

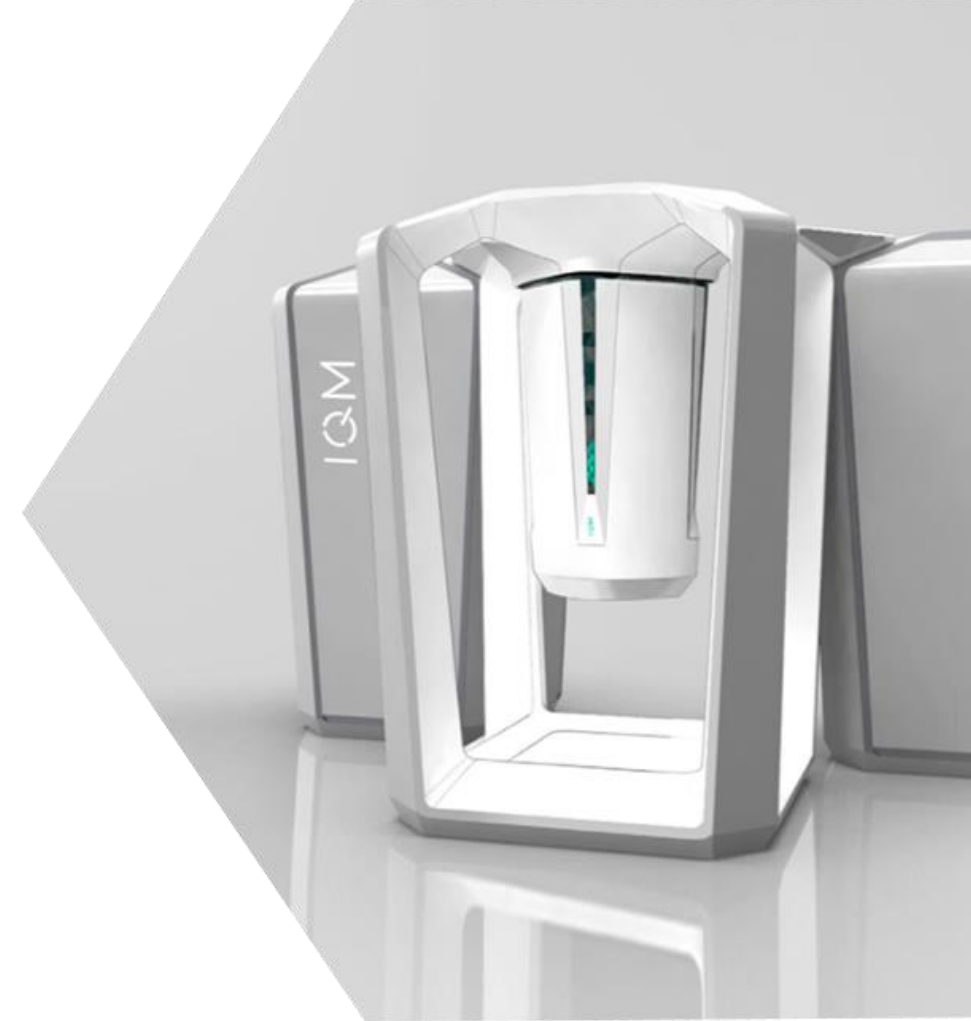


WE BUILD QUANTUM COMPUTERS

Jan Goetz

Lecture notes on PHYS-C0254 Quantum Circuits

www.meetiqm.com



About me: From science to entrepreneurship





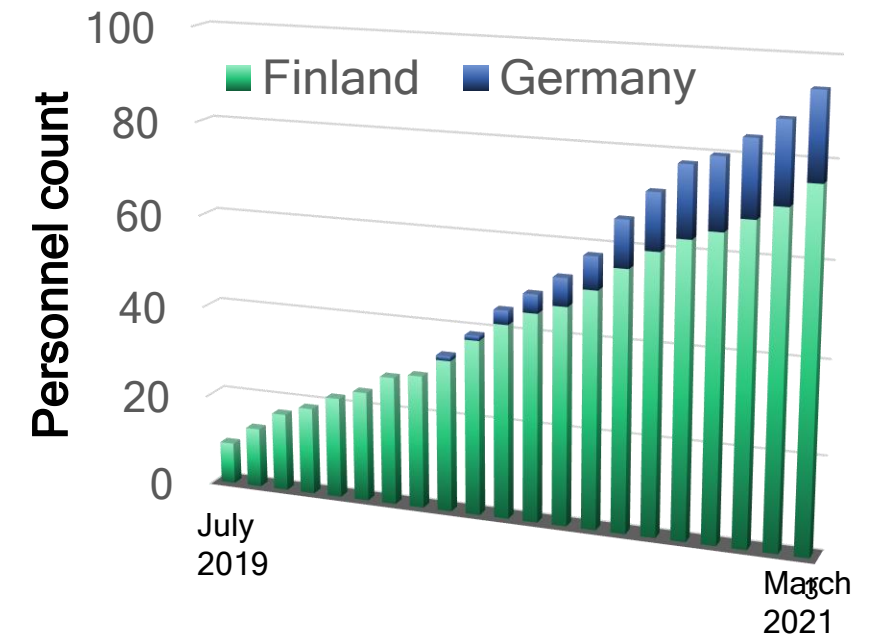
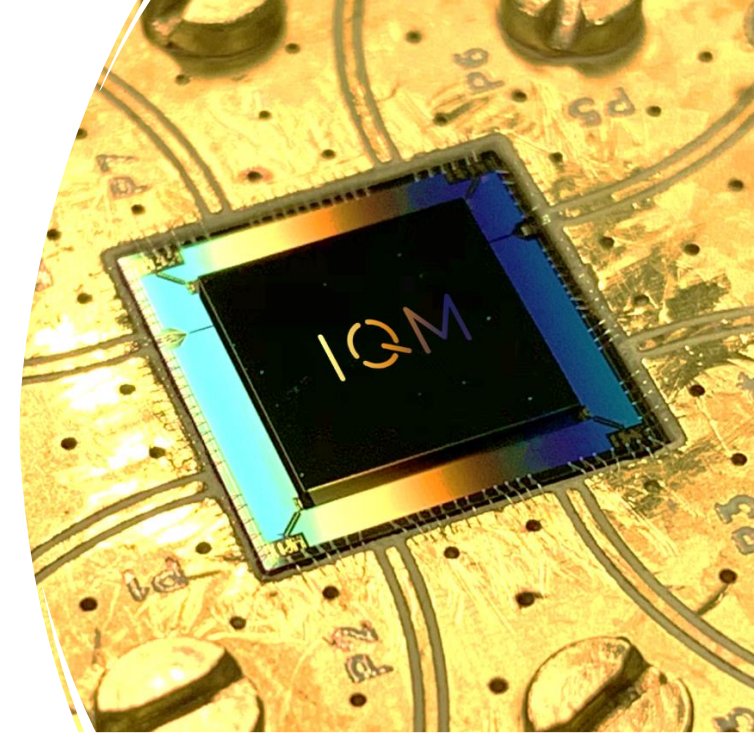
IQM in brief

Quantum computing scale-up

- Spinout of Aalto University and VTT in July 2019
- Build quantum computers based on superconducting technology
- Co-design approach for application-specific quantum computers
- Secured 2 rounds of private investment funding
- Involved in several public projects
- Sold one system in Finland

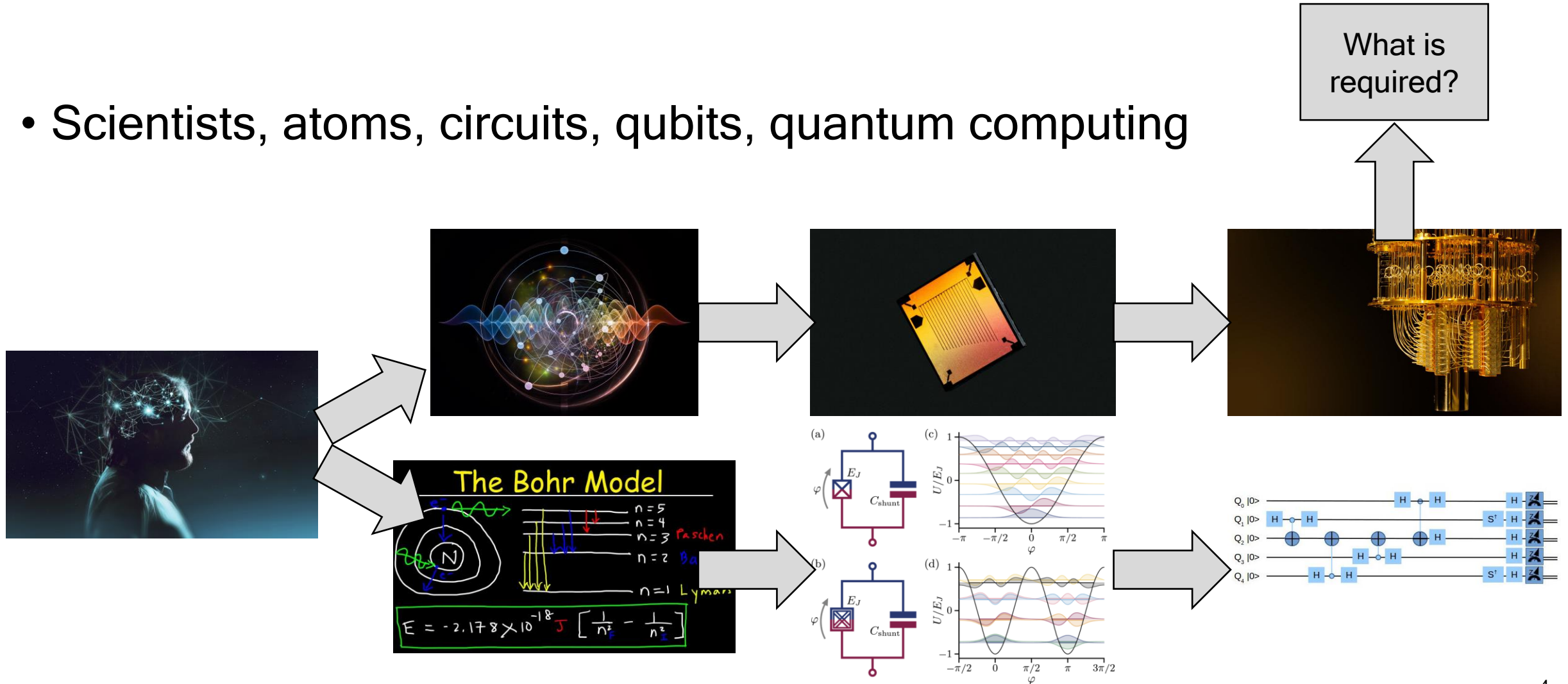
Employees (Mar. 2021)

- Finland: 76 employees
- Germany: 18 employees



How does everything fit into the big picture?

- Scientists, atoms, circuits, qubits, quantum computing



Di Vincenzo Criteria and where you can find them in this course

Statement of the criteria

1. A scalable physical system with well characterized qubit
2. The ability to initialize the state of the qubits to a simple fiducial state
3. Long relevant decoherence times
4. A "universal" set of quantum gates
5. A qubit-specific measurement capability



Agenda for lectures 7-12

7. Quantization of electrical networks
 - a. Harmonic oscillator: Lagrangian, eigenfrequency
 - b. Transfer step: LC oscillator, Legendre transform to Hamiltonian
 - d. Quantization of oscillators
8. Superconducting quantum circuits
 - a. Qubits: Transmon qubit, Charge qubit, Flux qubit 1st DiVincenzo criteria
 - b. Circuit-QED: Rabi model
 - c. Rotating Wave approximation: Jaynes-Cummings model
9. Single-qubit operations:
 - a. Initialization 2nd DiVincenzo criteria
 - b. Control: T1, T2 measurements, Randomized benchmarking 3rd DiVincenzo criteria
 - c. Readout 5th DiVincenzo criteria
10. Two-qubit operations: Architectures for 2-qubit gates 4th DiVincenzo criteria
 - a. iSWAP
 - b. cPhase
 - c. cNot
 - d. maybe: Google supremacy experiment
11. Quantum algorithms
 - a. VQE
 - b. Grover search
12. Challenges in quantum computing
 - a. Scaling
 - b. SW-HW gap
 - c. Error-correction

Agenda for today

7. Quantization of electrical networks

- a. **Harmonic oscillator: Lagrangian, eigenfrequency**
- b. Transfer step: LC oscillator, Legendre transform to Hamiltonian
- d. Quantization of oscillators

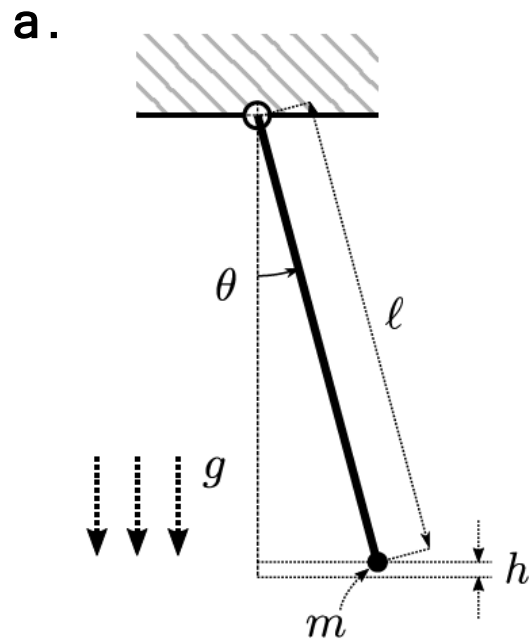


Figure 1: Classical pendulum.

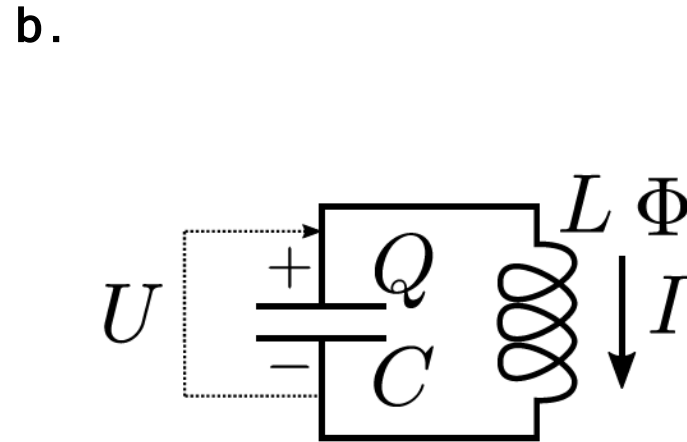
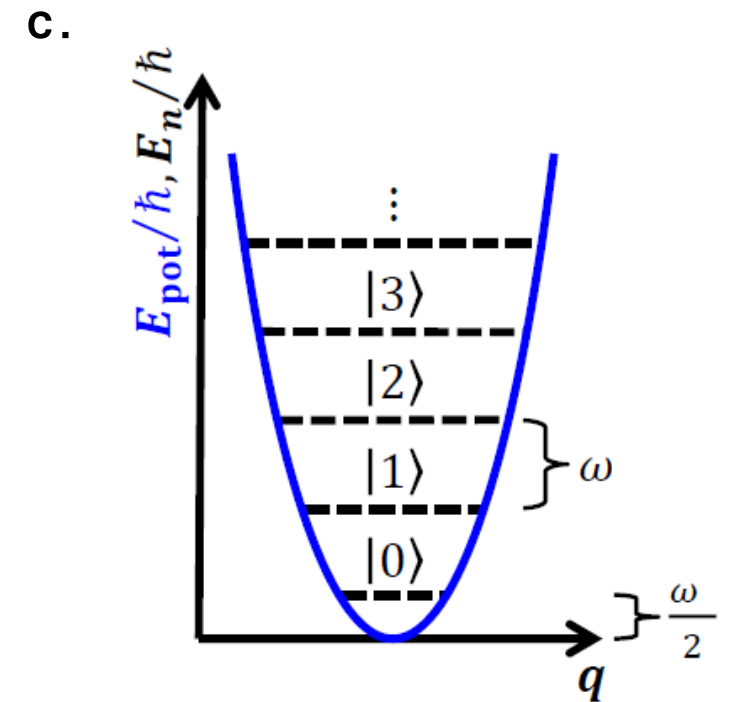


Figure 2: Superconducting LC oscillator.



General note: Harmonic oscillators

- General note: In physics, many phenomena can be explained by **harmonic oscillators**. They are the standard tool in our physics toolbox.
- Usually, there are two important variables involved like **position and momentum**, x and p .
- One can often find **analogies** where two system variables are equivalent to x and p . For example, in an **LC oscillator** these are charge and flux.

Short review: Lagrangian & Hamiltonian

- During this course, **Lagrangian** and **Hamiltonian** mechanics are used for analyzing quantum computing circuits.
- Recall that the Lagrangian is defined as the kinetic energy T **minus** the potential energy V :

$$L \equiv T - V .$$

- Quite often the Hamiltonian is representing the **total energy** of the system:

$$H = T + V .$$

Legendre transformation



Short review: Classical oscillator*

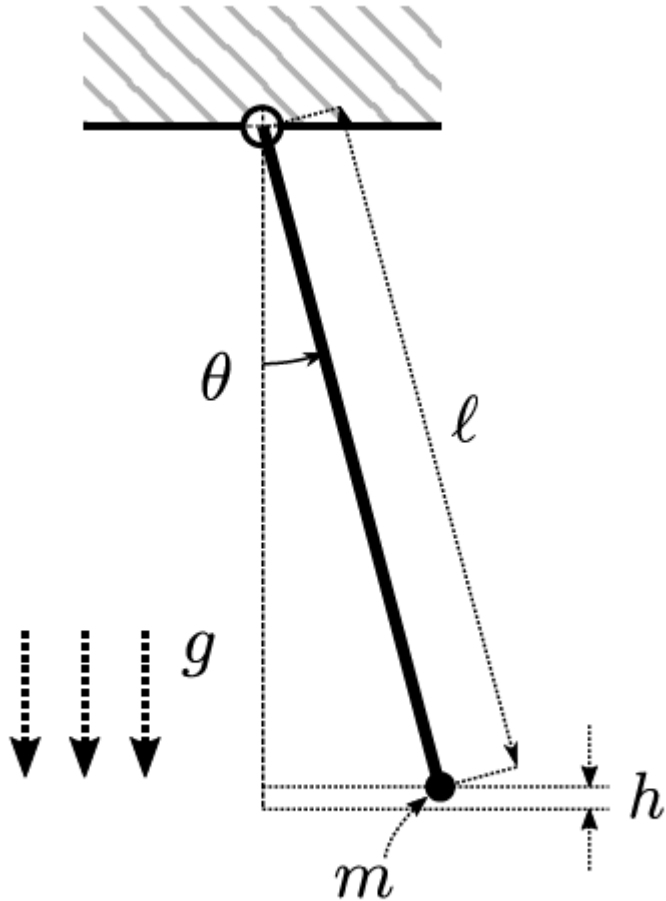


Figure 1: Classical pendulum.

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

The potential energy is

$$V = mgh = mg\ell(1 - \cos\theta) \approx \frac{1}{2}mg\ell\theta^2$$

$$L \equiv T - V.$$

We introduce generalized coordinates p and q

$$q = \ell\theta,$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \approx \frac{\partial}{\partial \dot{q}} \left(\frac{1}{2}m\dot{q}^2 - \frac{mg}{2\ell}q^2 \right) = m\dot{q} = m\ell\dot{\theta}$$

Short review: Classical oscillator*

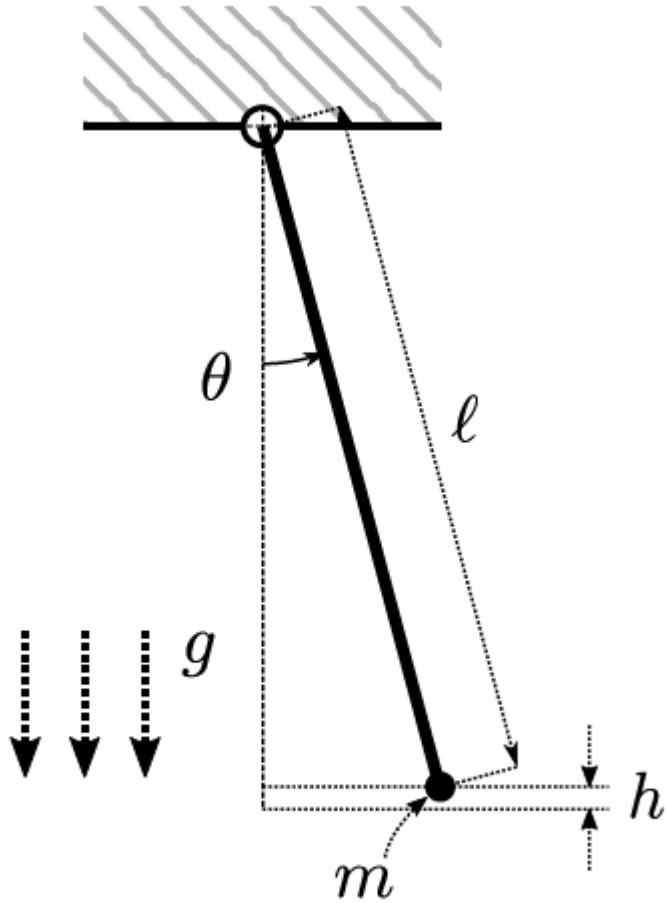


Figure 1: Classical pendulum.

The Euler-Lagrange equation states

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$$

Since T does not depend on q

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q}$$

Since $p = \partial L / \partial \dot{q}$ we obtain Newton's equation of motion

$$\frac{dp}{dt} = -\frac{\partial V}{\partial q}$$

Short review: Classical oscillator*

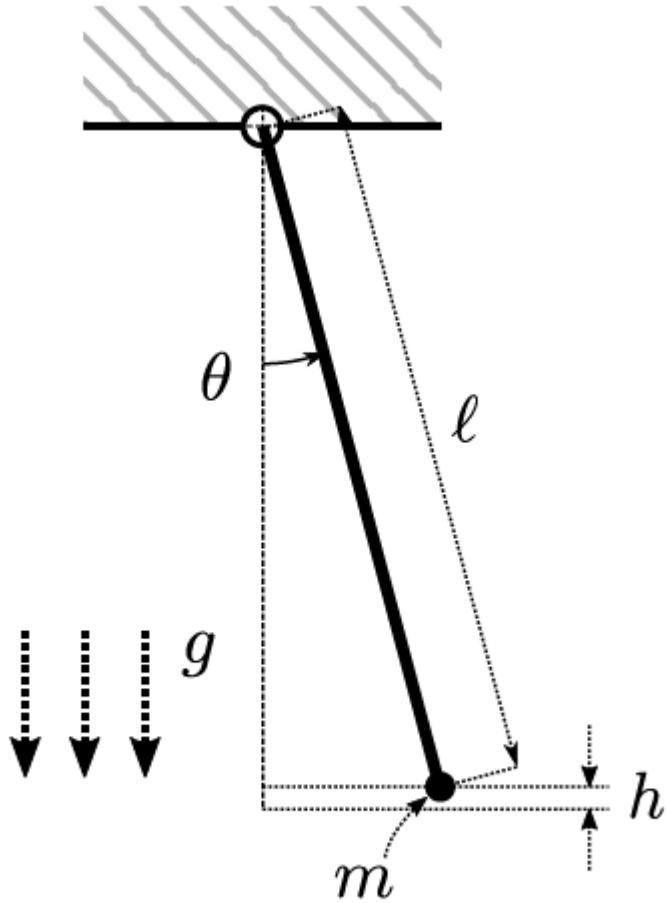


Figure 1: Classical pendulum.

Applying our example to the Euler-Lagrange equation gives

$$\dot{p} = -mg\theta$$

In addition, we can independently differentiate p wrt time:

$$\dot{p} = m\ell\ddot{\theta}$$

Together this yields:

$$m\ell\ddot{\theta} + mg\theta = 0,$$

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0.$$

Short review: Classical oscillator*

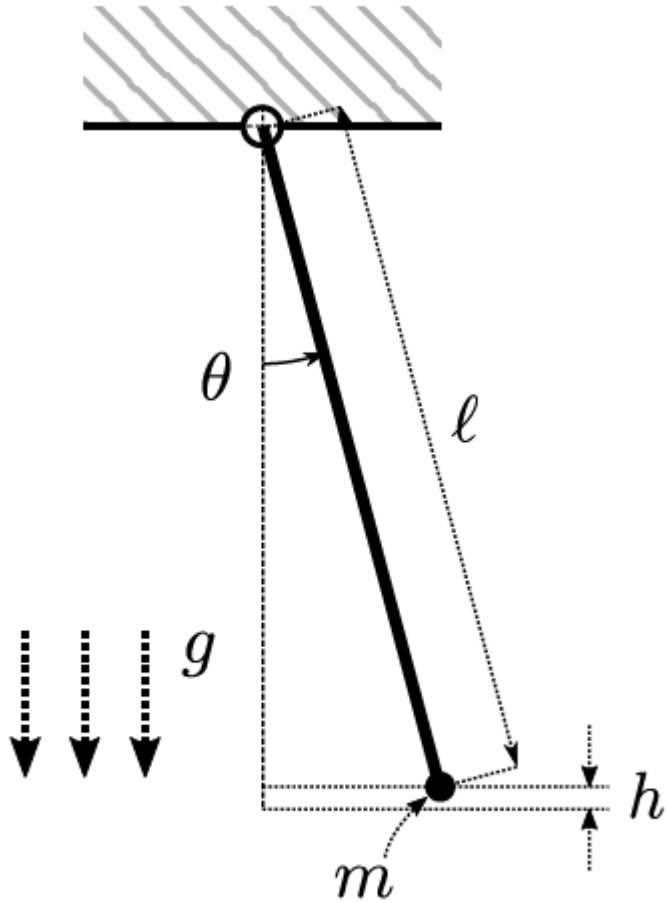


Figure 1: Classical pendulum.

Because we are smart, we chose the trial function

$$\theta = C \exp(i\omega t)$$

Inserting this function into differential equation yields:

$$i^2\omega^2 C \exp(i\omega t) + \frac{g}{\ell} C \exp(i\omega t) = 0$$

Solving this equation provides the well-known result:

$$\omega = \sqrt{g/\ell}$$

Key takeaway: Starting from energy considerations, we can derive the **eigenfrequency** of the system

Agenda for today

7. Quantization of electrical networks

- a. Harmonic oscillator: Lagrangian, eigenfrequency
- b. **Transfer step: LC oscillator, Legendre transform to Hamiltonian**
- d. Quantization of oscillators

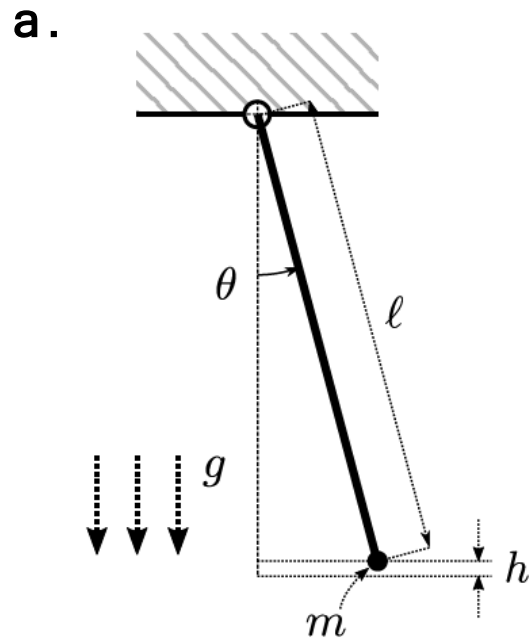


Figure 1: Classical pendulum.

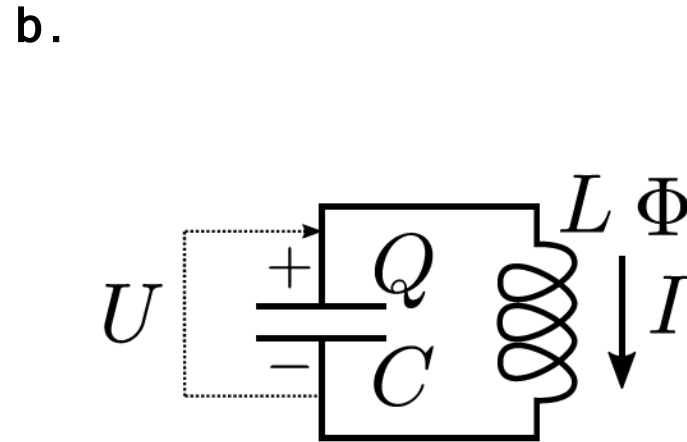
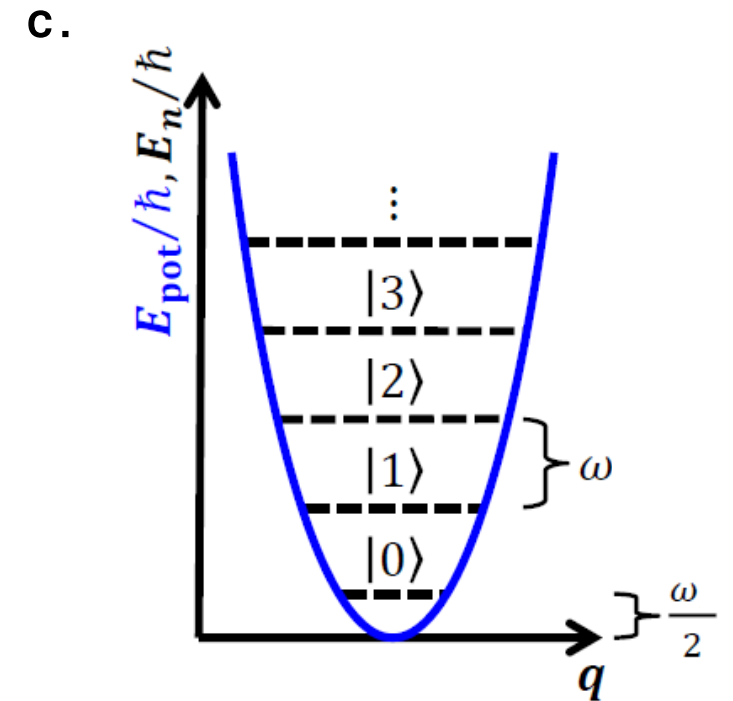


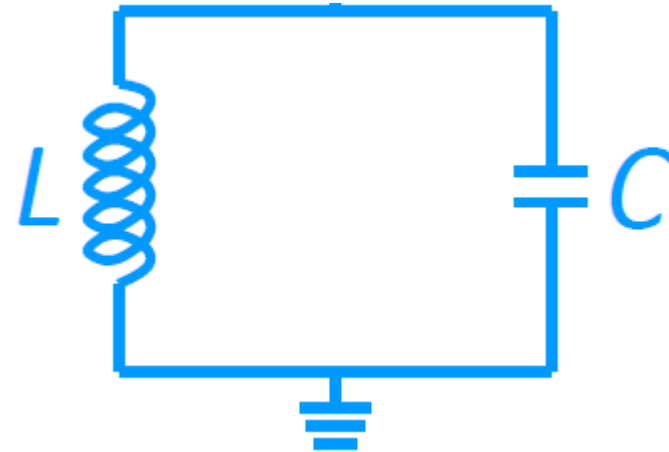
Figure 2: Superconducting LC oscillator.



General note: LC oscillators

- General note: Once you understand the harmonic oscillator, you can easily apply the concept to any other oscillator.

Momentum \hat{p}	\leftrightarrow Charge \hat{q}
Position \hat{x}	\leftrightarrow Flux $\hat{\Phi}$
Mass m	\leftrightarrow Capacitance C
Resonance frequency ω_r	$\leftrightarrow \omega_r = 1/\sqrt{LC}$



Transfer step: LC oscillator*

We assume an electrical circuit consisting of inductance L and capacitance C . The charge stored in the capacitor is

$$Q = CU$$

The power fed into the circuit is $P = UI$ and consequently

$$P = U\dot{Q}$$

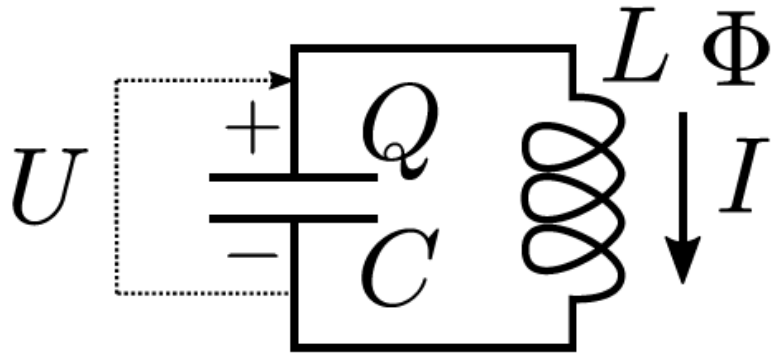


Figure 2: Superconducting LC oscillator.

Hence, the potential energy stored in the system is

$$V = \int_{t_0}^{t_1} P \, dt = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QU = \frac{1}{2} CU^2$$

Transfer step: LC oscillator*

For the magnetic flux Φ in a coil, it holds that

$$\Phi = LI$$

Lenz law tells us that

$$\dot{\Phi} = U$$

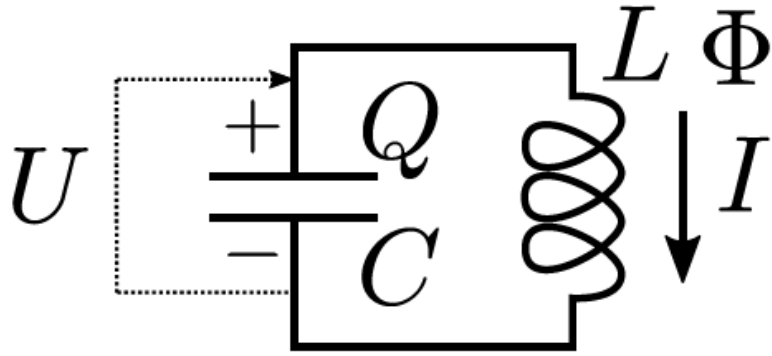


Figure 2: Superconducting LC oscillator.

Hence, the kinetic energy stored in the system is

$$T = \int_{t_0}^{t_1} P \, dt = \int_{t_0}^{t_1} UI \, dt = \frac{1}{2} LI^2 = \frac{\Phi^2}{2L} .$$

Transfer step: LC oscillator*

To apply **Lagrangian mechanics**, we use the previous results

$$V = \frac{1}{2}CU^2 = \frac{Q^2}{2C} , \quad T = \frac{1}{2}LI^2 = \frac{1}{2}L\dot{Q}^2 ,$$

allowing us to write the Lagrangian as

$$L = T - V = \frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C} .$$

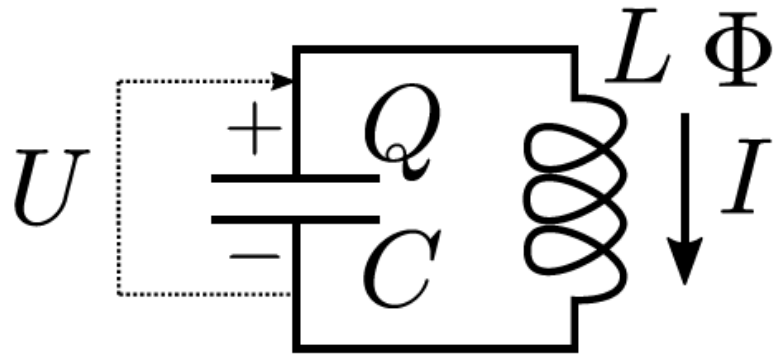


Figure 2: Superconducting LC oscillator.

To derive the **equation of motion**, we again introduce generalized coordinates

$$q = Q ,$$
$$p \equiv \frac{\partial L}{\partial \dot{q}} = L\dot{Q} = -LI = -\Phi .$$

Transfer step: LC oscillator*

Remind yourself again of Euler-Lagrange:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$$

Using the above results gives the equation of motion for charge:

$$\ddot{Q} + \frac{1}{LC} Q = 0$$

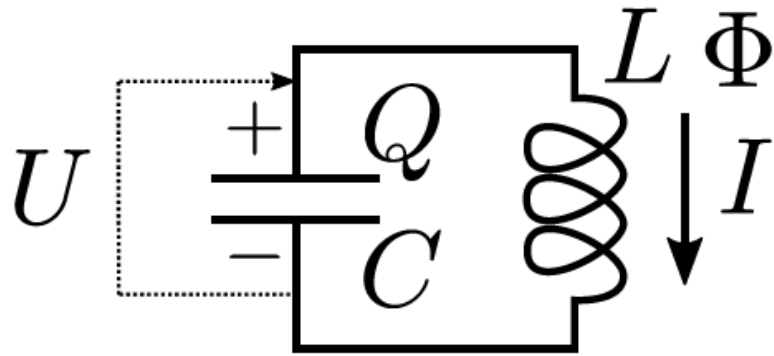


Figure 2: Superconducting LC oscillator.

Using a similar ansatz for the trial function yields the resonance frequency

$$\omega = \frac{1}{\sqrt{LC}} \longleftrightarrow \text{Pendulum: } \omega = \sqrt{g/\ell}$$

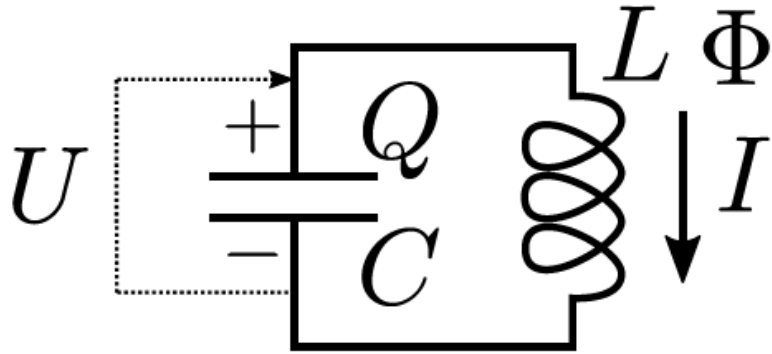
Key takeaway: Starting from energy considerations, we can derive the eigenfrequency of the system

Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

The general definition for a Hamiltonian is

$$H \equiv \dot{q}p - L .$$



We take the **total** time derivative to analyze the system dynamics:

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \dot{L}$$

Figure 2: Superconducting LC oscillator.

To solve this equation, we use

$$p \equiv \partial L / \partial \dot{q}$$

$$p = p(t) \text{ only, so } dp/dt = \dot{p}$$

Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

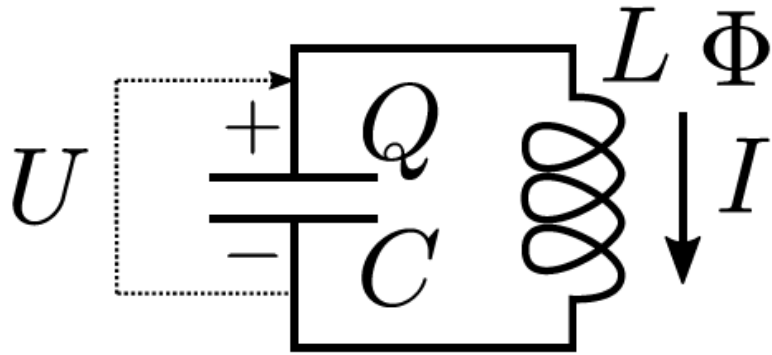


Figure 2: Superconducting LC oscillator.

Using the above terms in the total time derivative yields

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - p\ddot{q} - \dot{L}$$

Simplifying this formula further results in

$$\frac{dH}{dt} = \dot{q} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right] - \dot{L}$$

The Lagrangian is time independent and due to Euler Lagrange, the parentheses are also zero, hence

$$\frac{dH}{dt} = 0$$

We find that the Hamiltonian is a constant of motion, i.e. energy is conserved in the system

Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

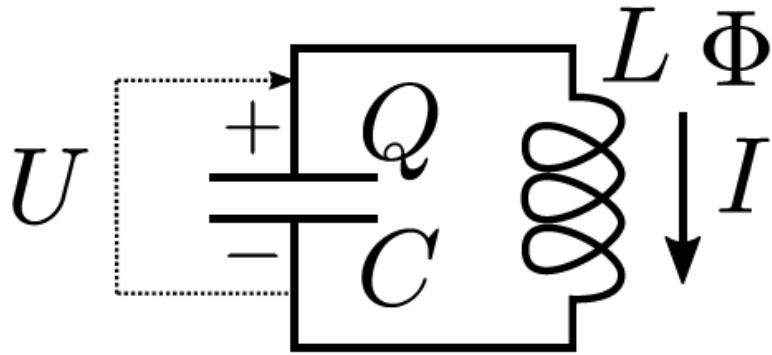


Figure 2: Superconducting LC oscillator.

We can use the general definition for the Hamiltonian to find

$$H = \dot{Q}(L\dot{Q}) - \left(\frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C} \right) = \frac{1}{2}L\dot{Q}^2 + \frac{Q^2}{2C}$$

Inserting the standard definitions, we find

$$H = \frac{1}{2}LI^2 + \frac{Q^2}{2C} = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}$$

Hence, the Hamiltonian represents the total energy of the system

$$H = T + V .$$

Key takeaway: Starting from Lagrangian, we can derive the **total energy** of the system. This is necessary to derive energy quantization.

Agenda for today

7. Quantization of electrical networks

- a. Harmonic oscillator: Lagrangian, eigenfrequency
- b. Transfer step: LC oscillator, Legendre transform to Hamiltonian
- d. **Quantization of oscillators**

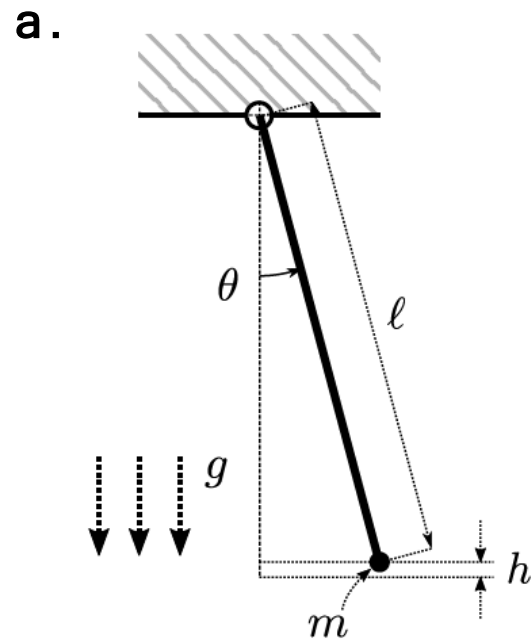


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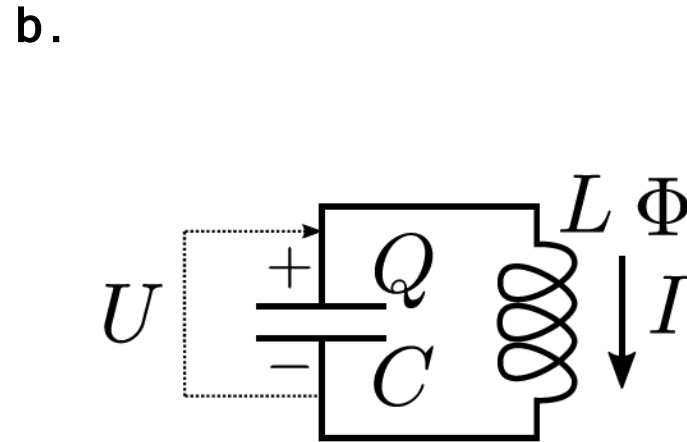
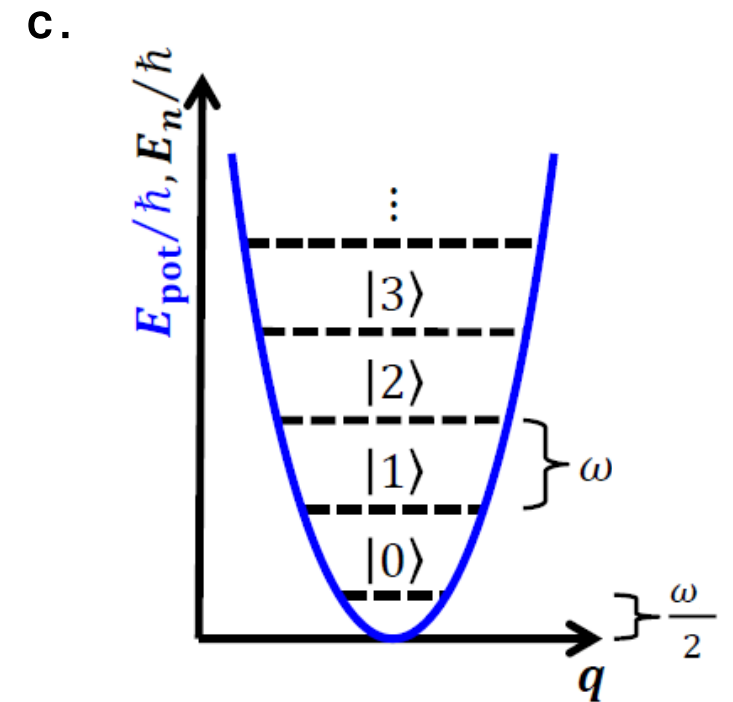
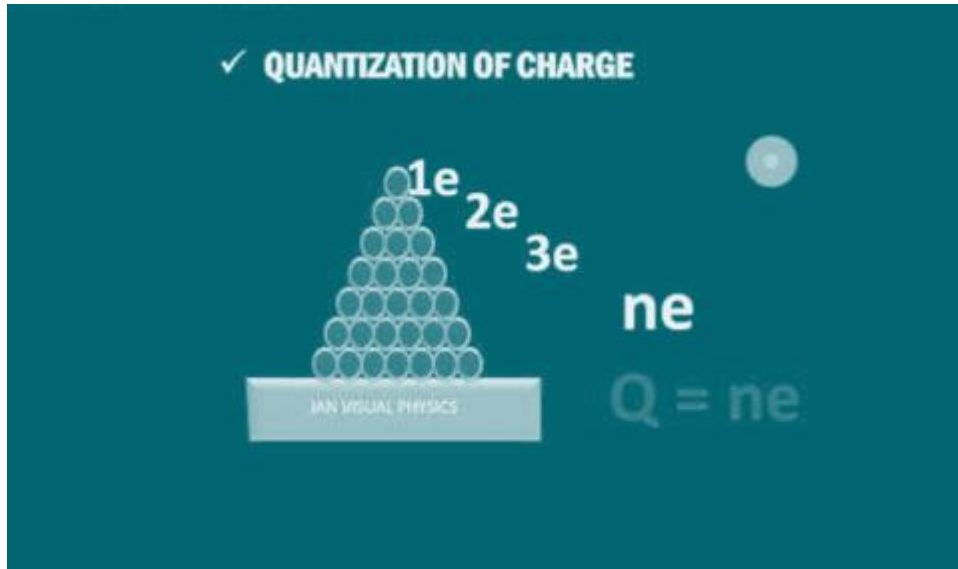


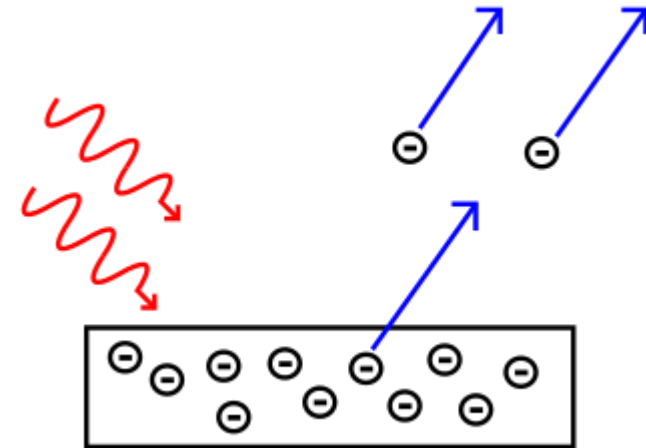
Figure 2: Superconducting LC oscillator.



Classical → Quantum



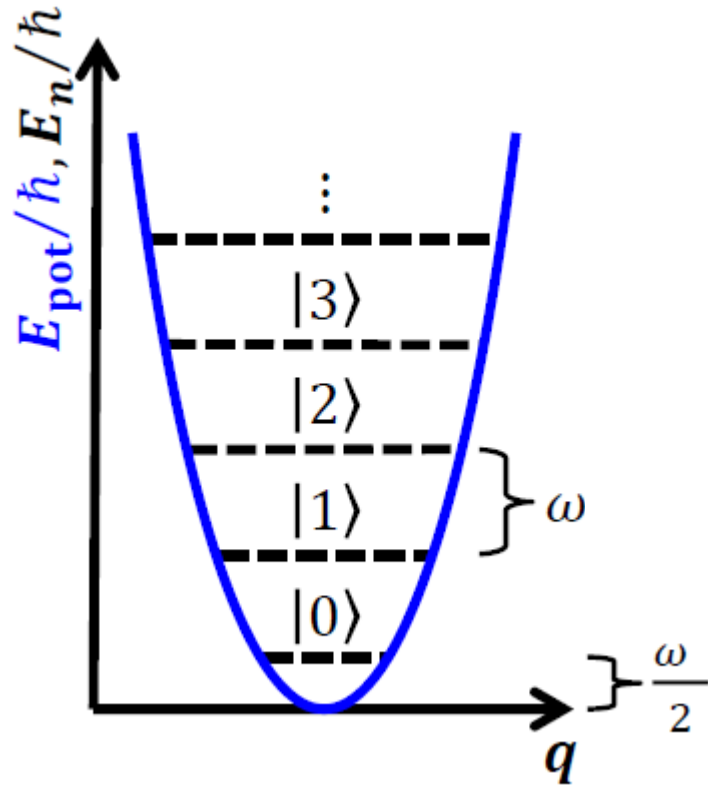
Photoelectric effect →
Electromagnetic field is quantized →
Energy is quantized: $E = \hbar\omega$.



General note: Quantization of oscillators

- General note: **Quantization** means we see the effect of single particles or excitations.
- In a harmonic oscillator, the energy is quantized **equidistantly**.
- Energy quantization can be seen as counting the **number of photons** stored in the oscillator.

Quantization of an oscillator



In quantum mechanics, variables are replaced by operators:

$$q \rightarrow \hat{q} : \mathcal{H} \rightarrow \mathcal{H} ,$$

$$p \rightarrow \hat{p} : \mathcal{H} \rightarrow \mathcal{H} ,$$

For practical reasons, we often use matrix representations

$$q_{kl} = \langle e_k | \hat{q} | e_l \rangle$$

Two conjugate variables follow the commutation relation

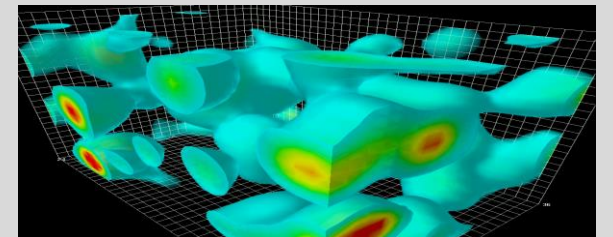
$$[\hat{p}, \hat{q}] \equiv \hat{p}\hat{q} - \hat{q}\hat{p} = -i\hbar$$

Examples

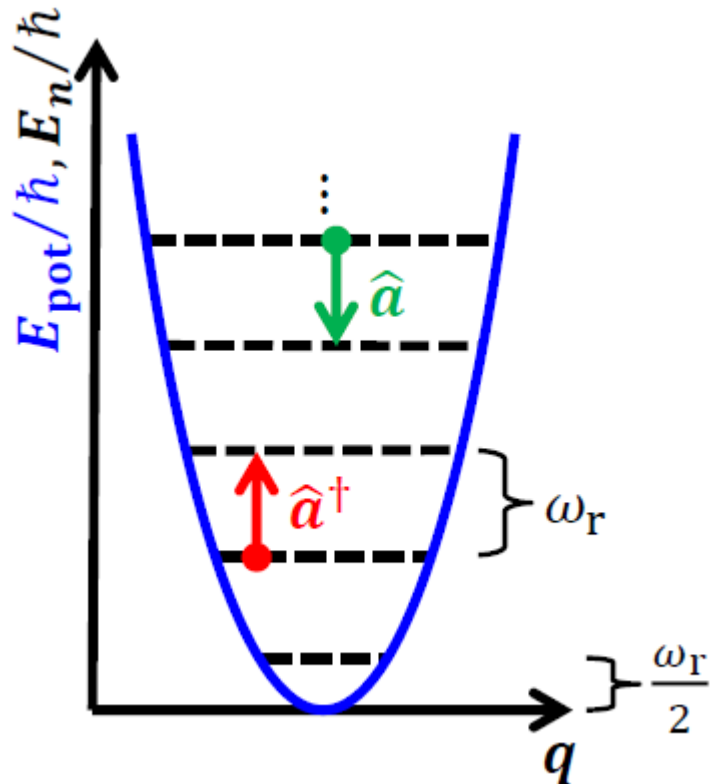
Charge \hat{q}
Flux $\hat{\Phi}$

$$\begin{pmatrix} (O\psi)_1 \\ (O\psi)_2 \\ \vdots \\ (O\psi)_i \\ \vdots \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} & \dots & O_{1j} & \dots \\ O_{21} & O_{22} & \dots & O_{2j} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_{i1} & O_{i2} & \dots & O_{ij} & \dots \\ \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_j \\ \vdots \end{pmatrix}$$

Vacuum fluctuations:



Quantization of an oscillator



For pedagogical reasons, it is convenient to transform systems into the basis of number states (give matrix representation of a)

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

These can be interpreted as ladder operators raising and lowering the excitation number

$$\hat{a} \equiv \frac{\omega_r C \hat{\Phi} + i \hat{q}}{\sqrt{2 \omega_r C \hbar}}$$

is the **annihilation operator**

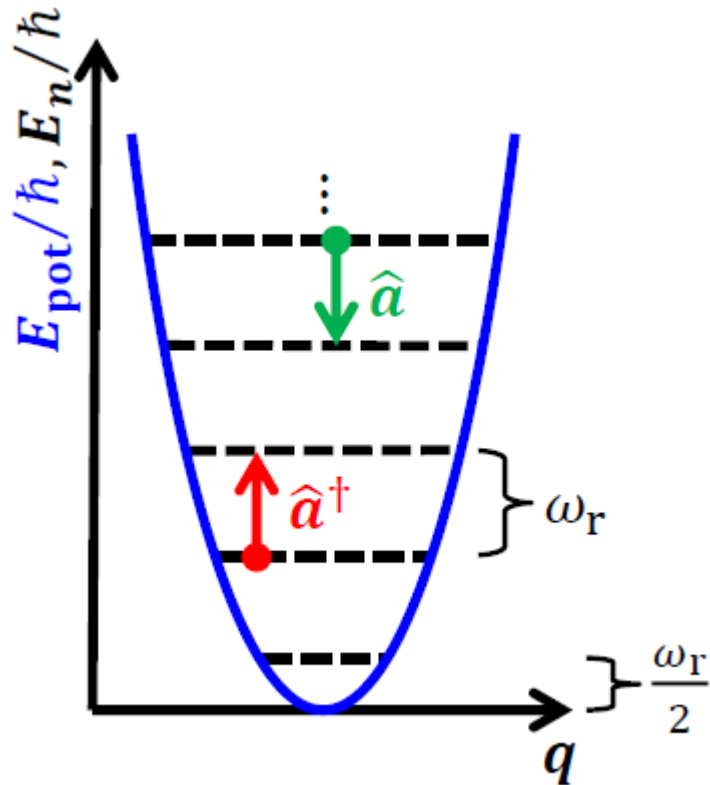
$$\hat{a}^\dagger \equiv \frac{\omega_r C \hat{\Phi} - i \hat{q}}{\sqrt{2 \omega_r C \hbar}}$$

is the **creation operator**

Quantization of an oscillator

Their product gives the excitation number of a system

$$\hat{n} \equiv \hat{a}^\dagger \hat{a}$$



Quantization of the LC oscillator

For the superconducting resonator, we have

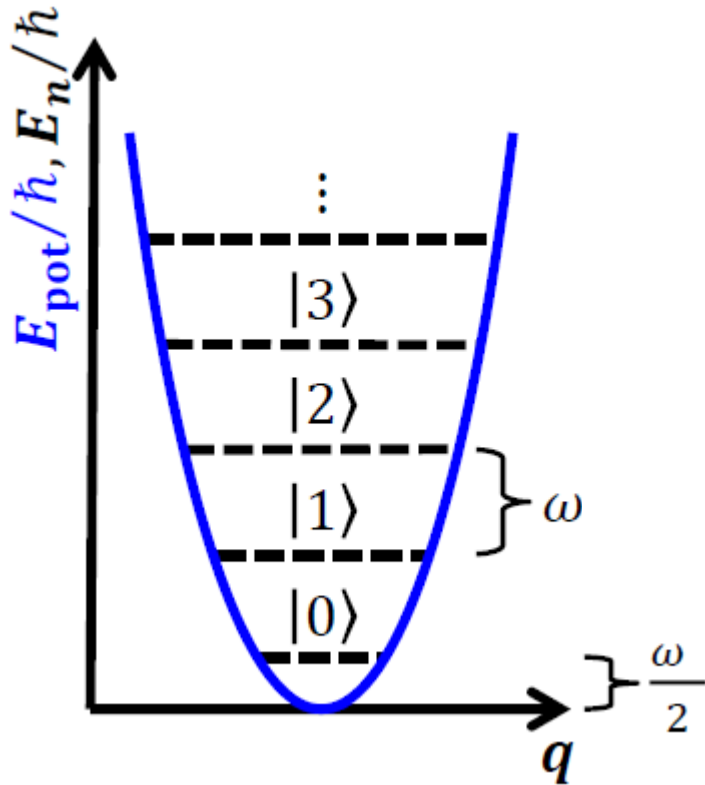
$$\hat{H} = \frac{\hat{p}^2}{2L} + \frac{\hat{q}^2}{2C} .$$

We aim to diagonalize H into a form involving only one operator. This can be done by a change of variables.

$$\hat{p} = \sqrt{\frac{\hbar\omega L}{2}} \left(\hat{a} + \hat{a}^\dagger \right) , \quad \hat{q} = \sqrt{\frac{\hbar\omega C}{2}} i \left(\hat{a} - \hat{a}^\dagger \right) .$$

Here ω is a free scalar parameter, which we will choose later. The square root factors have been inserted for convenience.

$(\hat{a} + \hat{a}^\dagger)$ and $i(\hat{a} - \hat{a}^\dagger)$ are Hermitian and independent.



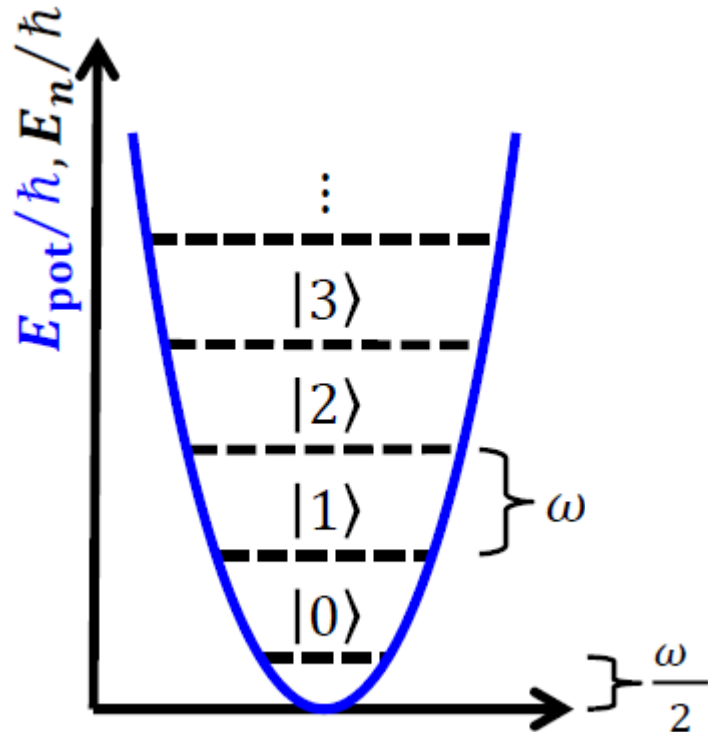
Quantization of the LC oscillator

The previous Hamiltonian becomes

$$\begin{aligned}\hat{H} &= \frac{\left[\sqrt{\frac{\hbar\omega L}{2}} (\hat{a} + \hat{a}^\dagger) \right]^2}{2L} + \frac{\left[\sqrt{\frac{\hbar\omega C}{2}} i (\hat{a} - \hat{a}^\dagger) \right]^2}{2C} \\ &= \frac{\hbar\omega}{4} (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ &= \frac{\hbar\omega}{2} (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) .\end{aligned}$$

Using $[\hat{a}, \hat{a}^\dagger] = 1$ it follows that $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$$



Key takeaway: The **total energy** of the system is given by vacuum fluctuations (+1/2) and the number of photons stored at frequency ω

Agenda for today (done)

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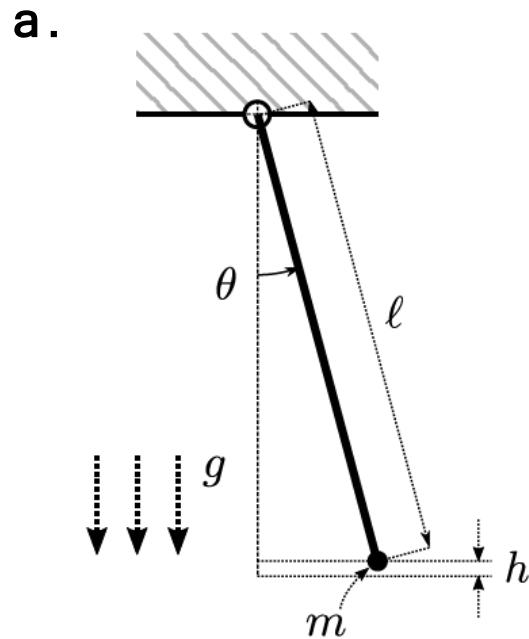


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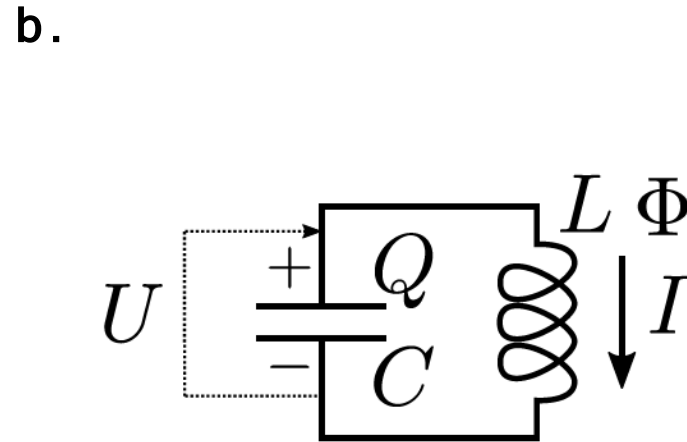
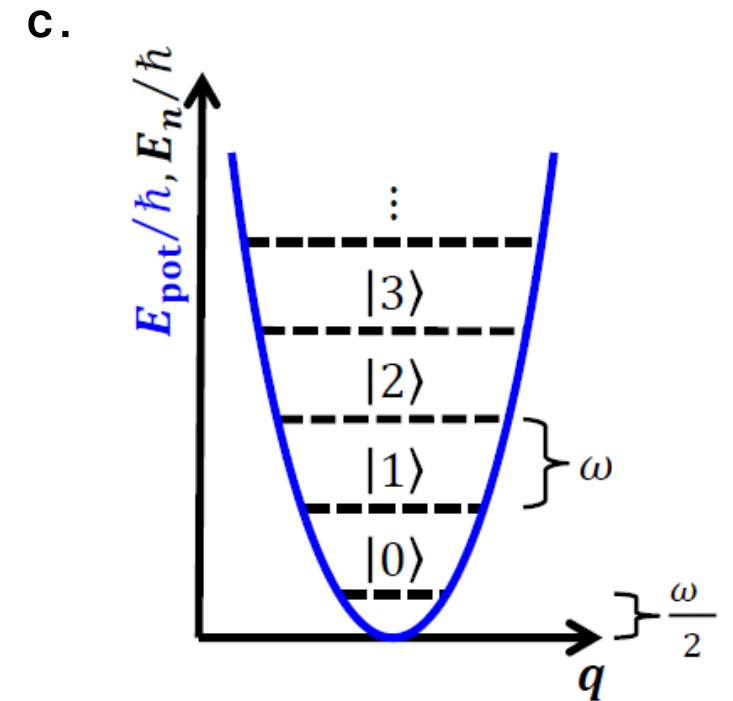


Figure 2: Superconducting LC oscillator.



Add-on: Vacuum fluctuations & thermal photons (if time allows)

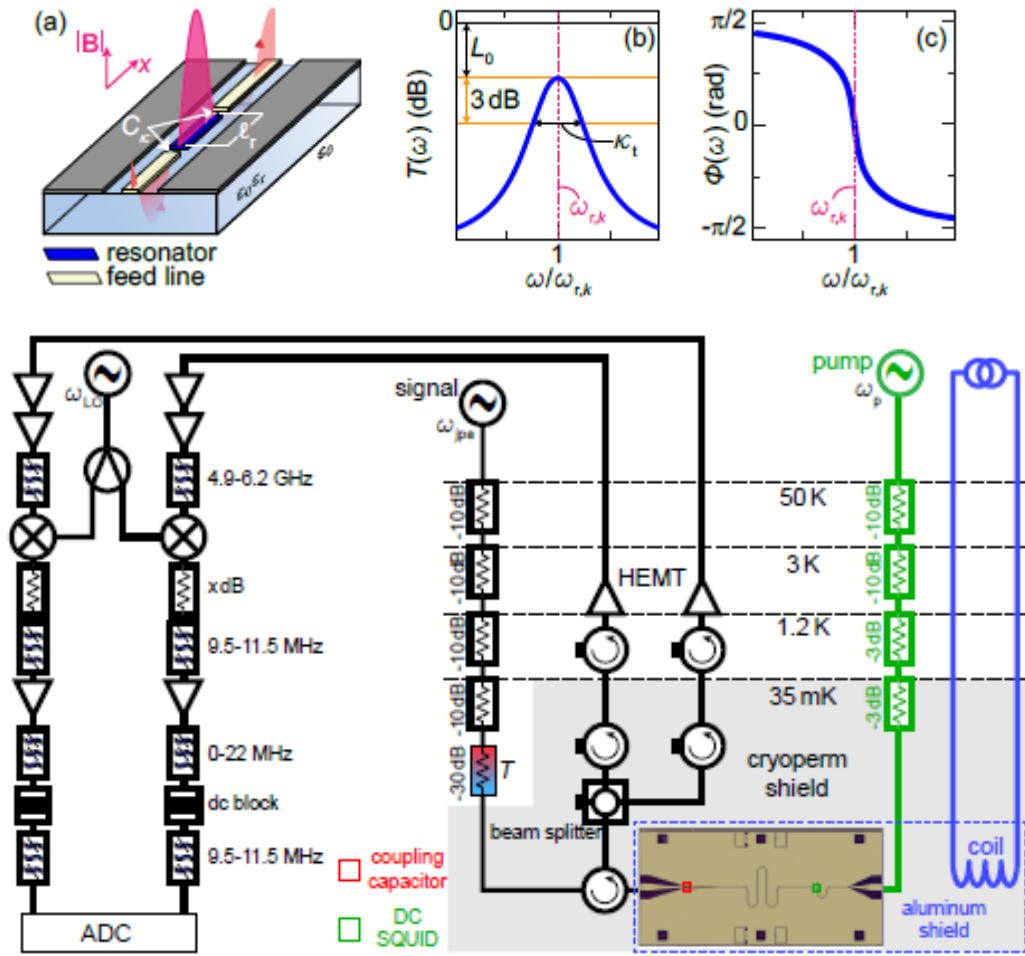


Figure 4.38: Schematics of the dual-path setup.

