which brings the *Bernoulli constraints*

$$\gamma_{xy} = \frac{dv}{dx} - \psi = 0 \text{ and } \gamma_{xz} = \frac{dw}{dx} + \theta = 0.$$

Due to the more severe assumptions, the modeling error of the Bernoulli model is larger than that of the Timoshenko beam model!

TIMOSHENKO BEAM BENDING $(x,z)-$plane



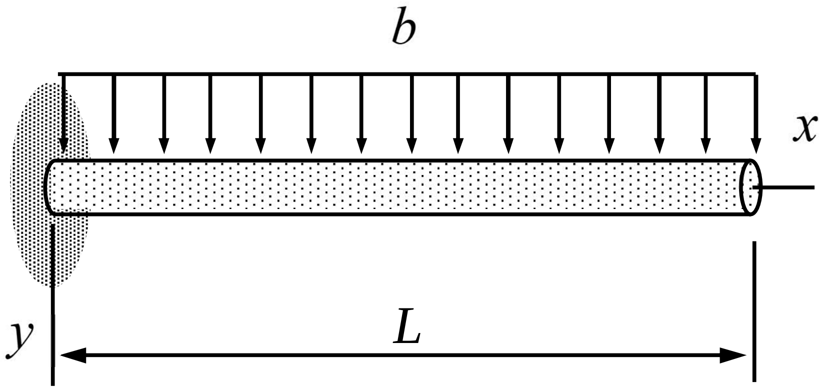
Equilibrium eqs. : $\dfrac{dQ_z}{dx}+b_z=0$ and $\dfrac{dM_y}{dx}-Q_z+c_y=0$ in $(0,L)$

Constitutive eqs. : $Q_z=GA(\dfrac{dw}{dx}+\theta)$ and $M_y=EI\dfrac{d\theta}{dx}$ in $(0,L)$

Natural boundary condition: $M_y=\underline{M}_y$ and $Q_z=\underline{F}_z$ at $x=L$

Essential boundary condition: $\theta=0$ and $w=0$ at $x=0$

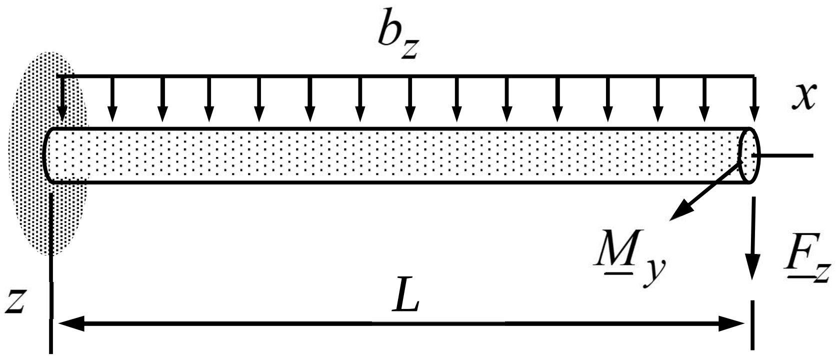
EXAMPLE 4.1 Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I and the loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Timoshenko beam model.



"Timoshenko effect"
$\sim 1+(t/L)^2$

Answer $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}\dfrac{4EI+GAL^2}{GAL^2}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

BERNOULLI BEAM BENDING $(x,z)-$plane



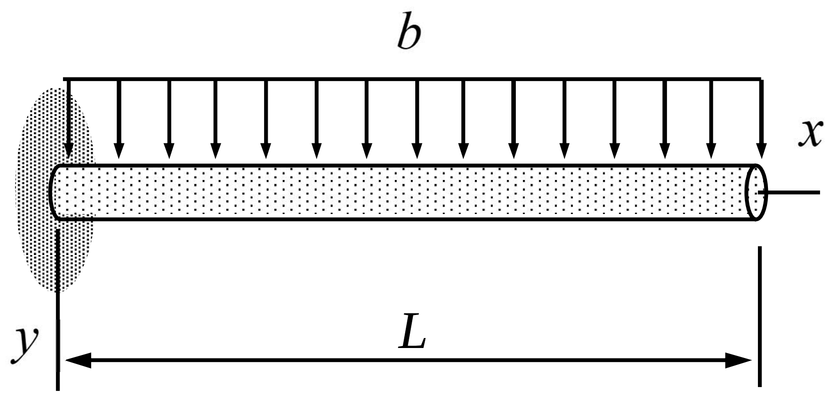
Equilibrium eqs. : $\dfrac{d^2 M_y}{dx^2} + b_z = 0$ and $Q_z = \dfrac{dM_y}{dx}$ in $(0, L)$

Constitutive eqs. : $M_y = -EI \dfrac{d^2 w}{dx^2}$ in $(0, L)$ (Bernoulli constraint $\gamma_{xz} = \dfrac{dw}{dx} + \theta = 0$)

Natural boundary condition: $M_y = \underline{M}_y$ and $Q_z = \underline{F}_z$ at $x = L$

Essential boundary condition: $w = 0$ and $\theta = -\dfrac{dw}{dx} = 0$ at $x = 0$

EXAMPLE 4.2 Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$ and I, and loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Bernoulli beam equations.



Answer $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

MOMENTS OF AREA

Cross-section geometry of a beam influences the constitutive equations through the moments of area (material is assumed to be homogeneous):

Zero moment: $A = \int dA$

First moments: $S_z = \int ydA$ and $S_y = \int zdA$

Second moments: $I_{zz} = \int y^2 dA$, $I_{yy} = \int z^2 dA$, and $I_{zy} = I_{yz} = \int yzdA$

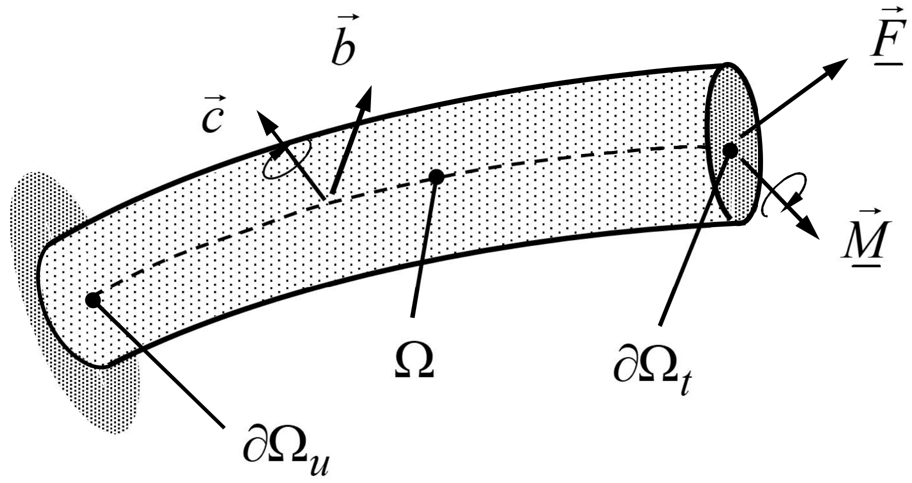
Polar moment: $I_{rr} = \int y^2 + z^2 dA = I_{zz} + I_{yy}$

The moments depend on the material coordinate system. For the simplest representation, position of the *x*-axis and orientation of the *y*-axis should result into $S_z = S_y = I_{yz} = 0$.

4.1 BEAM EQUATIONS

Virtual work expression of beam, principle of virtual work, integration by parts, and the fundamental lemma of variation calculus imply the equations:

$$\left.\begin{aligned}
&\frac{d\vec{F}}{ds}+\vec{b}=0 \quad \text{in} \quad \Omega, \\
&\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c}=0 \quad \text{in} \quad \Omega, \\
&n\vec{F}-\underline{\vec{F}}=0 \quad \text{or} \quad \vec{u}-\underline{\vec{u}}=0 \quad \text{on} \quad \partial\Omega, \\
&n\vec{M}-\underline{\vec{M}}=0 \quad \text{or} \quad \vec{\theta}-\underline{\vec{\theta}}=0 \quad \text{on} \quad \partial\Omega.
\end{aligned}\right\}$$



Constitutive equations $\vec{M}=\vec{M}(\vec{u},\vec{\theta})$, $\vec{F}=\vec{F}(\vec{u},\vec{\theta})$ (Bernoulli or Timoshenko) are needed in displacement analysis and in statically indeterminate cases!

Curvilinear (s,n,b) system represents a generic system. In terms of the stress and external force resultant, virtual work densities of the beam model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \text{and} \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

in which the strain measures $\vec{\varepsilon} = \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0$ and $\vec{\kappa} = \dfrac{d\vec{\theta}_0}{ds}$.

Integration by parts in the virtual work expression $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ gives a more convenient form for deducing the beam equations (the simple form of integration by parts formula applies):

$$\delta W = \int_{\Omega} -[\vec{F} \cdot (\frac{d\delta\vec{u}_0}{ds} + \vec{e}_s \times \delta\vec{\theta}_0) - \vec{M} \cdot \frac{d\delta\vec{\theta}_0}{ds}]ds +$$

$$\int_\Omega (\delta\vec{u}_0\cdot\vec{b}+\delta\vec{\theta}_0\cdot\vec{c})ds+\sum_{\partial\Omega}(\delta\vec{u}_0\cdot\underline{\vec{F}}+\delta\vec{\theta}_0\cdot\underline{\vec{M}}) \quad\Leftrightarrow$$

$$\delta W=\int_\Omega [\frac{d\vec{F}}{ds}\cdot\delta\vec{u}_0+(\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F})\cdot\delta\vec{\theta}_0]ds-\sum_{\partial\Omega}(\delta\vec{u}_0\cdot n\vec{F}+\delta\vec{\theta}_0\cdot n\vec{M})+$$

$$\int_\Omega (\delta\vec{u}_0\cdot\vec{b}+\delta\vec{\theta}_0\cdot\vec{c})ds+\sum_{\partial\Omega}(\delta\vec{u}_0\cdot\underline{\vec{F}}+\delta\vec{\theta}_0\cdot\underline{\vec{M}}) \quad\Leftrightarrow$$

$$\delta W=\int_\Omega [(\frac{d\vec{F}}{ds}+\vec{b})\cdot\delta\vec{u}_0+(\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c})\cdot\delta\vec{\theta}_0]ds+$$

$$\sum_{\partial\Omega}[(-n\vec{F}+\underline{\vec{F}})\cdot\delta\vec{u}_0+(-n\vec{M}+\underline{\vec{M}})\cdot\delta\vec{\theta}_0].$$

According to the principle of virtual work $\delta W=0 \ \ \forall(\delta\vec{u}_0,\delta\vec{\theta}_0)\in U$. First, if $\delta\vec{u}_0$ and $\delta\vec{\theta}_0$ are chosen to vanish on $\partial\Omega$, the fundamental lemma of variation calculus implies

$$\frac{d\vec{F}}{ds}+\vec{b}=0 \quad \text{and} \quad \frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c}=0 \quad \text{in } \Omega. \quad \leftarrow$$ equilibrium equations

Second, if $\delta\vec{u}_0$ and $\delta\vec{\theta}_0$ are varied without any restrictions on the boundary (the equilibrium equations are used to simplify the virtual work expression), the fundamental lemma of variation calculus gives

$$n\vec{F} - \underline{\vec{F}} = 0 \quad \text{and} \quad n\vec{M} - \underline{\vec{M}} = 0 \quad \text{on} \quad \partial\Omega. \quad \leftarrow \qquad \text{natural boundary conditions}$$

Third, the boundary terms vanish also if $\delta\vec{u}_0 = 0$ and/or $\delta\vec{\theta}_0 = 0$ on $\partial\Omega$ by definition of U. Then one may not deduce the condition above. However, $\delta\vec{u}_0 = 0$ and $\delta\vec{\theta}_0 = 0$ on $\partial\Omega_u$ imply that $\vec{u}_0 - \underline{\vec{u}}_0 = 0$ and $\vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0$ on $\partial\Omega_u$.

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the cross-section. If the kinetic assumptions are embedded in the elasticity tensor of the beam model

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA = \int \begin{bmatrix} \overleftrightarrow{E} & -\overleftrightarrow{E}\times\vec{\rho} \\ \vec{\rho}\times\overleftrightarrow{E} & -\vec{\rho}\times\overleftrightarrow{E}\times\vec{\rho} \end{bmatrix} dA\cdot\begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{bmatrix} \overleftrightarrow{A} & \overleftrightarrow{C} \\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B} \end{bmatrix}\cdot\begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix},$$

constitutive equation

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} JdA,$$

external distributed force and moment

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA$$

external point force and moment

where $J = 1 - n\kappa$ and $\overleftrightarrow{E} = E\vec{e}_s\vec{e}_s + G\vec{e}_n\vec{e}_n + G\vec{e}_b\vec{e}_b$ for an isotropic material.

4.2 CARTESIAN COORDINATE SYSTEM

Timoshenko beam model equilibrium and constitutive equations in component forms

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z \dfrac{d\psi}{dx} + ES_y \dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx} - \psi) - GS_y \dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx} + \theta) + GS_z \dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx} - \psi) + GS_z(\dfrac{dw}{dx} + \theta) + GI_{rr} \dfrac{d\phi}{dx} \\ ES_y \dfrac{du}{dx} - EI_{zy} \dfrac{d\psi}{dx} + EI_{yy} \dfrac{d\theta}{dx} \\ -ES_z \dfrac{du}{dx} + EI_{zz} \dfrac{d\psi}{dx} - EI_{yz} \dfrac{d\theta}{dx} \end{Bmatrix}.$$

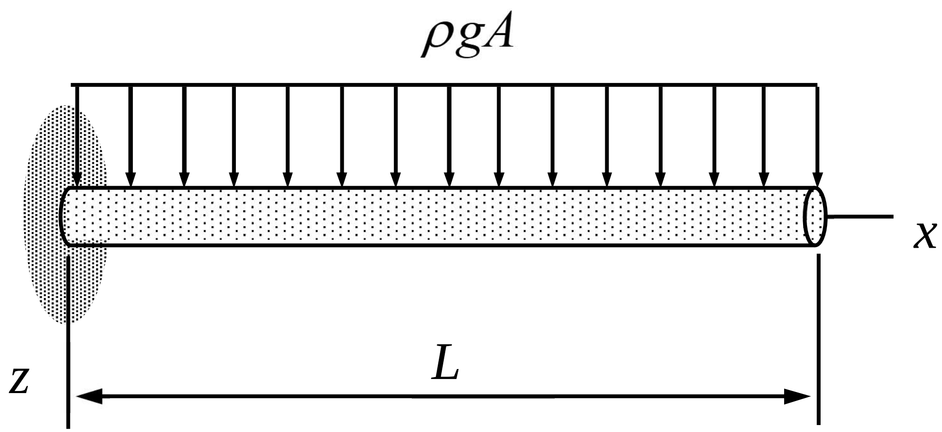
If the $x-$axis of the material coordinate system is aligned with the geometrical axis, the Cartesian system component representations of displacement, rotation, force resultant, moment resultant, elasticity tensor of beam and the relative position vector

$$\vec{u}_0 = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix}, \quad \vec{\theta}_0 = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \phi(x) \\ \theta(x) \\ \psi(x) \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N(x) \\ Q_y(x) \\ Q_z(x) \end{Bmatrix}, \quad \vec{M} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} T(x) \\ M_y(x) \\ M_z(x) \end{Bmatrix},$$

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ and } \vec{\rho} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix}.$$

What remains is just finding the component representations of equilibrium and constitutive equations by substituting the expression above.

EXAMPLE 4.3 Consider a beam loaded by its own weight and clamped at its left end (figure). Determine $\vec{F}$ and $\vec{M}$ as functions of x by using the beam equations $d\vec{F}/dx + \vec{b} = 0$ and $d\vec{M}/dx + \vec{i} \times \vec{F} + \vec{c} = 0$ and the boundary conditions $\vec{F} = 0$ and $\vec{M} = 0$ at the free end.



Answer $N(x) = 0,\ Q_z(x) = -\rho g A(x - L),\ M_y(x) = -\rho g A \frac{1}{2}(x - L)^2$

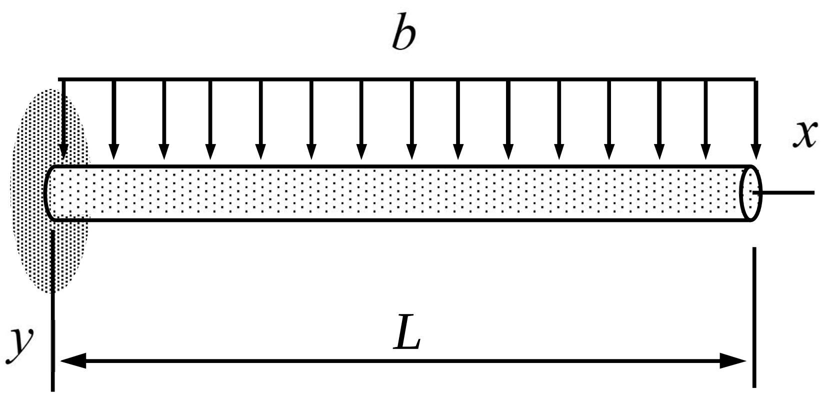
In a statically determinate case one may solve the beam equations for the stress resultants no matter the material. The non-zero loading component $b_z = \rho g A$. Equilibrium equations and the boundary condition at the free end (let us consider only the equations of the planar problem) give

$$\frac{dN}{dx} = 0 \ \text{ in } \ (0, L) \quad \text{and} \quad N(L) = 0 \quad \Rightarrow \quad N(x) = 0,$$

$$\frac{dQ_z}{dx} + \rho g A = 0 \ \text{ in } \ (0, L) \quad \text{and} \quad Q_z(L) = 0 \quad \Rightarrow \quad Q_z(x) = -\rho g A(x - L),$$

$$\frac{dM_y}{dx} + \rho g A(x - L) = 0 \ \text{ in } \ (0, L) \quad \text{and} \quad M_y(L) = 0 \quad \Rightarrow \quad M_y(x) = -\rho g A \frac{1}{2}(x - L)^2.$$

EXAMPLE Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$ and I, and loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Bernoulli beam equations.

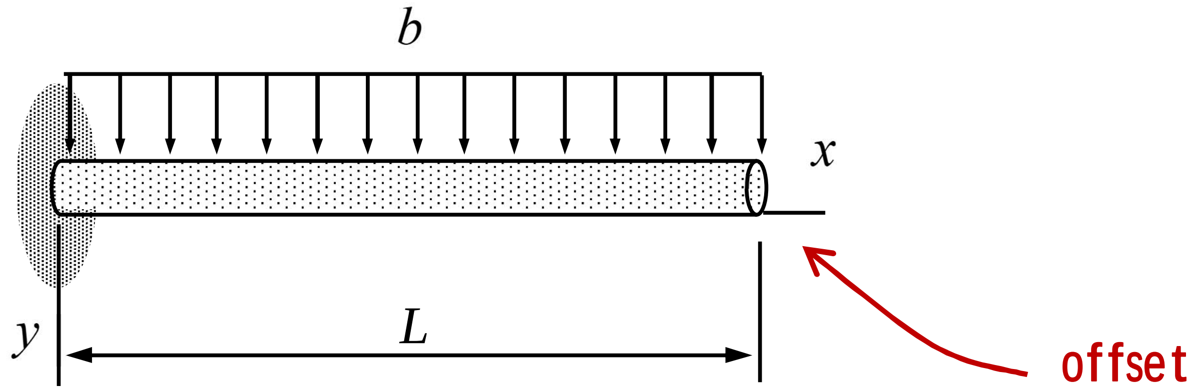


Answer (Mathematica notebook) $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

In the Bernoulli model, Bernoulli constraints $\gamma_{xy} = dv/dx - \psi = 0$ and $\gamma_{xz} = dw/dx + \theta = 0$ are used to eliminate the rotation components θ and ψ from the constitutive equations of the Timoshenko beam model. Then shear force components Q_y and Q_z become constraint forces whose values follow from the equilibrium equations. Assuming that $S_y = S_z = I_{yz} = 0$, one may just replace the constitutive equations for Q_y and Q_z by Bernoulli constraints.

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx} - \psi \\ \dfrac{dw}{dx} + \theta \end{Bmatrix}, \quad \begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

EXAMPLE 4.4 Consider the beam of the figure of length L. Material properties E and G, and loading b are constants. Due to the offset of the x-axis, cross-section properties are given by A, $S=-rA$ and $I+r^2A$, in which I is the second moment with respect to the symmetry axis and r is the radius of the cross-section. Determine the axial displacement, deflection, and rotation at the free end (at the x-axis) according to the Bernoulli beam model.



Answer (Mathematica notebook) $u(L)=-\dfrac{bL^3r}{6EI}$, $v(L)=\dfrac{bL^4}{8EI}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

4.3 CURVILINEAR COORDINATE SYSTEM

Assuming that $S_n = S_b = I_{nb} = 0$, the equilibrium and constitutive equations are

$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0,$$

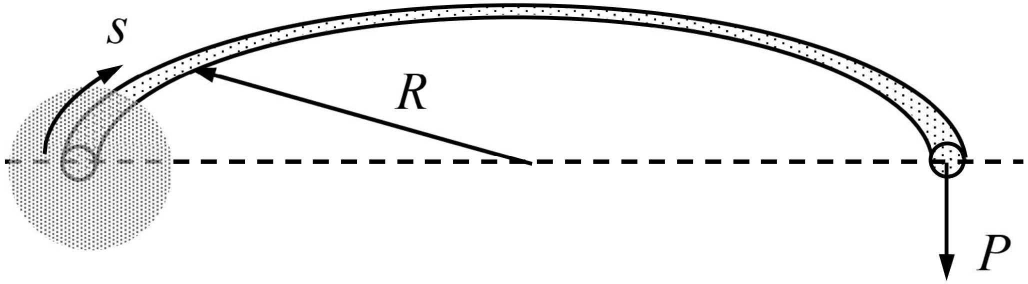
$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\dfrac{du}{ds} - v\kappa) \\ GA(\dfrac{dv}{ds} + u\kappa - w\tau - \psi) \\ GA(\dfrac{dw}{ds} + v\tau + \theta) \end{Bmatrix}, \text{ and } \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GI_{rr}(\dfrac{d\phi}{ds} - \theta\kappa) \\ EI_{nn}(\dfrac{d\theta}{ds} + \phi\kappa - \psi\tau) \\ EI_{bb}(\dfrac{d\psi}{ds} + \theta\tau) \end{Bmatrix}.$$

The constitutive equations assume that the $(s,n,b)-$coordinate system is chosen so that $S_n = S_b = I_{nb} = 0$ to simplify the generic forms of the constitutive equations

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\frac{du}{ds} - v\kappa) + ES_n(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\frac{d\psi}{ds} + \theta\tau) \\ GA(\frac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\frac{d\phi}{ds} - \theta\kappa) \\ GA(\frac{dw}{ds} + v\tau + \theta) + GS_b(\frac{d\phi}{ds} - \theta\kappa) \end{Bmatrix},$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}.$$

EXAMPLE 4.5 Consider the planar beam loaded perpendicularly to its plane and clamped at its end as shown. Mid-curve of the beam is a half-circle of radius R. Write down the equilibrium equations of the curved beam and solve for the stress resultants as functions of s. Curvature and torsion of a circular mid-curve $\kappa = 1/R$ and $\tau = 0$ are constants.



Answer $Q_b = P$, $T = PR(1+\cos\frac{s}{R})$, and $M_n = -PR\sin(\frac{s}{R})$

Boundary conditions define the external forces or moments acting on the boundaries or their work conjugate displacements and rotations. The number of conditions need to match the number of equations of the first order representation i.e. 12. In a statically determined case, the equilibrium equations of beam and force/moment conditions at the free end suffice

equil. eqns.

$$\left.\begin{array}{lll} \dfrac{dN}{ds}-\dfrac{1}{R}Q_n=0 & \dfrac{dQ_n}{ds}+\dfrac{1}{R}N=0 & \dfrac{dQ_b}{ds}=0 \\[2ex] \dfrac{dT}{ds}-\dfrac{1}{R}M_n=0 & \dfrac{dM_n}{ds}+\dfrac{1}{R}T-Q_b=0 & \dfrac{dM_b}{ds}+Q_n=0 \end{array}\right\} \quad \text{in } (0,L)$$

B.C:s

$$\left.\begin{array}{lll} N=0 & Q_n=0 & Q_b=P \\[2ex] T=0 & M_n=0 & M_b=0 \end{array}\right\} \quad \text{at } \ s=L$$

The equations can be integrated, e.g., in the following order ($L = \pi R$). First

$$\frac{dQ_b}{ds} = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad Q_b = P \quad \text{at} \quad s = L \quad \Rightarrow \quad Q_b(s) = P,$$

then

$$\frac{dN}{ds} - \frac{1}{R} Q_n = 0, \quad \frac{dQ_n}{ds} + \frac{1}{R} N = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N = Q_n = 0 \quad \text{at} \quad s = L \quad \Rightarrow$$

$$\frac{d^2 Q_n}{ds^2} + \frac{1}{R^2} Q_n = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad Q_n = \frac{dQ_n}{ds} = 0 \quad \text{at} \quad s = L \quad \Rightarrow \quad Q_n(s) = 0$$

$$\frac{d^2 N}{ds^2} + \frac{1}{R^2} N = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N = \frac{dN}{ds} = 0 \quad \text{at} \quad s = L \quad \Rightarrow \quad N(s) = 0,$$

then

$$\frac{dM_b}{ds} = 0 \text{ in } (0,L) \text{ and } M_b = 0 \text{ at } s = L \quad \Rightarrow \quad M_b(s) = 0,$$

then

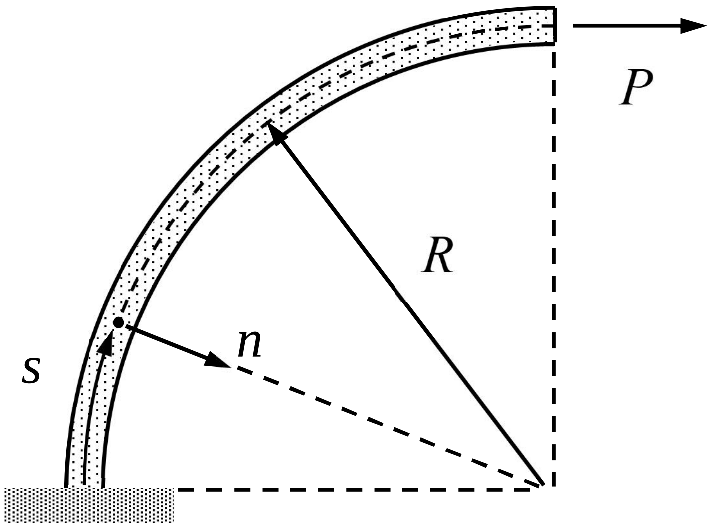
$$\frac{dT}{ds} - \frac{1}{R}M_n = 0, \; \frac{dM_n}{ds} + \frac{1}{R}T - P = 0 \text{ in } (0,L) \text{ and } T = M_n = 0 \text{ at } s = L \quad \Rightarrow$$

$$R\frac{d^2T}{ds^2} + \frac{1}{R}T - P = 0 \text{ in } (0,L) \text{ and } T = \frac{dT}{ds} = 0 \text{ at } s = L \quad \Rightarrow$$

$$T = PR(\cos\frac{s}{R} + 1) \quad \Rightarrow \quad M_n = -PR\sin(\frac{s}{R}).$$

When hand calculations become tedious, the Mathematica notebook of MEC-E8003 homepage helps.

EXAMPLE 4.6 Consider the curved beam shown. Determine the displacement and rotation components u, v and ψ at the free end according to the Bernoulli beam theory. The moments of cross-section A, $S=0$, and I. Material parameters E, G, curvature $\kappa=1/R$ and torsion $\tau=0$ are constants.



Answer

$$u(L)=\frac{PR}{E}\frac{I\pi+A(-8+3\pi)R^2}{4AI},\ v(L)=\frac{PR}{E}\frac{AR^2-I}{2AI},\text{ and }\psi(L)=\frac{PR^2}{EI}\frac{\pi-2}{2}$$

When writing the beam equations, it is convenient to write the equilibrium equations, constitutive equations and boundary conditions “as is” without any eliminations (notice that the constitutive equation for the shear force has been replaced by its “work conjugate” Bernoulli constraint):

$$\frac{dN}{ds} - \frac{1}{R} Q_n = 0, \quad \frac{dQ_n}{ds} + \frac{1}{R} N = 0, \text{ and } \quad \frac{dM_b}{ds} + Q_n = 0 \quad \text{in} \quad (0, L),$$

$$N = EA\left(\frac{du}{ds} - \frac{1}{R} v\right), \quad \frac{dv}{ds} - \psi + \frac{1}{R} u = 0, \quad \text{and} \quad M_b = EI \frac{d\psi}{ds} \quad \text{in} \quad (0, L),$$

$$u = 0, \quad v = 0, \quad \text{and} \quad \psi = 0 \quad \text{at} \quad s = 0,$$

$$N = P, \quad Q_n = 0, \text{ and } M_b = 0 \quad \text{at} \quad s = L.$$

In a statically determined case, it is possible to solve for the stress resultants first. Elimination is used in the connected first order equations to end up with second order non-connected equations. With $L = \pi R / 2$

$$\frac{d^2 N}{ds^2} + \frac{1}{R^2} N = 0 \text{ in } (0, L) \quad \text{and} \quad \frac{dN}{ds} = 0,\; N = P \;\text{ at }\; s = L \Rightarrow$$

$$N(s) = P \sin(\frac{s}{R}) \;,\;\; Q_n(s) = P \cos(\frac{s}{R}),$$

$$\frac{dM_b}{ds} + P \cos(\frac{s}{R}) = 0 \;\text{ in } (0, L) \text{ and } \; M_b = 0 \;\text{ at }\; s = L \Rightarrow \; M_b(s) = PR[1 - \sin(\frac{s}{R})].$$

After that, one may continue with the constitutive equations. Again, elimination is used in the connected first order equations to end up with second order non-connected equations

$$\frac{d\psi}{ds}=\frac{PR}{EI}[1-\sin(\frac{s}{R})] \text{ in } (0,L) \text{ and } \psi=0 \text{ at } s=0 \Rightarrow \psi=\frac{PR^2}{EI}[\frac{s}{R}+\cos(\frac{s}{R})-1],$$

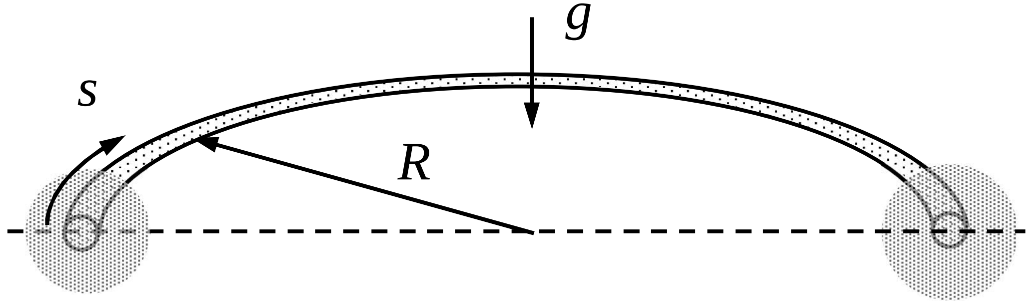
$$\frac{d^2v}{ds^2}+\frac{1}{R^2}v=\frac{PR}{EI}-(\frac{P}{EA}+\frac{PR^2}{EI})\frac{1}{R}\sin(\frac{s}{R}) \text{ in } (0,L) \text{ and } v=0,\ \frac{dv}{ds}=0 \text{ at } s=0 \Rightarrow$$

$$v(s)=\frac{PR^3}{EI}+(\frac{P}{EA}+\frac{PR^2}{EI})[\frac{s}{2}\cos(\frac{s}{R})-\frac{R}{2}\sin(\frac{s}{R})],$$

$$u(s)=R(\psi-\frac{dv}{ds})=\frac{PR^3}{EI}[\frac{s}{R}+\cos(\frac{s}{R})-1]+(\frac{P}{EA}+\frac{PR^2}{EI})[\frac{s}{2}\sin(\frac{s}{R})].$$

Notice that the missing boundary condition for the second order problems, obtained through the elimination, is given by the original first order equations!

EXAMPLE 4.7 Consider the semi-circular beam loaded perpendicularly to its plane and clamped at its ends shown. Write down the Timoshenko beam boundary value problem and solve for the vertical deflection at the mid-point with the Mathematica notebook of the course. The cross-section is circular with properties A, $I_{nn} = I_{bb} = I$, $I_{rr} = 2I$ and $S_n = S_b = 0$. The material properties E, G, ρ and curvature $\kappa = 1/R$ are constants (torsion $\tau = 0$). Finally, consider the Bernoulli limit.



Answer $w = \dfrac{\rho A g[16G(-2+\pi)+E(-16+8\pi-4\pi^2+\pi^3)]R^4}{16EGI\pi}$

The parameters of the Timoshenko beam equations are in this case $b_s = b_n = 0$, $b_b = b = \rho A g$, and $c_s = c_n = c_b = 0$. Therefore

$$\left.\begin{array}{lll} \dfrac{dN}{ds} - \dfrac{1}{R} Q_n = 0 & \dfrac{dQ_n}{ds} + \dfrac{1}{R} N = 0 & \dfrac{dQ_b}{ds} + b = 0 \\[2ex] \dfrac{dT}{ds} - \dfrac{1}{R} M_n = 0 & \dfrac{dM_n}{ds} + \dfrac{1}{R} T - Q_b = 0 & \dfrac{dM_b}{ds} + Q_n = 0 \end{array}\right\} \begin{array}{l} \text{equil. eqns.} \\ \text{in } (0, \pi R) \end{array}$$

$$\left.\begin{array}{lll} N = EA\left(\dfrac{du}{ds} - \dfrac{1}{r} v\right) & Q_n = GA\left(\dfrac{dv}{ds} + \dfrac{1}{R} u - \psi\right) & Q_b = GA\left(\dfrac{dw}{ds} + \theta\right) \\[2ex] T = 2GI\left(\dfrac{d\phi}{ds} - \dfrac{1}{R}\theta\right) & M_n = EI\left(\dfrac{d\theta}{ds} + \dfrac{1}{R}\phi\right) & M_b = EI\dfrac{d\psi}{ds} \end{array}\right\} \begin{array}{l} \text{const. eqns.} \\ \text{in } (0, \pi R) \end{array}$$

$$\left.\begin{array}{lll} u = 0 & v = 0 & w = 0 \\ \phi = 0 & \theta = 0 & \psi = 0 \end{array}\right\} \begin{array}{l} \text{B.C:s} \\ s \in \{0, \pi R\} \end{array}$$

In a statically non-determinate case all equations have to be solved simultaneously which may mean tedious calculations. Solution to the non-zero displacement and rotation components at the mid-point $s=\pi R/2$ as given by the Mathematica notebook are

$$\phi(\frac{\pi R}{2})=\rho g A R^3\frac{(\pi-2)(\pi E-4G)-4E}{4G\pi EI},$$

$$w(\frac{\pi R}{2})=\rho g A R^4\frac{16G(\pi-2)+(8\pi-4\pi^2+\pi^3-16)E}{16G\pi EI}.$$

The code finds first the Timoshenko solution. After that, the Bernoulli solution is obtained by enforcing the Bernoulli constraints.

4.4 CURVED BEAM KINEMATICS

The use of arc length s as the curve parameter is convenient. Curvature $\kappa = 1/R$ and torsion τ define the basis vector derivatives.

Mapping: $\vec{r}(s,n,b) = \vec{r}_0(s) + n\vec{e}_n(s) + b\vec{e}_b(s)$

Basis: $\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}_0 / \partial s \\ (\partial\vec{e}_s / \partial s) / \left|\partial\vec{e}_s / \partial s\right| \\ \vec{e}_s \times \vec{e}_n \end{Bmatrix}$ and $\dfrac{\partial}{\partial s}\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}$

Gradient: $\nabla = \vec{e}_s \dfrac{1}{J}[\dfrac{\partial}{\partial s} + \tau(b\dfrac{\partial}{\partial n} - n\dfrac{\partial}{\partial b})] + \vec{e}_n \dfrac{\partial}{\partial n} + \vec{e}_b \dfrac{\partial}{\partial b}$, where $J = 1 - n\kappa$

Volume element: $dV = JdAds$

Intrinsic $(s,n,b)-$coordinate system is curvilinear and orthonormal. The matrix of the basis vector derivatives is anti-symmetric and expressible in the form

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial s}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}$$

depending on torsion τ and curvature $\kappa=1/R$.

Using the generic expression of the gradient operator (the basis vectors are not orthogonal away from the axis and the simple expression based on the scaling coefficient does not apply)

$$\nabla=\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial_s\\ \partial_n\\ \partial_b\end{Bmatrix}=\frac{\vec{e}_s}{1-n\kappa}[\frac{\partial}{\partial s}+\tau(b\frac{\partial}{\partial n}-n\frac{\partial}{\partial b})]+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b}$$

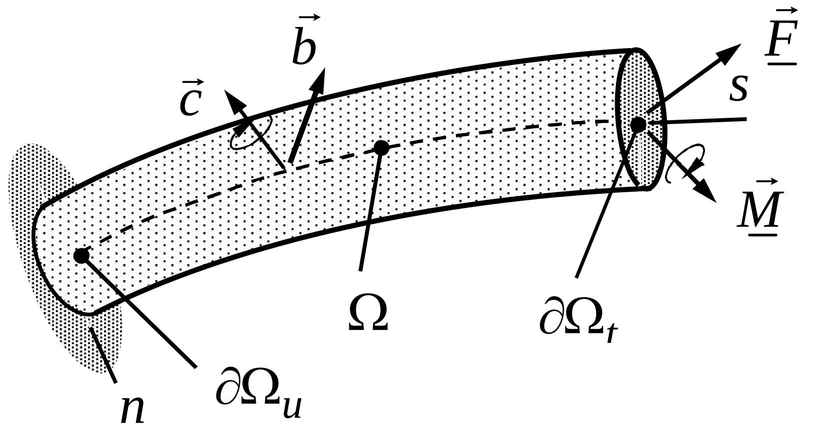
The expression of the volume element is

$$dV = (\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial n}) \cdot \frac{\partial \vec{r}}{\partial b} dndbds = [(\vec{e}_s + n\frac{d\vec{e}_n}{ds} + b\frac{d\vec{e}_b}{ds}) \times \vec{e}_n] \cdot \vec{e}_b dAds = (1 - n\kappa)dAds$$

The radius of curvature $R = 1/\kappa$ at a point is given by the best fitting circle and it is a geometric quantity. Torsion τ describes the rate by which $\vec{e}_n$ and $\vec{e}_b$ rotate around the mid-curve.

4.5 VIRTUAL WORK DENSITY

Virtual work densities can be expressed in terms of generalized forces (force, moment) and their work conjugate strains:



$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

where

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} J dA, \text{ and } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA.$$

In vector notation, the kinematic assumption of the beam model, gradient operator, gradient of the relative position vector and displacement gradient are ($J = 1 - n\kappa$)

$$\left.\begin{aligned}
&\vec{u}(s,n,b) = \vec{u}_0(s) + \vec{\theta}_0(s) \times \vec{\rho}(n,b) \quad \text{where} \quad \vec{\rho} = n\vec{e}_n + b\vec{e}_b, \\
&\nabla = \nabla_0 + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b} \quad \text{where} \quad \nabla_0 = \vec{e}_s \frac{1}{J}[\frac{\partial}{\partial s} + \tau(b\frac{\partial}{\partial n} - n\frac{\partial}{\partial b})], \\
&\nabla\vec{\rho} = \overleftrightarrow{I} - \frac{1}{J}\vec{e}_s\vec{e}_s \quad \text{where} \quad \overleftrightarrow{I} = \vec{e}_s\vec{e}_s + \vec{e}_n\vec{e}_n + \vec{e}_b\vec{e}_b, \\
&dV = JdAds.
\end{aligned}\right\}$$

The displacement gradient becomes ($\overleftrightarrow{I} \times \vec{\theta}$ is antisymmetric) as displacement components depend on s only

$$\nabla\vec{u} = \frac{1}{J}(\vec{e}_s \frac{d\vec{u}_0}{ds} + \vec{e}_s \frac{d\vec{\theta}_0}{ds} \times \vec{\rho}) - (\vec{\vec{I}} - \frac{1}{J}\vec{e}_s\vec{e}_s) \times \vec{\theta}_0 \quad \Leftrightarrow$$

$$\nabla\vec{u} = \frac{1}{J}\vec{e}_s(\vec{\varepsilon} + \vec{\kappa} \times \vec{\rho}) - \vec{\vec{I}} \times \vec{\theta}_0 \quad \text{where} \quad \vec{\varepsilon} = \frac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \quad \text{and} \quad \vec{\kappa} = \frac{d\vec{\theta}_0}{ds}.$$

Assuming the symmetry of stress $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$, virtual work of internal forces per unit volume becomes (vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ is needed in the derivation) with notation $\vec{\sigma} = \vec{e}_s \cdot \vec{\vec{\sigma}} = \vec{\vec{\sigma}} \cdot \vec{e}_s$

$$\delta w_V^{\mathrm{int}} = -\vec{\vec{\sigma}}_{\mathrm{c}} : \delta\nabla\vec{u} = -\frac{1}{J}\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix}.$$

As $dV = JdAds$ and $ds = d\Omega$, the virtual work expression of internal forces takes the form

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega -\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} d\Omega, \quad \text{where} \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA. \;\Leftarrow$$

If the surface forces are acting on the end surfaces only (just to simplify the derivation), the virtual works of external volume and surface forces are (vector identity $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ and $dV = JdAd\Omega$ are used again)

$$\delta W_V^{\text{ext}} = \int_V (\delta\vec{u}\cdot\vec{f})dV = \int_\Omega \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} d\Omega, \text{ where } \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} JdA. \;\Leftarrow$$

$$\delta W_A^{\text{ext}} = \int_A (\delta\vec{u}\cdot\vec{t})dA = \sum_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA. \;\Leftarrow$$

4.6 CONSTITUTIVE EQUATIONS

Constitutive equations follow from the generalized Hooke's law (taking into accout the kinetic assumptions $\sigma_{nn} = \sigma_{bb} = 0$), gradient of the displacement for the beam model, and definitions of stress resultants:

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA = \begin{bmatrix} \overleftrightarrow{A} & \overleftrightarrow{C} \\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B} \end{bmatrix} \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{Bmatrix} \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \dfrac{d\vec{\theta}_0}{ds} \end{Bmatrix} \text{ and}$$

$$\begin{bmatrix} \overleftrightarrow{A} & \overleftrightarrow{C} \\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B} \end{bmatrix} = \int \begin{bmatrix} \overleftrightarrow{E} & -\overleftrightarrow{E} \times \vec{\rho} \\ \vec{\rho} \times \overleftrightarrow{E} & -\vec{\rho} \times \overleftrightarrow{E} \times \vec{\rho} \end{bmatrix} \frac{1}{J} dA, \text{ where } \overleftrightarrow{E} = \vec{e}_s \cdot \overset{\leftrightarrow}{\overleftrightarrow{E}} \cdot \vec{e}_s \text{ and } \vec{\rho} = n\vec{e}_n + b\vec{e}_b.$$

The three tensors $\overleftrightarrow{A}$, $\overleftrightarrow{B}$, and $\overleftrightarrow{C}$ define the constitutive equations.

Derivation uses stress resultant definitions, beam model elasticity tensor, and displacement gradient. Displacement gradient was derived earlier when discussing the virtual work expression

$$\nabla\vec{u} = \frac{1}{J}\vec{e}_s(\vec{\varepsilon} + \vec{\kappa}\times\vec{\rho}) - \overleftrightarrow{I}\times\vec{\theta}_0 \ .$$

The stress-strain relationship of a linearly elastic material with elasticity tensor $\overset{\leftrightarrow}{\overleftrightarrow{E}}$ for the beam model gives with definition $\vec{\sigma} = \vec{e}_s \cdot \overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}\cdot\vec{e}_s$ (notice that $\overleftrightarrow{I}\times\vec{\theta}_0$ is antisymmetric and vanishes as elasticity tensor is symmetric with respect to the last index pair)

$$\overleftrightarrow{\sigma} = \overset{\leftrightarrow}{\overleftrightarrow{E}} : \nabla\vec{u} = \overset{\leftrightarrow}{\overleftrightarrow{E}}\cdot\vec{e}_s\cdot(\vec{\varepsilon} + \vec{\kappa}\times\vec{\rho})\frac{1}{J} \quad \Rightarrow \quad \vec{\sigma} = \vec{e}_s\cdot\overleftrightarrow{\sigma} = \overleftrightarrow{E}\cdot(\vec{\varepsilon} + \vec{\kappa}\times\vec{\rho})\frac{1}{J}$$

in which $\overleftrightarrow{E} = \vec{e}_s\cdot\overset{\leftrightarrow}{\overleftrightarrow{E}}\cdot\vec{e}_s$.

Finally, constitutive equations follow from the definitions of stress resultants. In a concise form, the constitutive equations and parameters taking into account the cross-section geometry and the material can be written as

$$\begin{Bmatrix}\vec{F}\\ \vec{M}\end{Bmatrix} = \int \begin{Bmatrix}\vec{\sigma}\\ \vec{\rho}\times\vec{\sigma}\end{Bmatrix} dA = \int \begin{Bmatrix}\overleftrightarrow{E}\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho})\\ \vec{\rho}\times\overleftrightarrow{E}\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho})\end{Bmatrix}\frac{1}{J}\,dA = \begin{bmatrix}\overleftrightarrow{A} & \overleftrightarrow{C}\\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B}\end{bmatrix}\cdot\begin{Bmatrix}\vec{\varepsilon}\\ \vec{\kappa}\end{Bmatrix} \quad \text{where}$$

may depend on s

$$\begin{bmatrix}\overleftrightarrow{A} & \overleftrightarrow{C}\\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B}\end{bmatrix} = \int \begin{bmatrix}\overleftrightarrow{E} & -\overleftrightarrow{E}\times\vec{\rho}\\ \vec{\rho}\times\overleftrightarrow{E} & -\vec{\rho}\times\overleftrightarrow{E}\times\vec{\rho}\end{bmatrix}\frac{1}{J}\,dA \quad \text{and} \quad \begin{Bmatrix}\vec{\varepsilon}\\ \vec{\kappa}\end{Bmatrix} = \begin{Bmatrix}\dfrac{d\vec{u}_0}{ds}+\vec{e}_s\times\vec{\theta}_0\\ \dfrac{d\vec{\theta}_0}{ds}\end{Bmatrix}.$$

Assuming an isotropic material, the kinetic assumption of the beam model $\sigma_{nn}=\sigma_{bb}=0$ gives the elasticity tensor expression

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{e}_s\vec{e}_s \\ \vec{e}_n\vec{e}_n \\ \vec{e}_b\vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s\vec{e}_s \\ \vec{e}_n\vec{e}_n \\ \vec{e}_b\vec{e}_b \end{Bmatrix} + \begin{Bmatrix} \vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s \\ \vec{e}_n\vec{e}_b + \vec{e}_b\vec{e}_n \\ \vec{e}_b\vec{e}_s + \vec{e}_s\vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s \\ \vec{e}_n\vec{e}_b + \vec{e}_b\vec{e}_n \\ \vec{e}_b\vec{e}_s + \vec{e}_s\vec{e}_b \end{Bmatrix} \Rightarrow$$

$$\overset{\leftrightarrow}{E} = \vec{e}_s \cdot \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} \cdot \vec{e}_s = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}.$$

If the material is further homogeneous and cross section geometry constant, component forms of $\overset{\leftrightarrow}{A}$, $\overset{\leftrightarrow}{B}$ and $\overset{\leftrightarrow}{C}$ of the constitutive equation take the forms

$$\overset{\leftrightarrow}{A} = \int \overset{\leftrightarrow}{E} \frac{1}{J} dA = \overset{\leftrightarrow}{E} \int \frac{1}{J} dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} AE & 0 & 0 \\ 0 & AG & 0 \\ 0 & 0 & AG \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix},$$

$$\overset{\leftrightarrow}{C} = -\int (\overset{\leftrightarrow}{E} \times \vec{\rho})\frac{1}{J}dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & S_n E & -S_b E \\ -S_n G & 0 & 0 \\ S_b G & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix},$$

$$\overset{\leftrightarrow}{B} = -\int \vec{\rho} \times \overset{\leftrightarrow}{E} \times \vec{\rho}\frac{1}{J}dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} (I_{nn} + I_{bb})G & 0 & 0 \\ 0 & I_{nn}E & -I_{nb}E \\ 0 & -I_{nb}E & I_{bb}E \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}$$

in which the moments of cross sections

$$A = \int \frac{1}{J}dA,\ \ S_n = \int b\frac{1}{J}dA,\ \ S_b = \int n\frac{1}{J}dA,\ \ I_{bn} = I_{nb} = \int nb\frac{1}{J}dA,$$

$$I_{nn} = \int b^2 \frac{1}{J}dA,\ \ I_{bb} = \int n^2 \frac{1}{J}dA,\ \text{and}\ I_{rr} = I_{nn} + I_{bb}.$$

depend on the geometry of the cross-section, curvature ($J = 1 - n\kappa$), and positioning of the material coordinate system.

Finally, the component representations of the constitutive equations for an isotropic and homogeneous material take the forms ($\vec{u}_0 = u\vec{e}_s + v\vec{e}_n + w\vec{e}_b,\ \vec{\theta}_0 = \phi\vec{e}_s + \theta\vec{e}_n + \psi\vec{e}_b$)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\dfrac{du}{ds} - v\kappa) + ES_n(\dfrac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\dfrac{d\psi}{ds} + \theta\tau) \\ GA(\dfrac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\dfrac{d\phi}{ds} - \theta\kappa) \\ GA(\dfrac{dw}{ds} + v\tau + \theta) + GS_b(\dfrac{d\phi}{ds} - \theta\kappa) \end{Bmatrix}, \quad \Leftarrow$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}. \Leftarrow$$

The cross-section properties are constants only in case of constant curvature and cross-section geometry. If the beam is thin compared to the radius of curvature $J \approx 1$, this effect can be omitted.

EXAMPLE In curved geometry, position of the neutral axis depends on the curvature as $J = 1 - n\kappa$. If the mid-curve is placed at the geometric centroid of the cross-section and curvature $\kappa = 1/R$, the (non-zero) moments for a circular cross-section of radius r are ($\varepsilon = r/R$):

$\varepsilon = r/R$	0	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	1
$A/(\pi r^2)$	1	1.00	1.01	1.02	1.04	1.07	1.11	1.17	1.25	1.39	2
$S_b/(\pi r^3)$	0	0.03	0.05	0.08	0.11	0.14	0.18	0.24	0.31	0.44	1
$I_{bb}/(\frac{1}{4}\pi r^4)$	1	1.00	1.02	1.05	1.09	1.15	1.23	1.36	1.56	1.94	4
$I_{nn}/(\frac{1}{4}\pi r^4)$	1	1.00	1.01	1.02	1.03	1.05	1.07	1.10	1.15	1.21	1.33

The moments of the cross section follow from the integrals (need to be evaluated numerically for each value of $\varepsilon = r / R$,

$$A = \pi r^2 \left(\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{s}{1 - s\varepsilon \cos\beta} ds d\beta \right),$$

$$S_b = \pi r^3 \left(\frac{1}{\pi} \int_0^{2\pi} \int_0^1 s \cos\beta \frac{s}{1 - s\varepsilon \cos\beta} ds d\beta \right),$$

$$I_{bb} = \frac{1}{4} \pi r^4 \left(\frac{4}{\pi} \int_0^{2\pi} \int_0^1 s^2 \cos^2\beta \frac{s}{1 - s\varepsilon \cos\beta} ds d\beta \right),$$

$$I_{nn} = \frac{1}{4} \pi r^4 \left(\frac{4}{\pi} \int_0^{2\pi} \int_0^1 s^2 \sin^2\beta \frac{s}{1 - s\varepsilon \cos\beta} ds d\beta \right).$$

Above, $n = sr\cos\beta$ and $b = sr\sin\beta$.