Problem Set 4

Due March 29, 16:00 PM

Lagrangian, Hamiltonian, and Legendre transformation

1. Lagrangian mechanics: Consider a pendulum with an elastic supporting wire. Its rest length is l_0 and spring constant k . Use the length of the wire l and the swinging angle θ as generalized coordinates. The Lagrangian of such a system can be written as,

$$L = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) + mgl\cos\theta - \frac{1}{2}k(l - l_0)^2. \tag{1}$$

where the kinetic energy $T = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)$ and the potential energy $V = -mgl\cos\theta + \frac{1}{2}k(l - l_0)^2$. Convince yourself why the Lagrangian has this form.

Derive the equation of motions (Euler-Lagrange) for generalized co-ordinates l, θ in the limit of a small angle θ.

2. Hamiltonian mechanics: Find the Hamiltonian of the system in (1) using Legendre transformation and specify the equation of motions.

Note: The advantage of going from Lagrangian to Hamiltonian formalism is that you get coupled first-order differential equations from the Hamiltonian formalism, instead of a second-order differential equation that you obtain from the Lagrangian formalism. It is much easier to solve a coupled first-order DE compared to second-order DE.

Quantization of the electrical networks

3. Show that if the state $|n\rangle$ is normalized, i.e. $\langle n| |n\rangle = 1$, then in order for the state $|n+1\rangle$ to also be normalized, the operator $\hat{a}^\dagger$ must have the property

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle\,. \tag{2}$$

(*Hint:* $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$, $\hat{a}^\dagger |0\rangle = |1\rangle$, $\hat{a}|0\rangle = 0$.)

4. Show that the operator $\hat{a}^\dagger$ can be represented as

$$\hat{a}^\dagger = \sum_{m=0}^{\infty} \sqrt{m+1}\, |m+1\rangle \langle m|\,. \tag{3}$$

Problem Set 4

Superconducting quantum circuits

5. Transmon qubit: A charge qubit provides an excellent anharmonicity to the energy levels, however the charge dispersion, i.e., the dependence of the energy on the gate charge, introduces a drastic charge noise. Thus, charge noise is the main source of decoherence in the charge qubit.

By adding an additional large shunt capacitance in parallel with the Josephson junction, we suppress the charge noise quite dramatically. This type of qubit is called transmon qubit. The Hamiltonian of the transmon qubit is of the same form as of the charge qubit,

$$H = E_{\mathrm{C}}(n - n_{\mathrm{g}})^2 - E_{\mathrm{J}} \cos \varphi. \tag{4}$$

However, for the transmon qubit the energy ratio is in the range $40 < \frac{E_{\mathrm{J}}}{E_{\mathrm{C}}} < 100$. The Josepshon coupling energy dominates the charging energy, thus suppressing the charge noise.

In this assignment, we will quantize the transmon qubit. We begin by setting the offset gate charge $n_{\mathrm{g}} = 0$, since it does not matter how we bias the transmon with the gate charge. Then follow these steps:

a Consider small angle approximation for cosine potential. Approximate (Taylor expansion) the cosine function to the fourth order term.

b Combine the quadratic terms together and express it in terms of harmonic oscillator equation (Eqn. (46) from the lecture material).

c Promote the classical variables n, φ to quantum operators and express them in terms of annihilation $\hat{b}$ and creation $\hat{b}^\dagger$ operators.

d Diagonalize the Hamiltonian by plugging the operators defined in step (c) to the Hamiltonian you have obtained from step (a).

e In the resulting fourth order term, invoke rotating wave approximation i.e., only keep the terms with equal number of $\hat{b}$ and $\hat{b}^\dagger$.

6. Jaynes-Cummings Hamiltonian: In this exercise, we study a coupled interaction between the resonator and the transmon qubit.

a. Form the total Hamiltonian of the transmon–resonator system, which contains the uncoupled diagonalized Hamiltonians of the resonator and the qubit and the interaction Hamiltonian. The interaction between the resonator and the qubit is modeled by the interation Hamiltonian,

$$\hat{H}_{\mathrm{int}} = -C_{\mathrm{G}} \hat{V}_{\mathrm{Q}} \otimes \hat{V}_{\mathrm{R}}. \tag{5}$$

where, the voltage operators take the form $\hat{V}_{\mathrm{Q}} = -\frac{2e}{C_\Sigma}\hat{n}$ and $\hat{V}_{\mathrm{R}} = \frac{\hat{q}}{C_{\mathrm{r}}}$ for the qubit and the resonator respectively. Here, $C_\Sigma = C_{\mathrm{G}} + C_{\mathrm{J}} + C_{\mathrm{S}}$ is the sum capacitance.

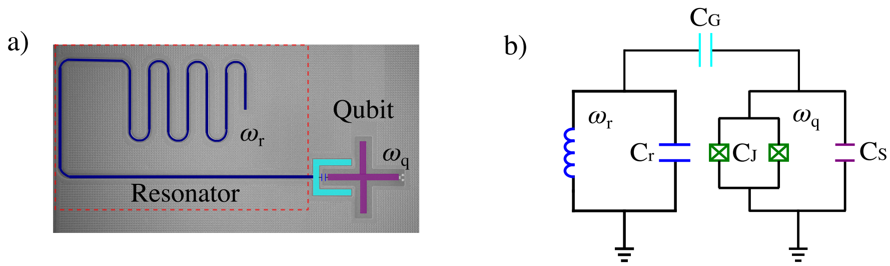

Figure 1: Coupled resonator and transmon qubit. a) Experimental realization, and b) Lumped-circuit model.

b. Invoke rotating wave approximation, and obtain the Jaynes-Cummings Hamiltonian

$$\hat{H}_{\mathrm{JC}} = \hbar\omega_{\mathrm{r}}\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_{\mathrm{q}}}{2}\hat{\sigma}_z + \hbar g(\hat{\sigma}_- \otimes \hat{a}^\dagger + \hat{\sigma}_+ \otimes \hat{a})\,, \tag{6}$$

where $\hat{\sigma}_- = |0\rangle\langle 1|$ and $\hat{\sigma}_+ = |1\rangle\langle 0|$.

c. Solve the Jaynes-Cummings Hamiltonian for the case i) $\Delta = \omega_{\mathrm{r}} - \omega_{\mathrm{q}} = 0$, ii) $\Delta = \omega_{\mathrm{r}} - \omega_{\mathrm{q}} = 0.5\,\mathrm{GHz}$, and iii) $\Delta = \omega_{\mathrm{r}} - \omega_{\mathrm{q}} = 1\,\mathrm{GHz}$. Initially, there are $n+1$ photons in the resonator and the qubit is in the ground state. Plot the probability density of the transmon being in the excited states for n = 1, 10 and 100 photons.