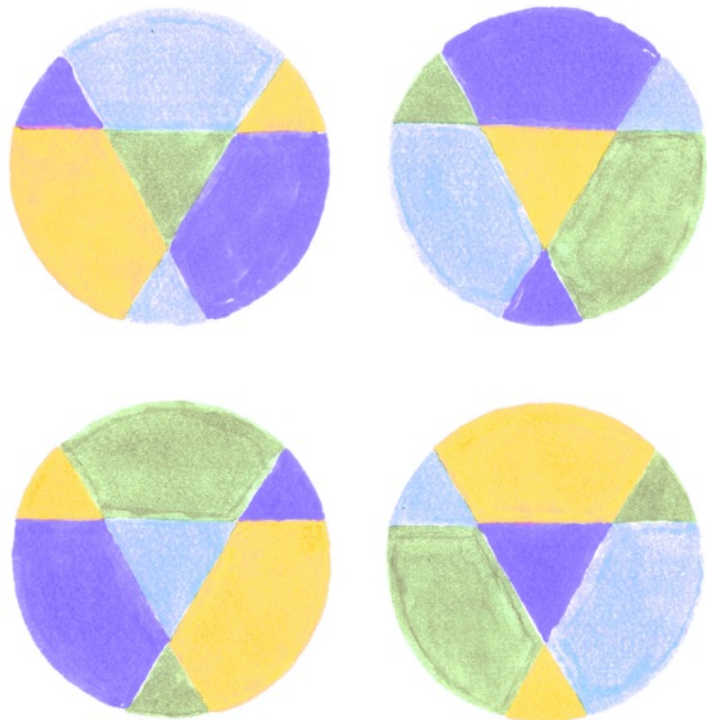
PROJECTIVE GEOMETRY
PART 5



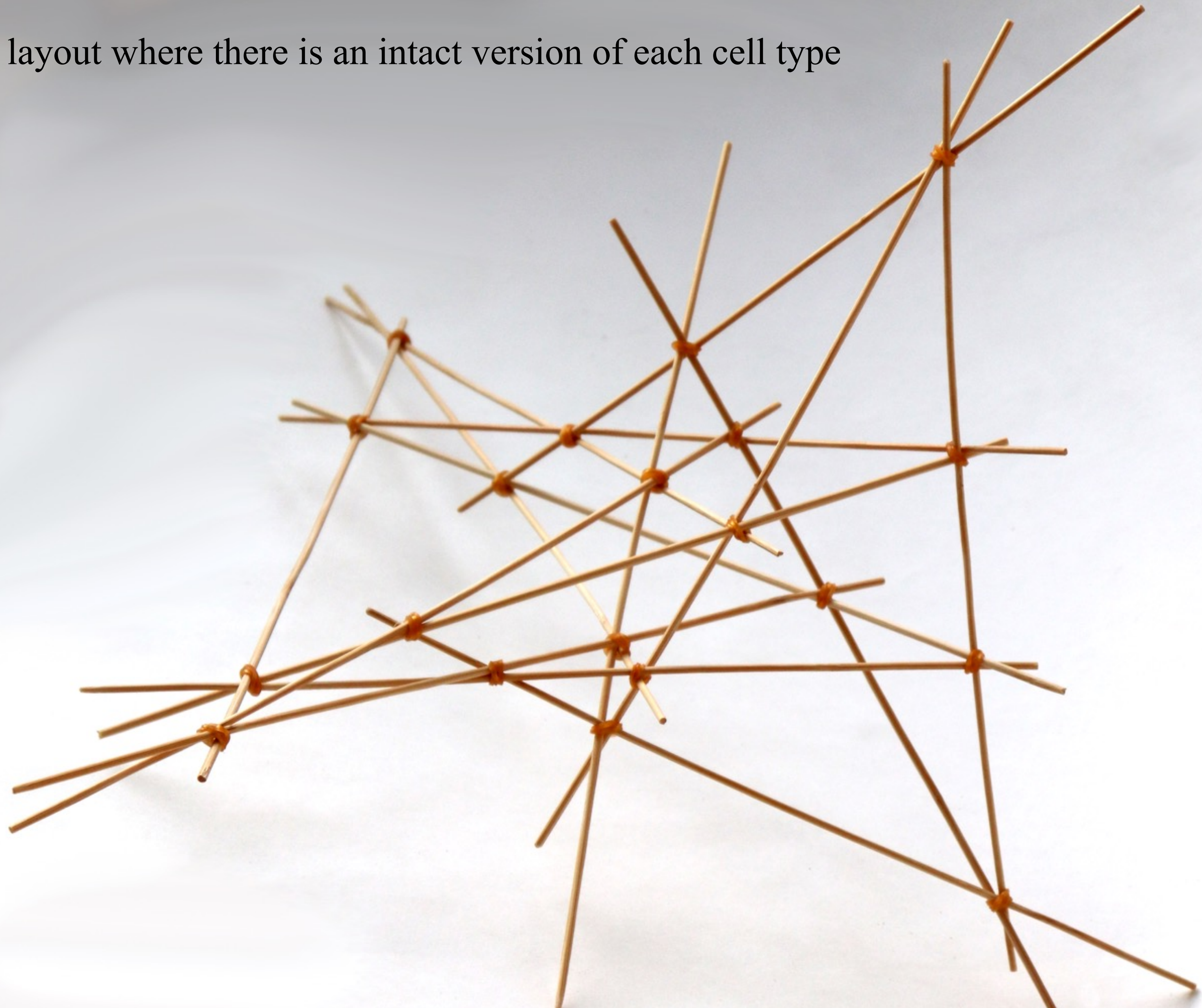
Taneli Luotoniemi

CRYSTAL FLOWER IN HALLS OF MIRRORS 2021

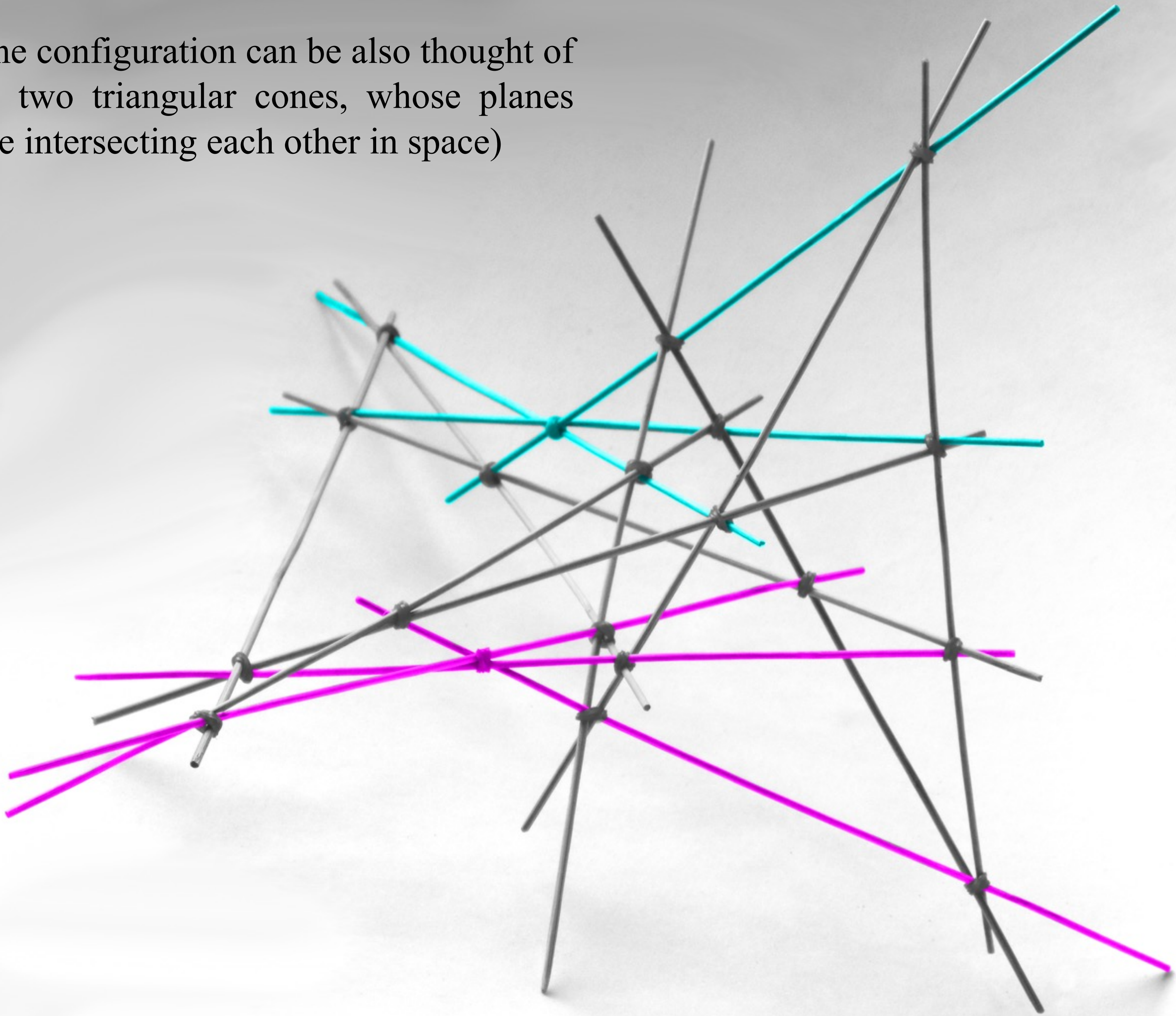
COMPLETE HEXAHEDRON

(SIX PLANES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE SPACE)

A layout where there is an intact version of each cell type

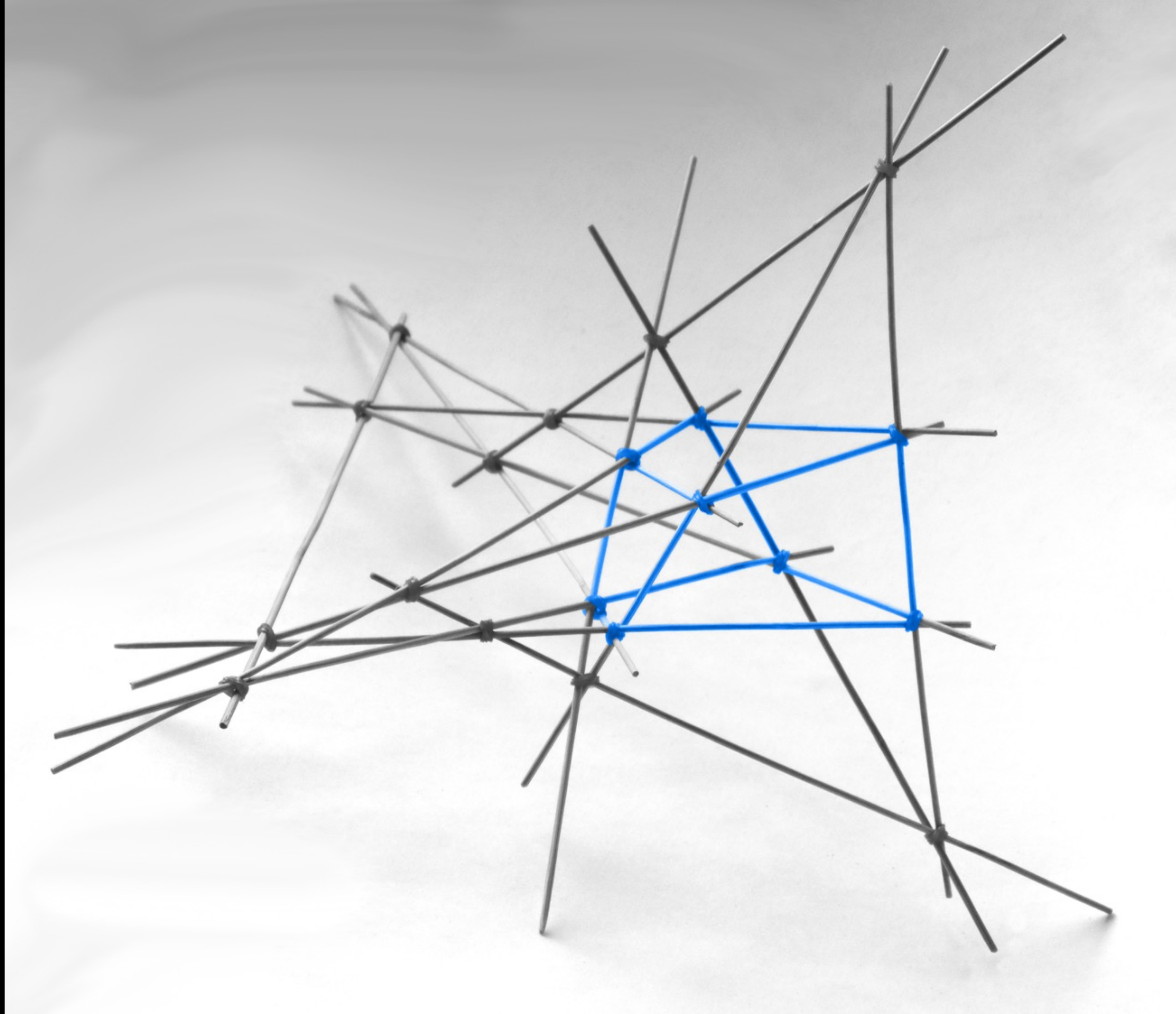


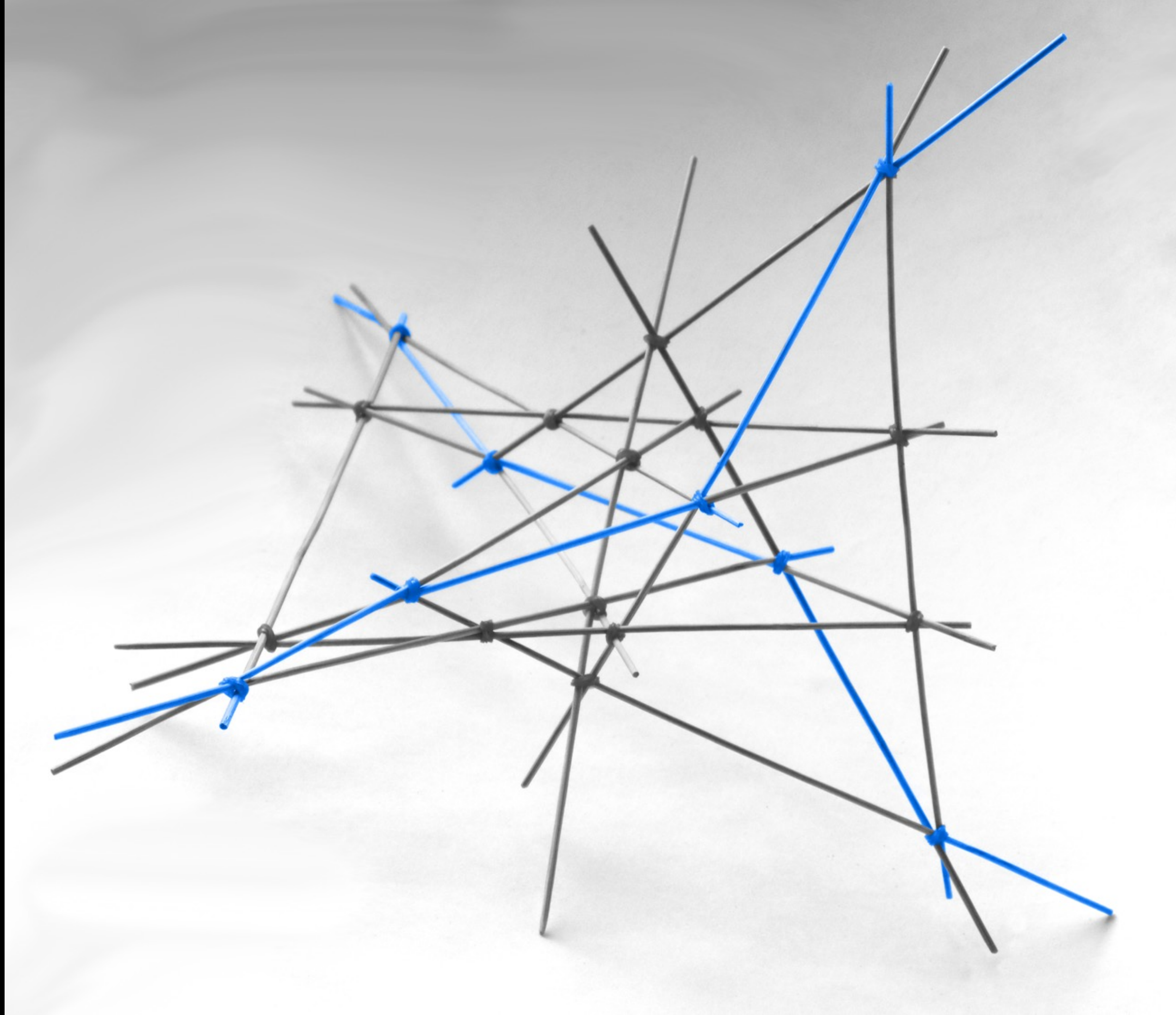
(the configuration can be also thought of as two triangular cones, whose planes are intersecting each other in space)



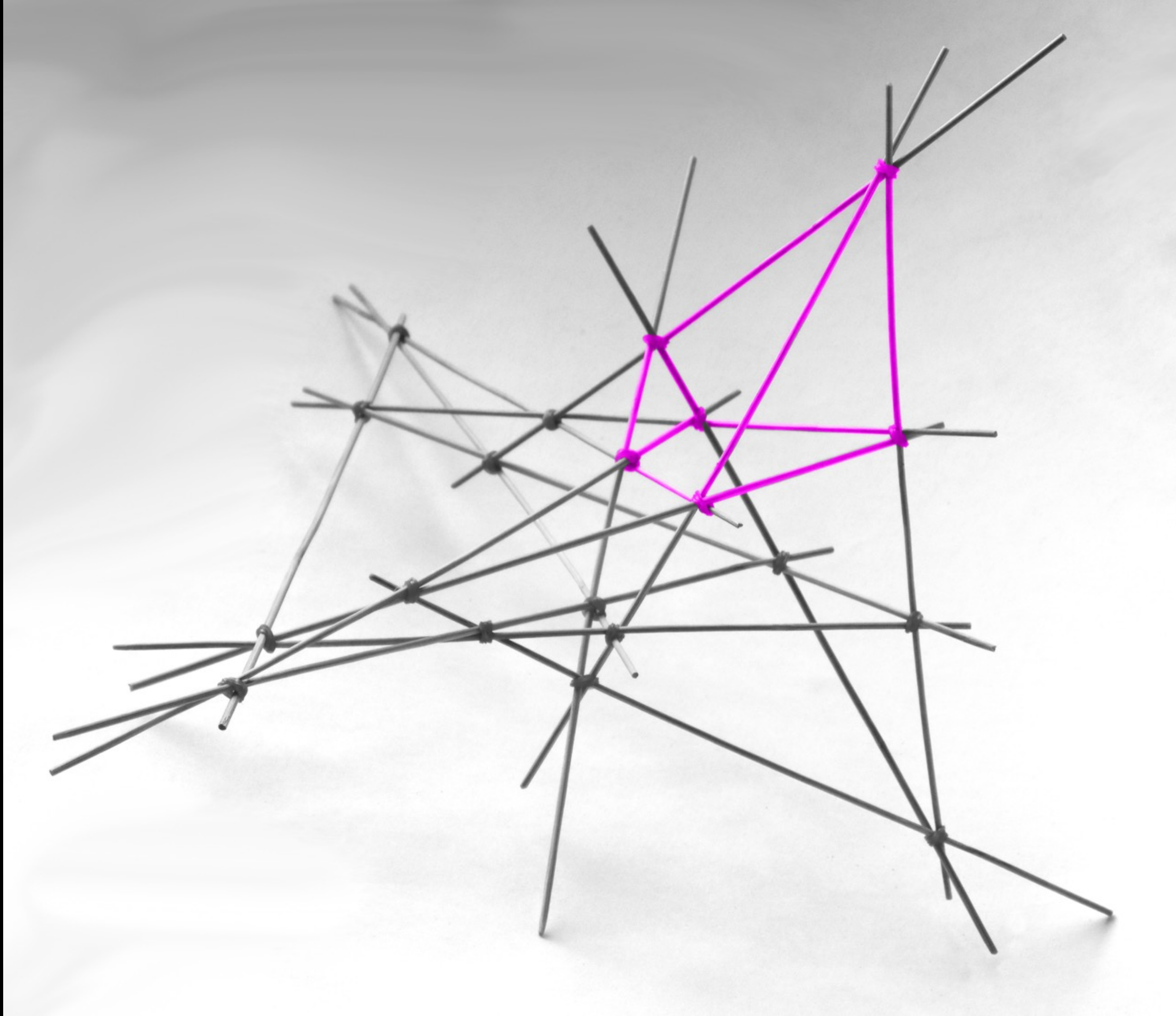
THREE-DIMENSIONAL REGIONS BOUNDED BY THE PLANES

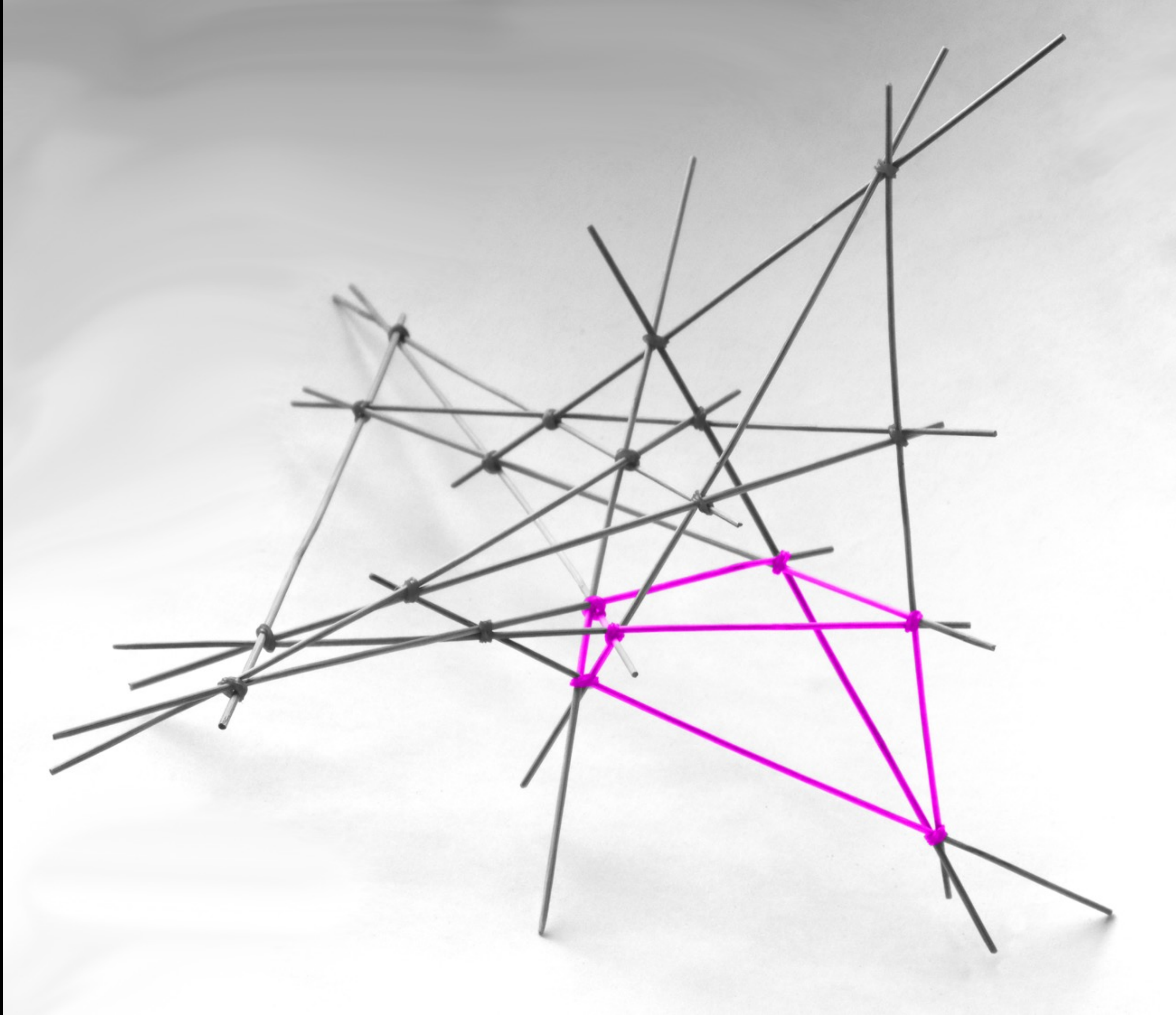
2 x CUBOIDAL CELLS

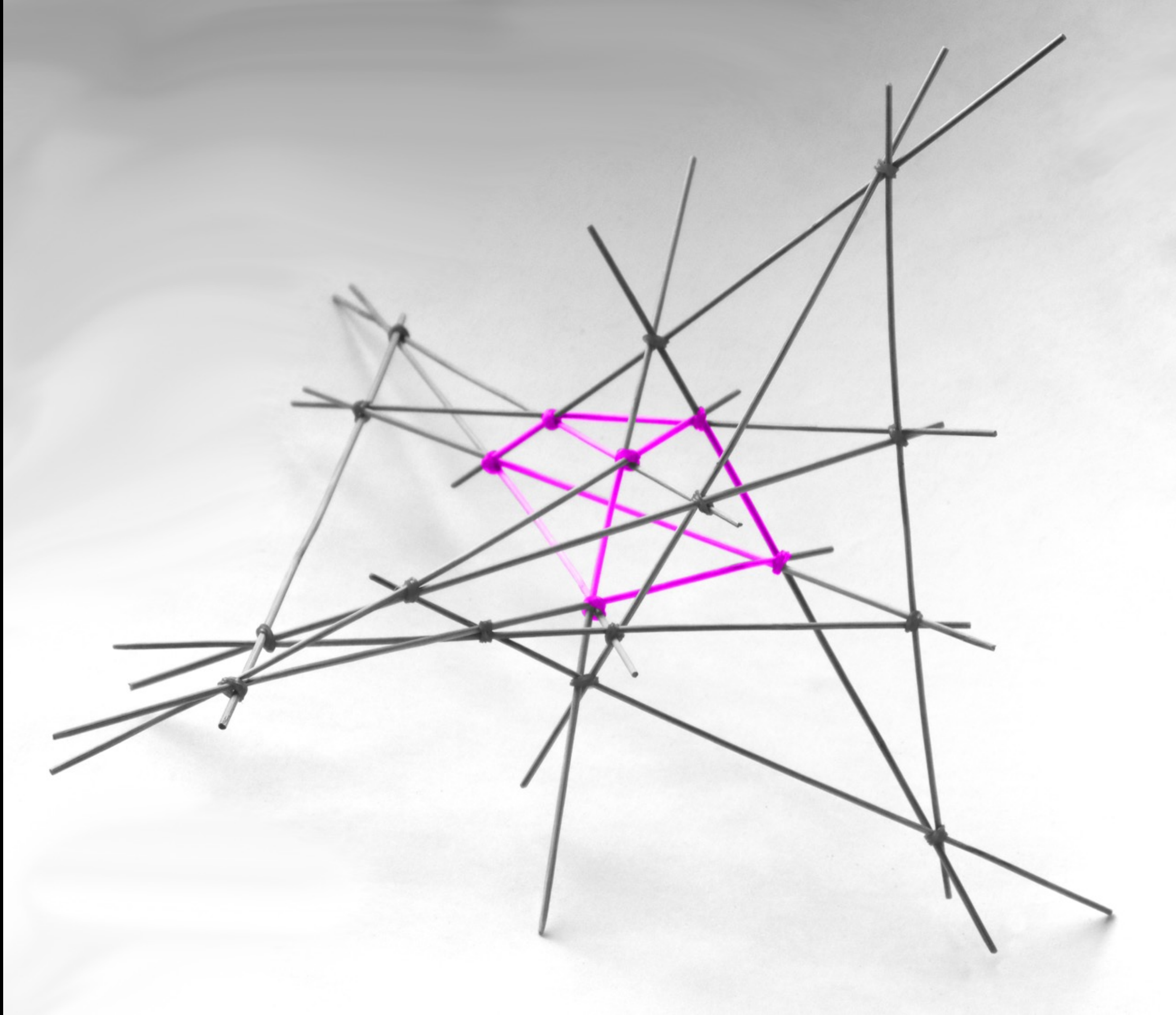


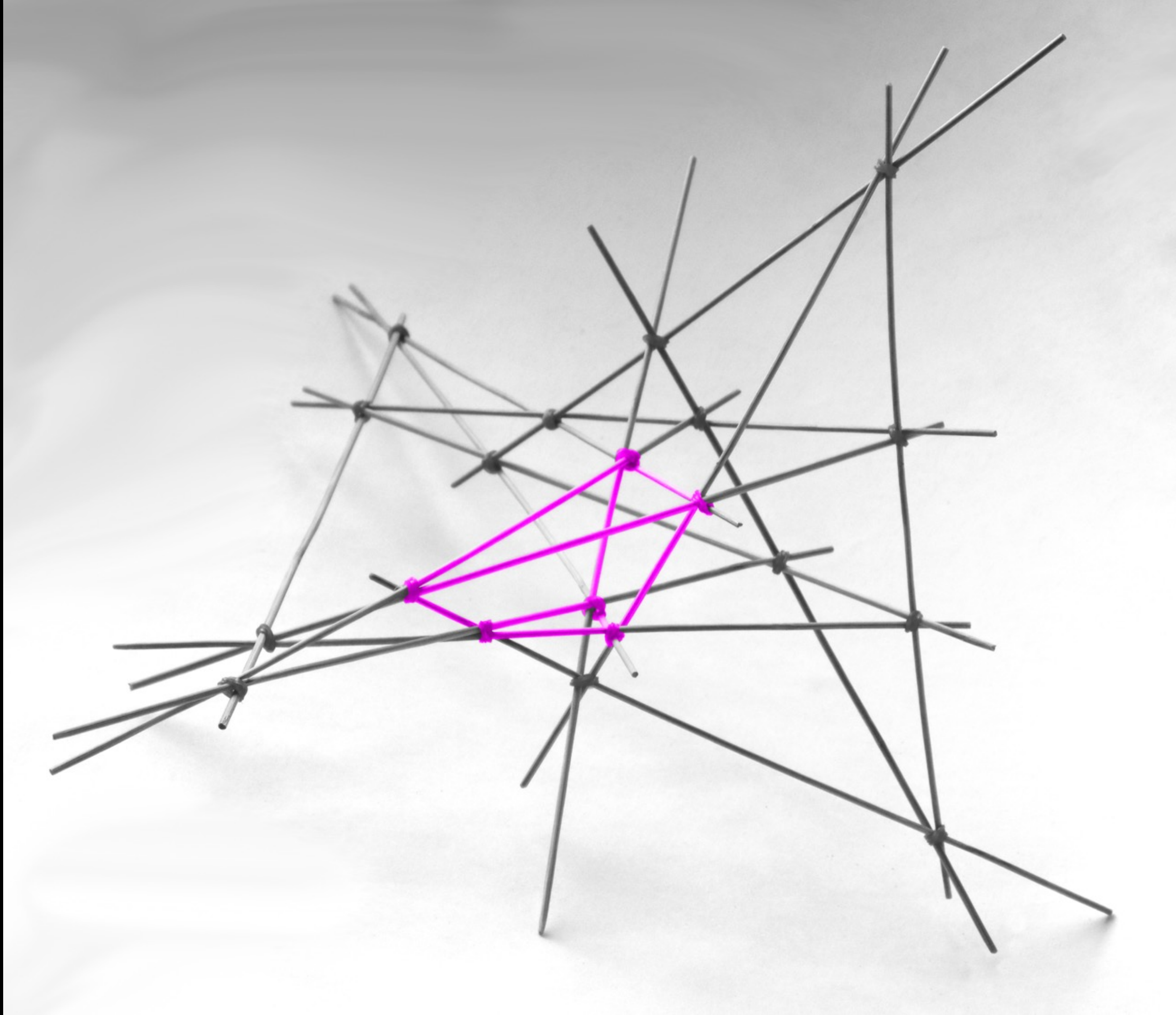


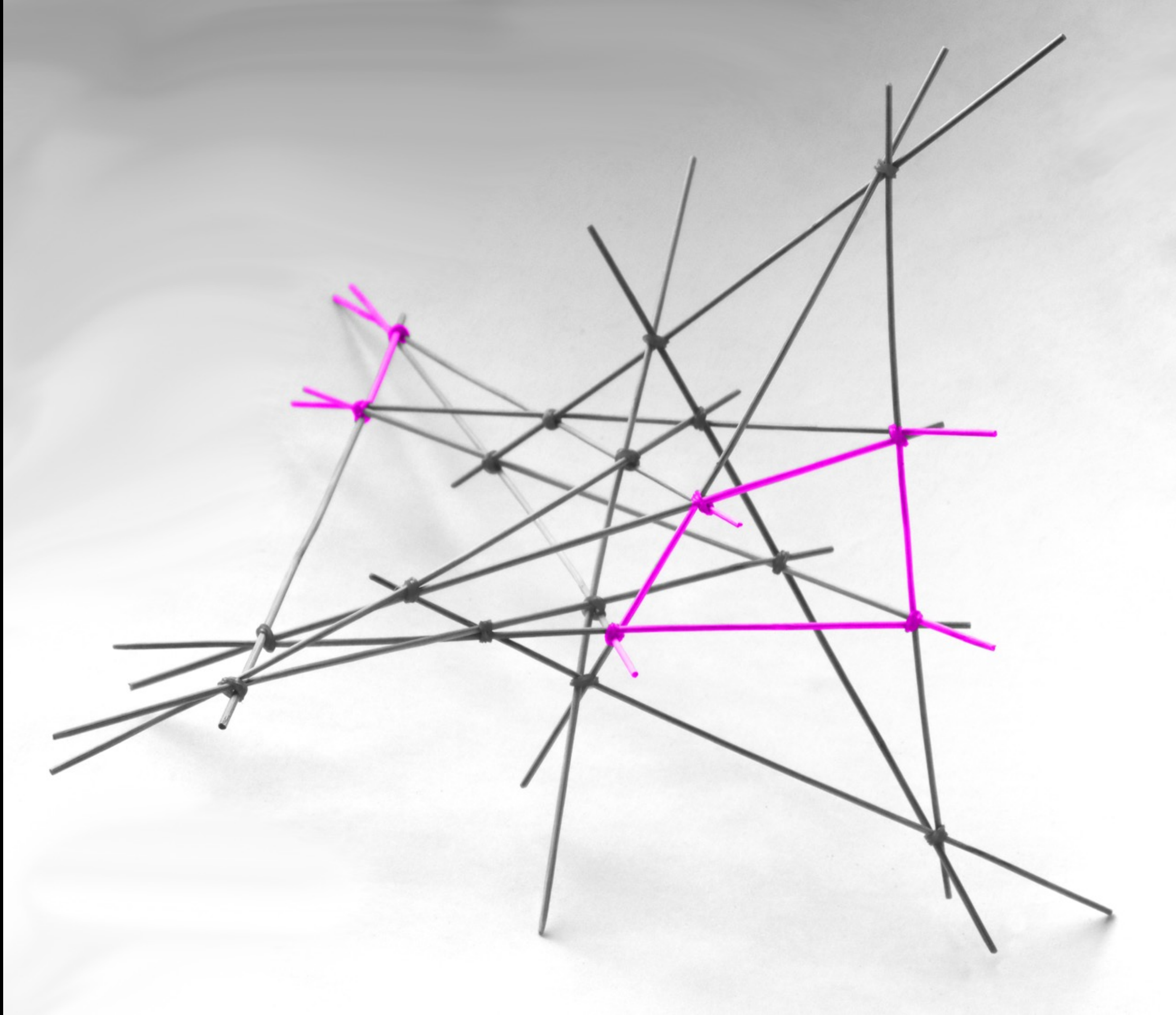
12 x PRISMATIC CELLS

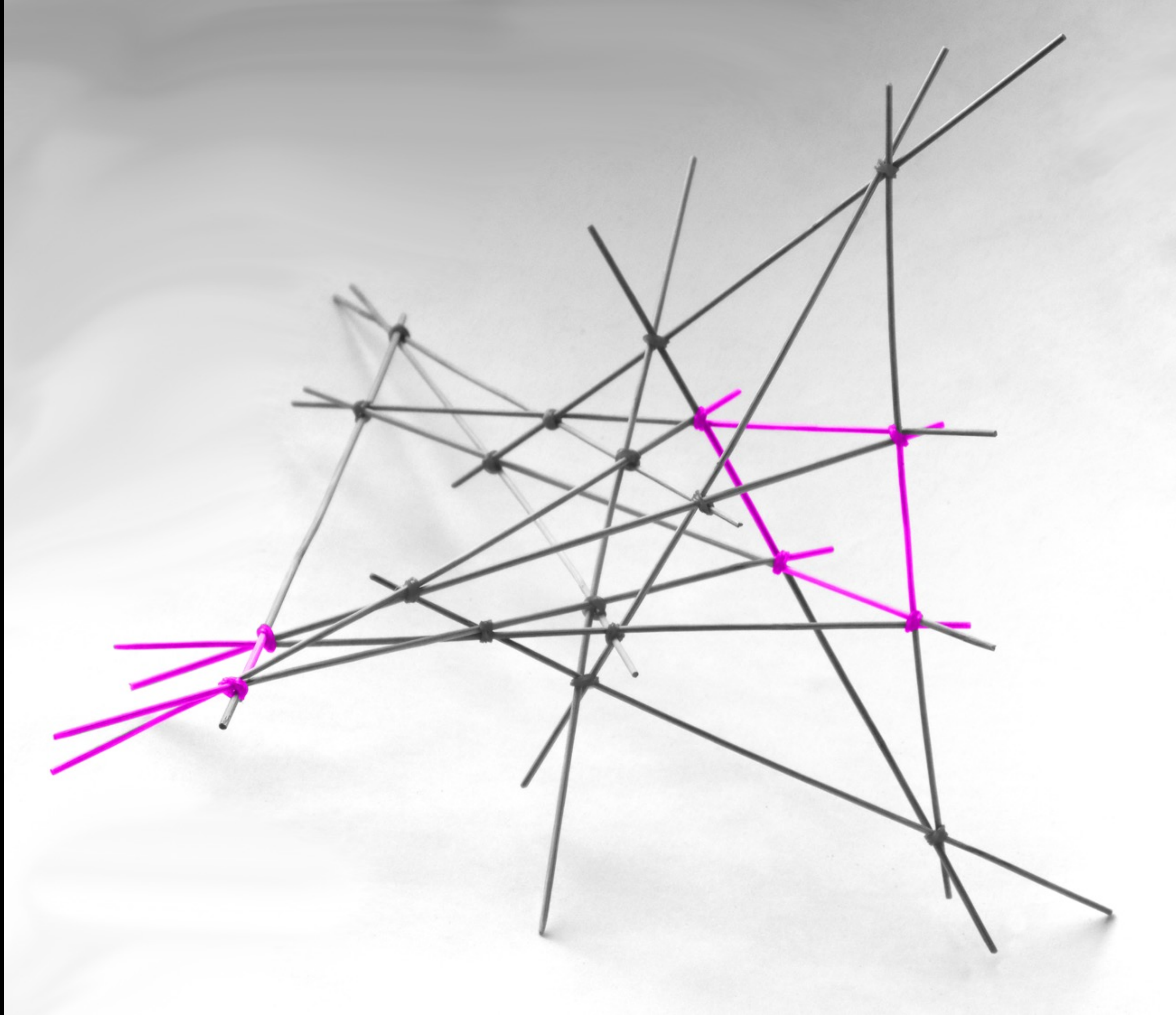


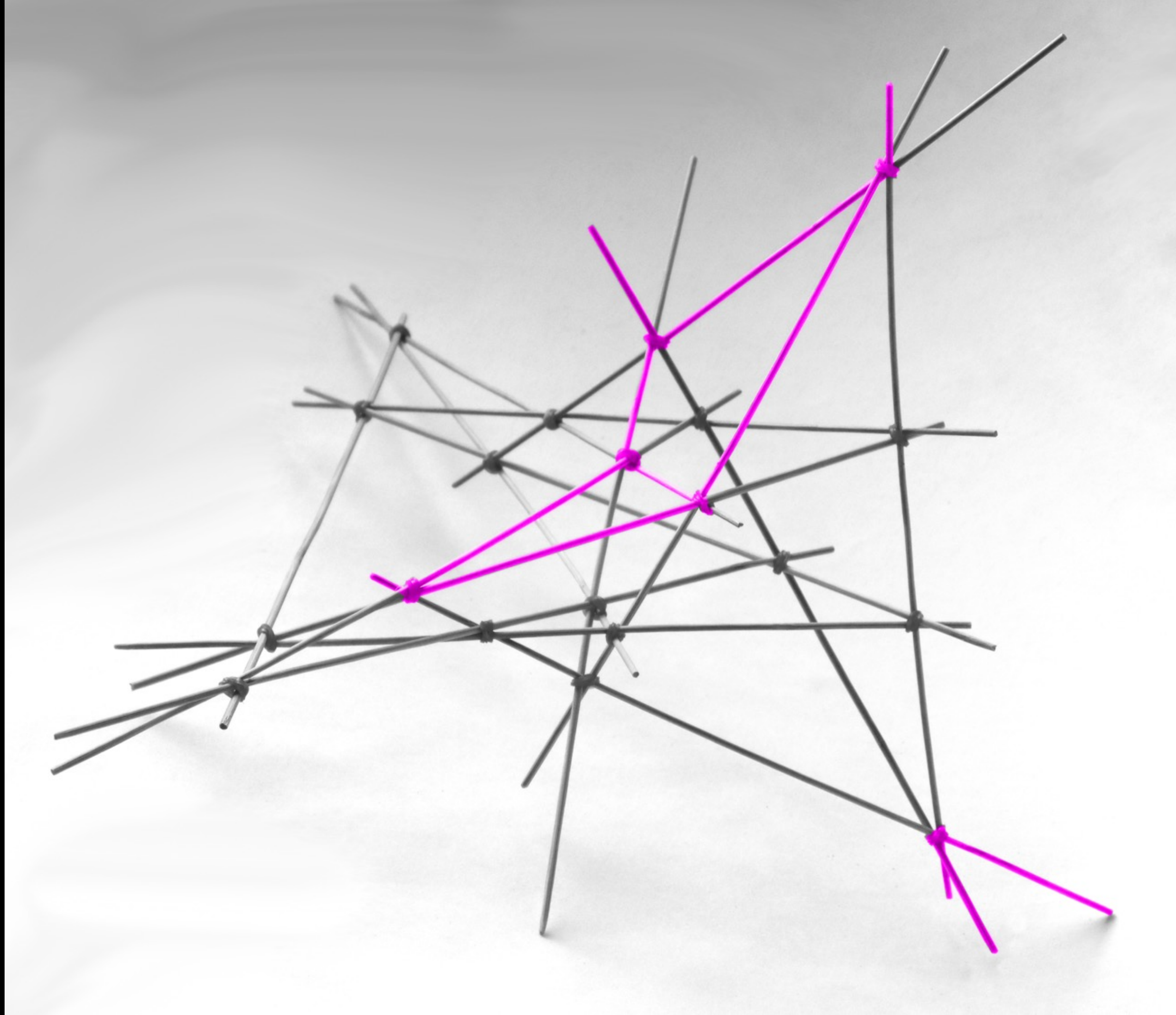


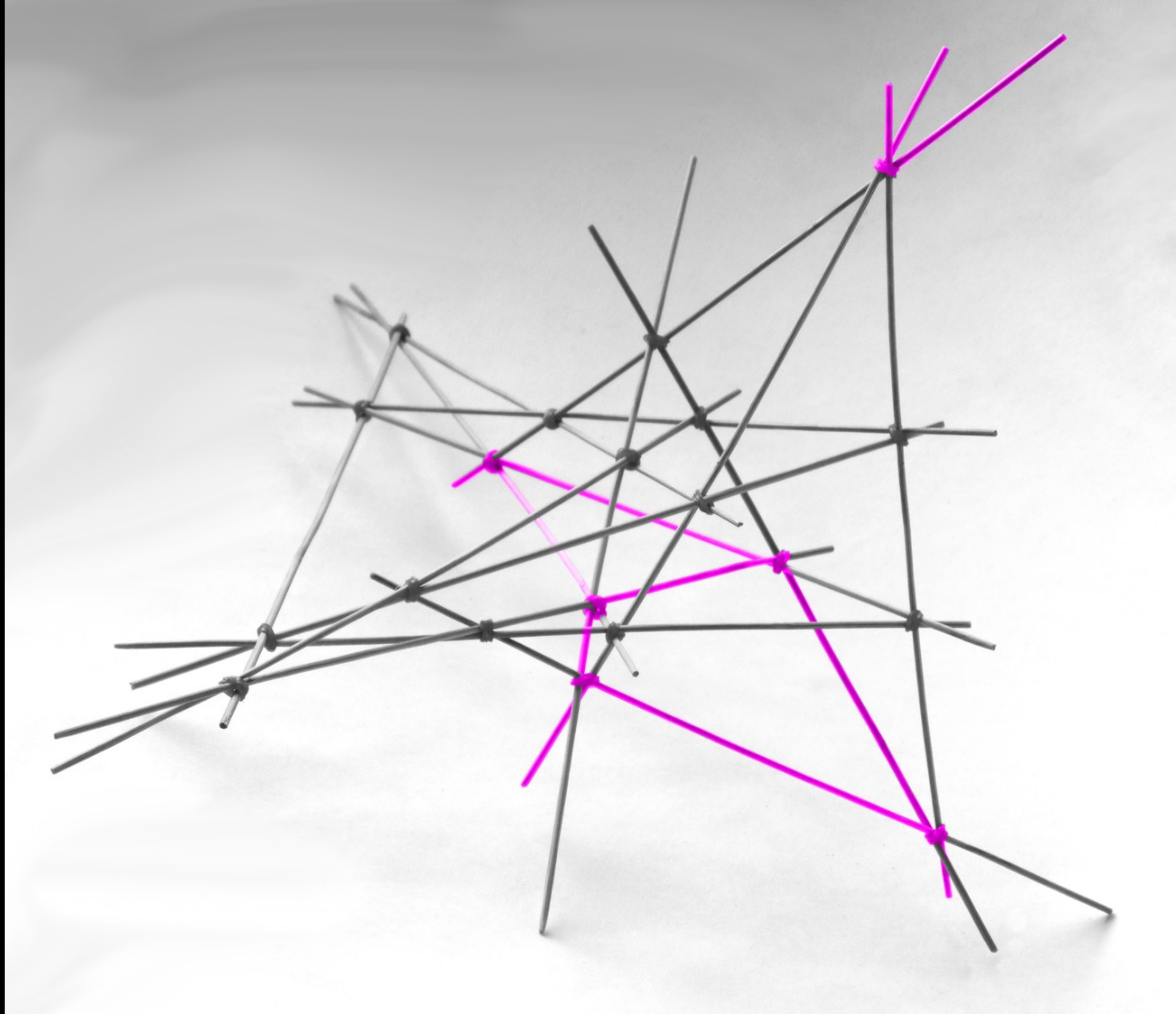


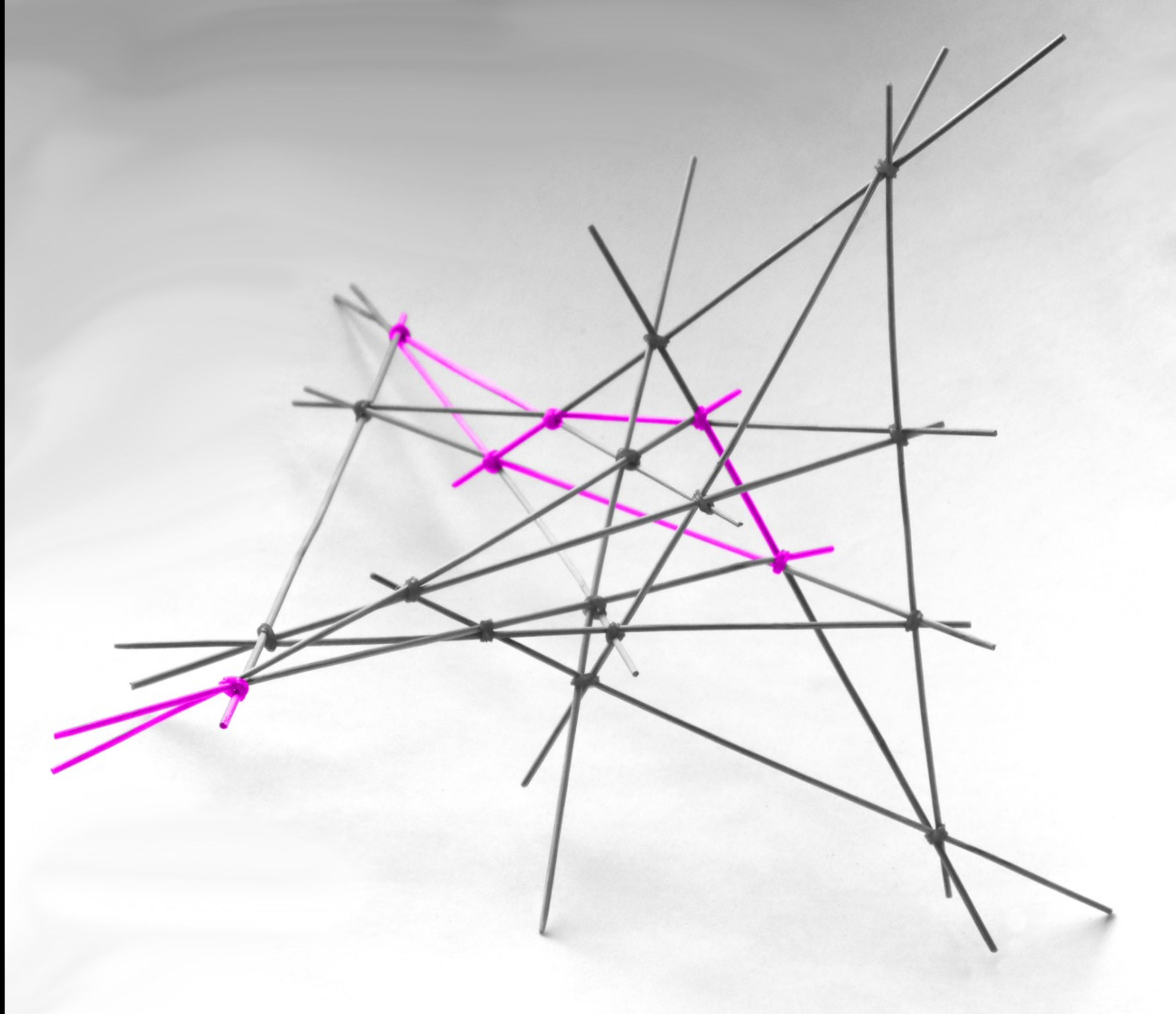


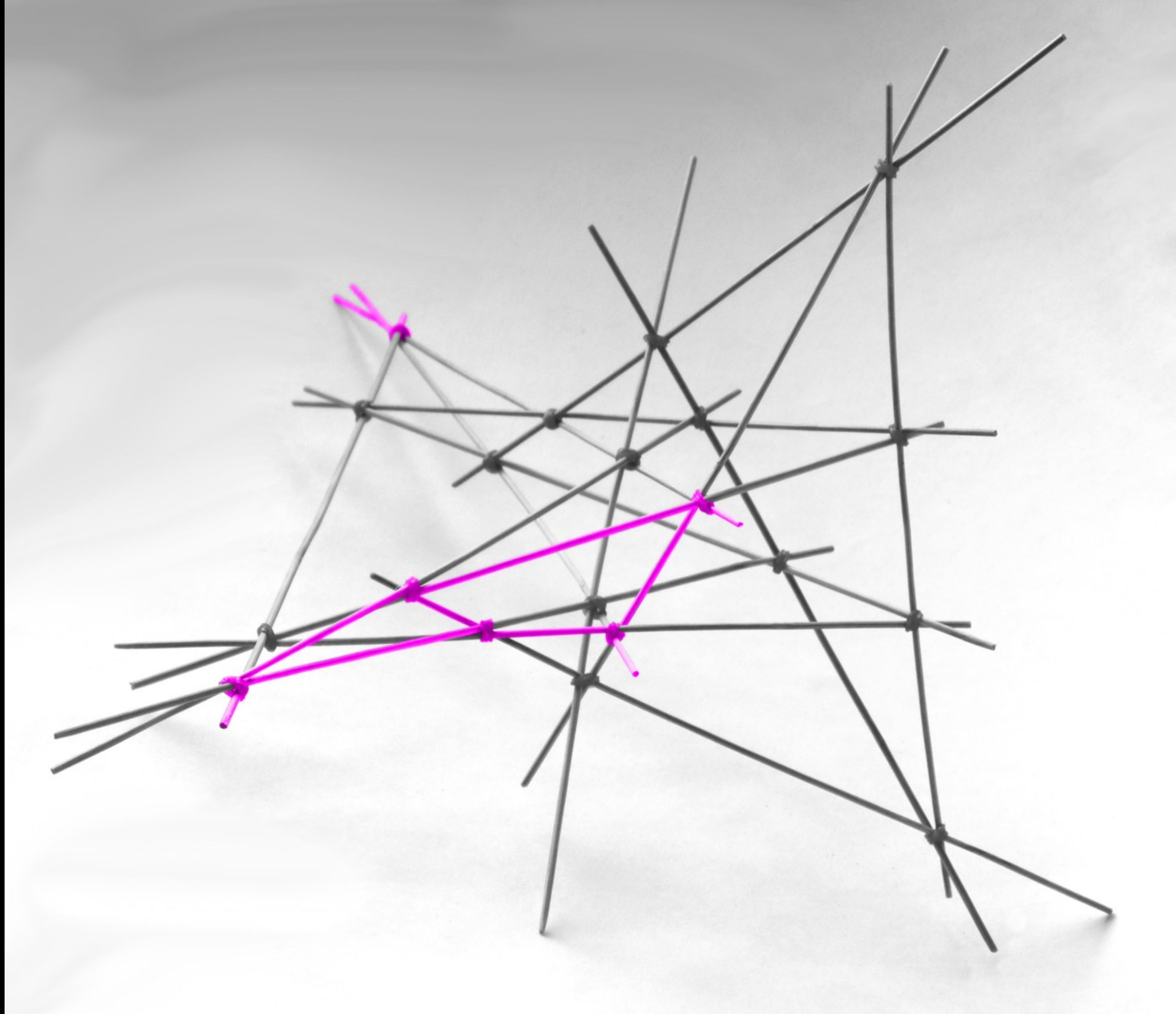


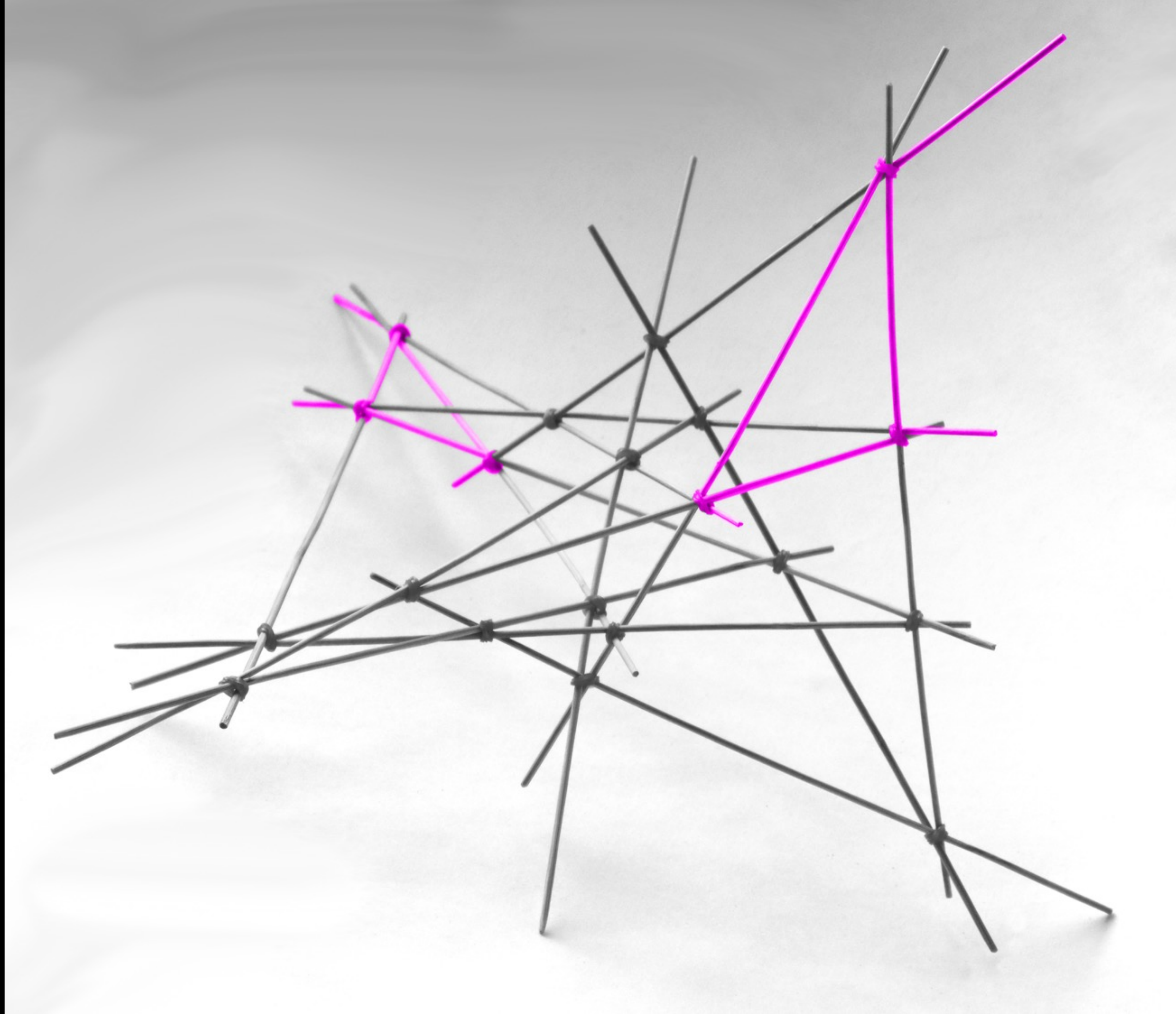


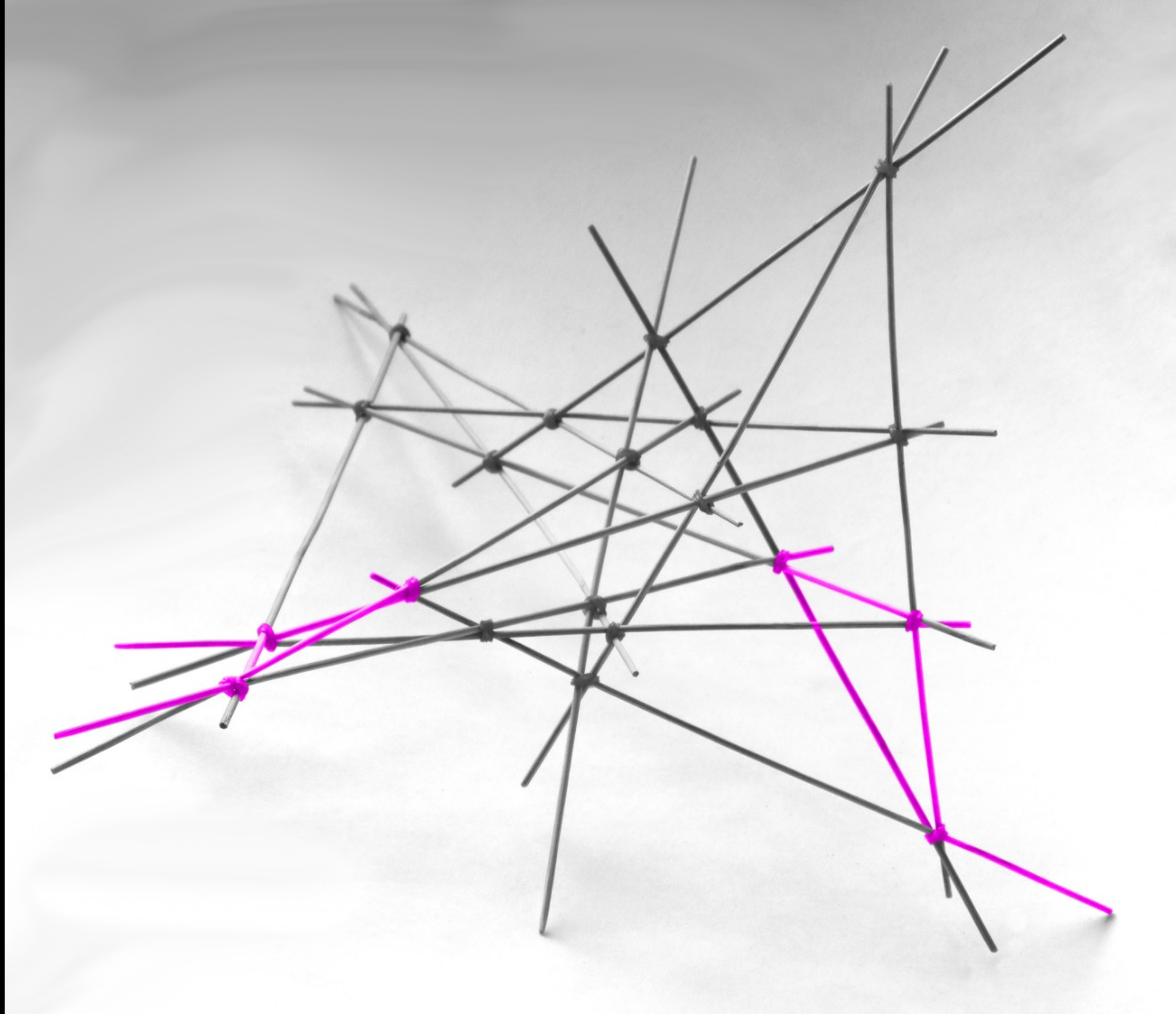




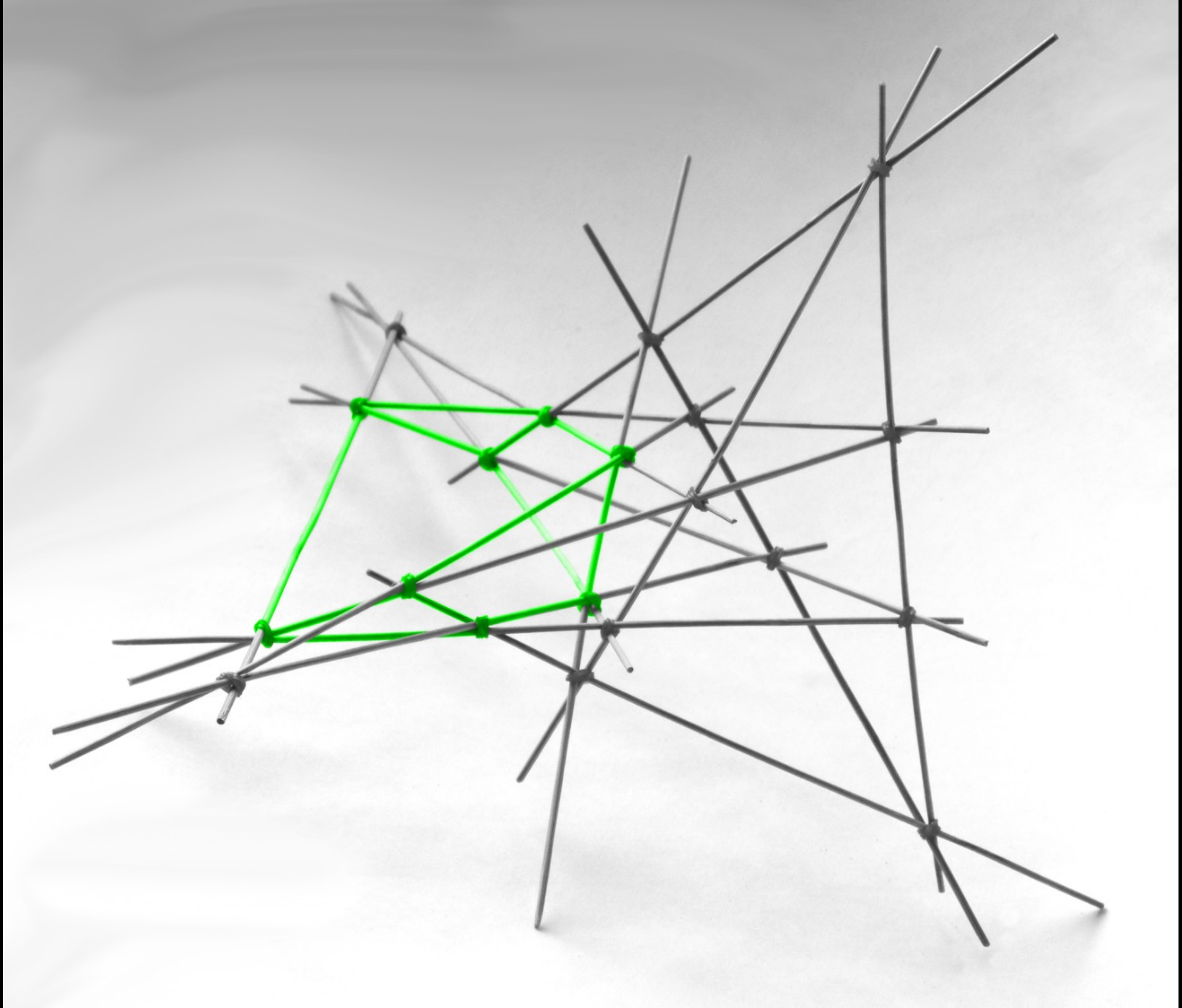


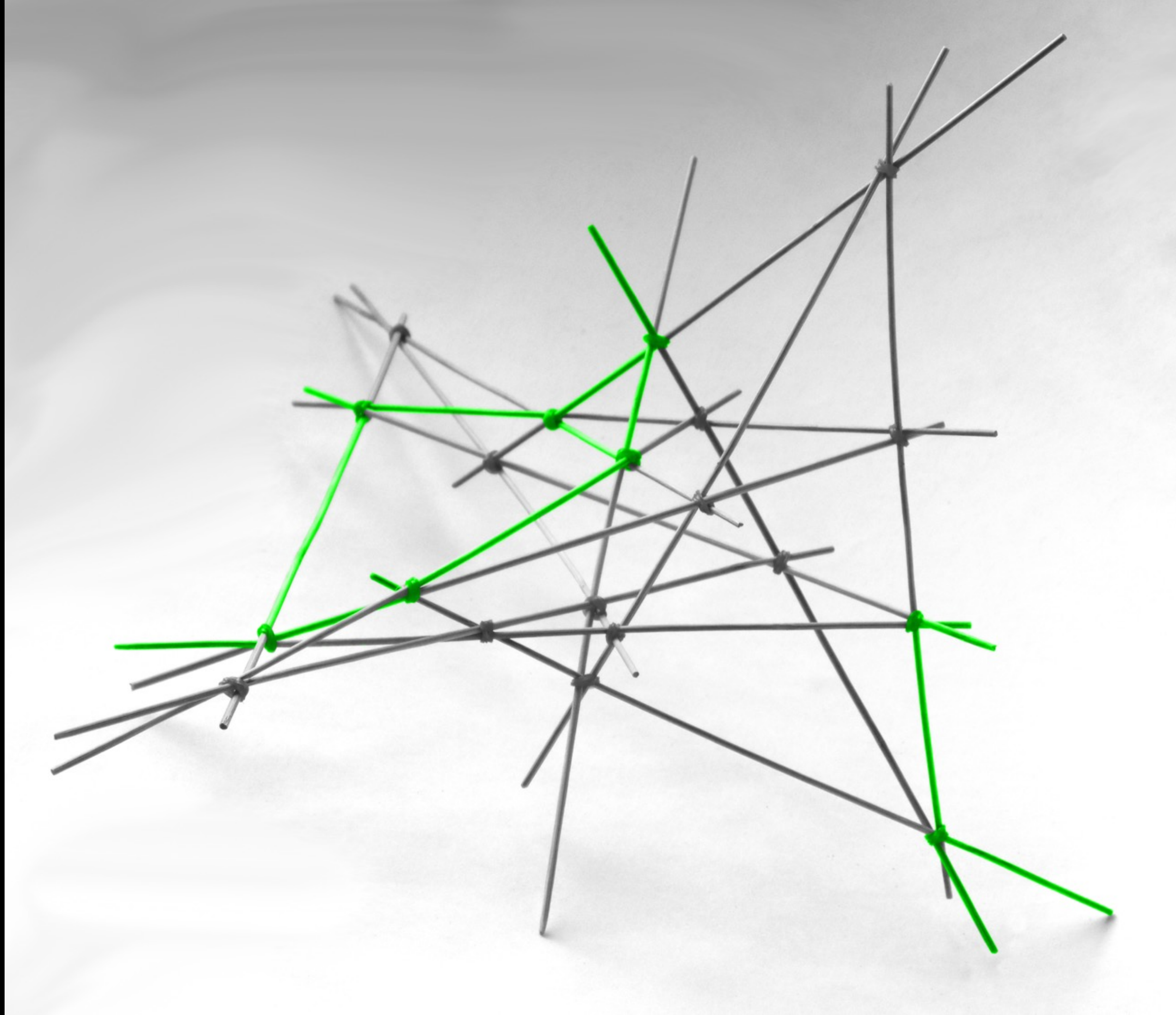


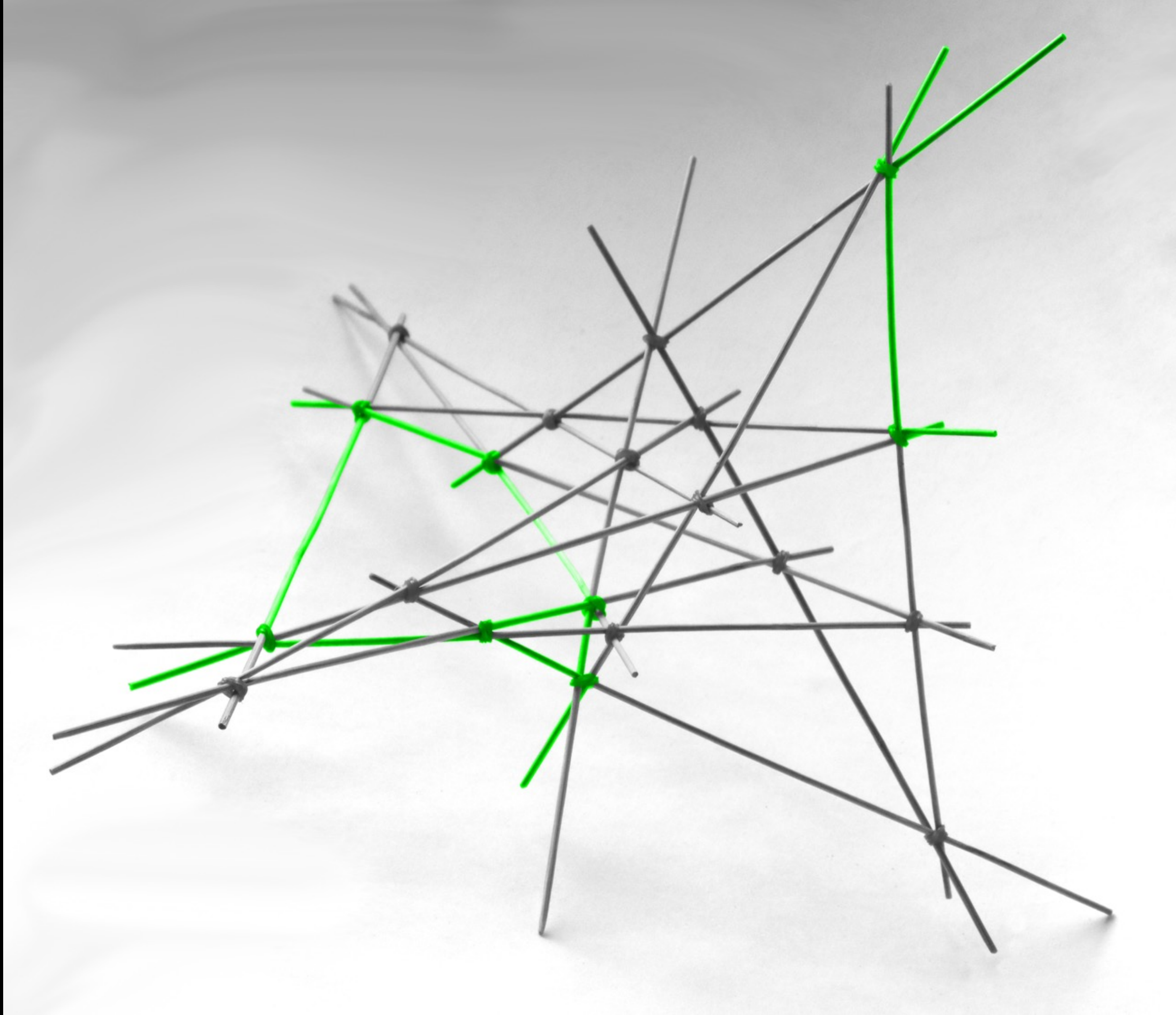


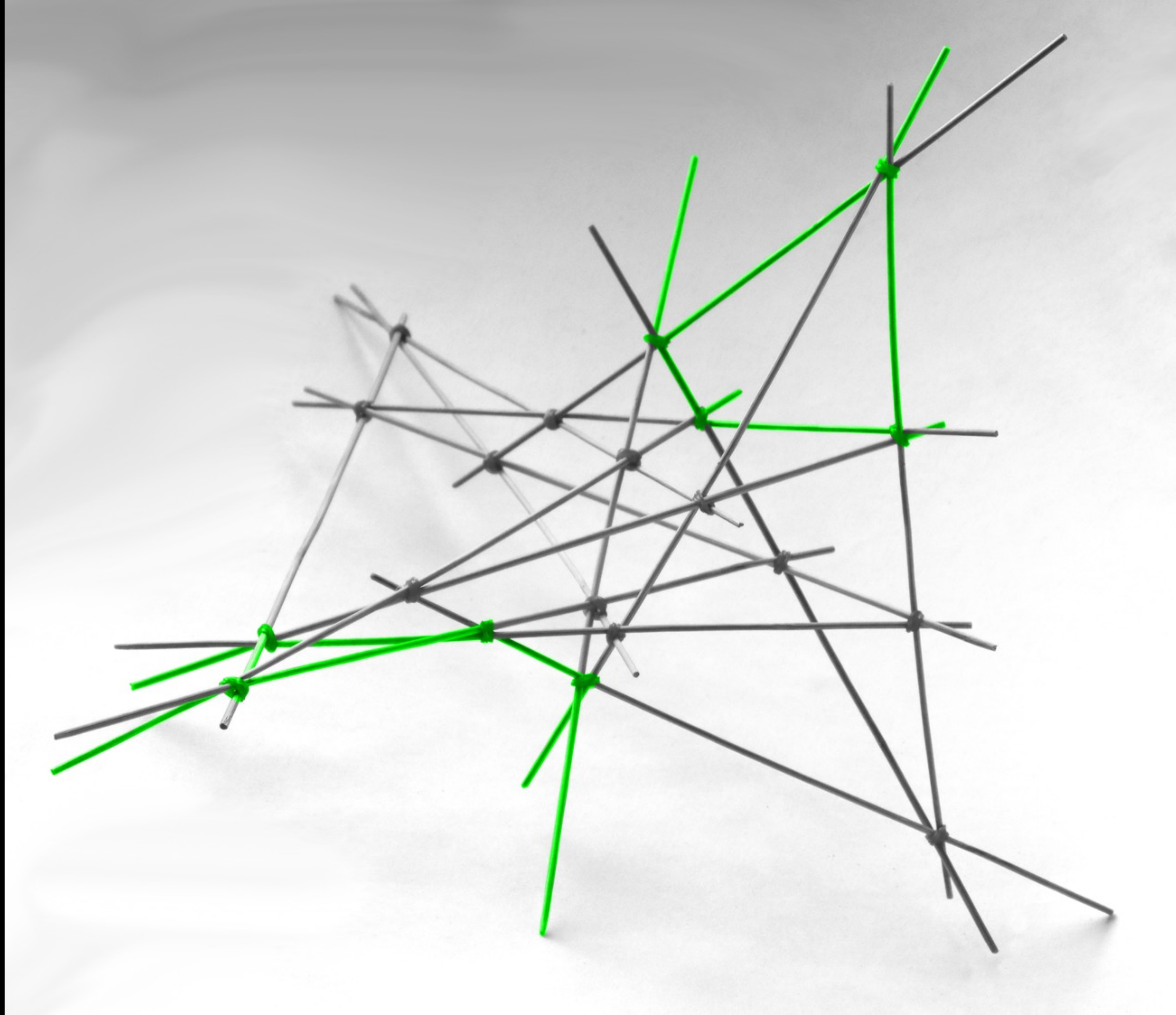


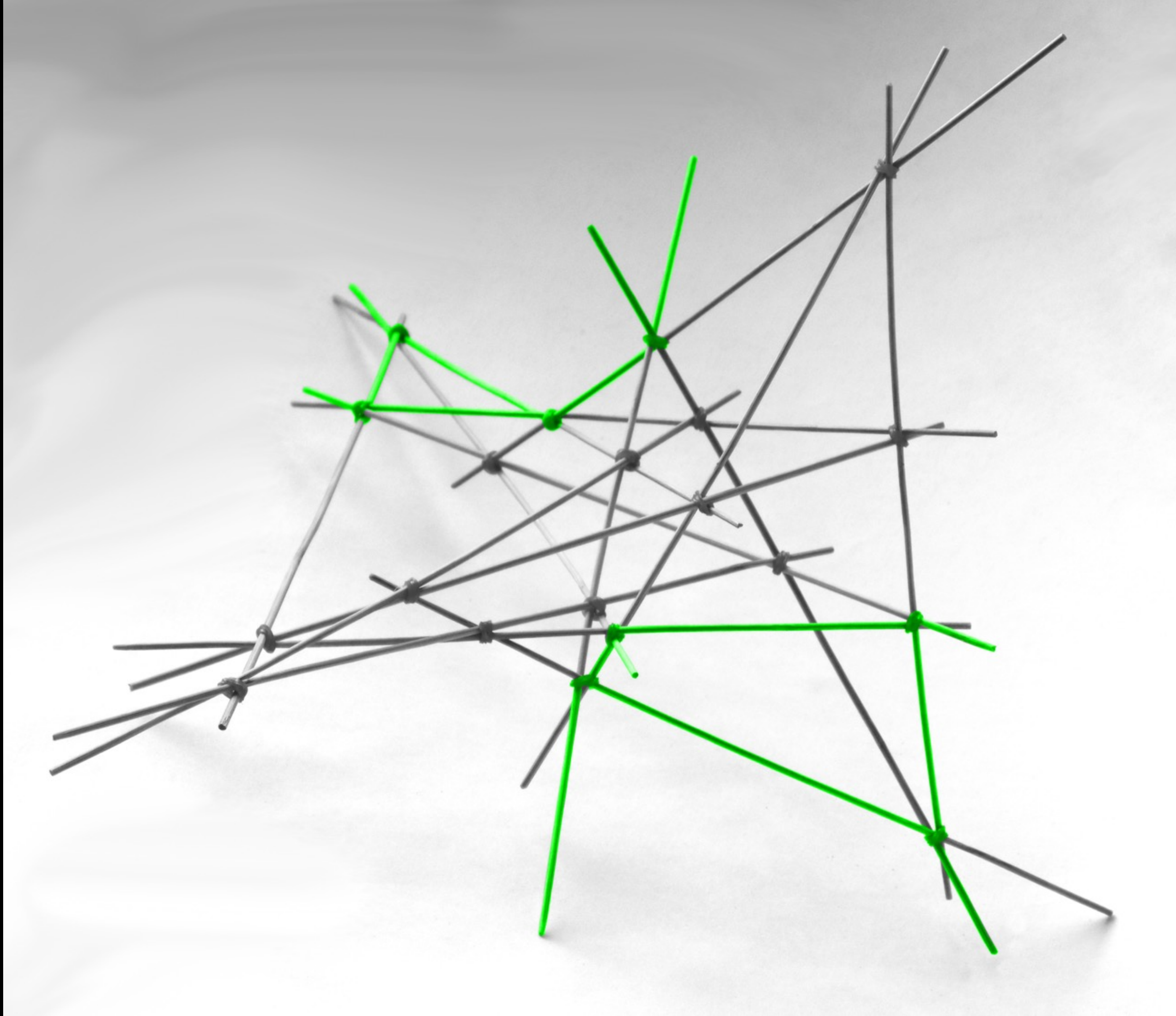
6 x PENTAGONAL 'WEDGE' CELLS

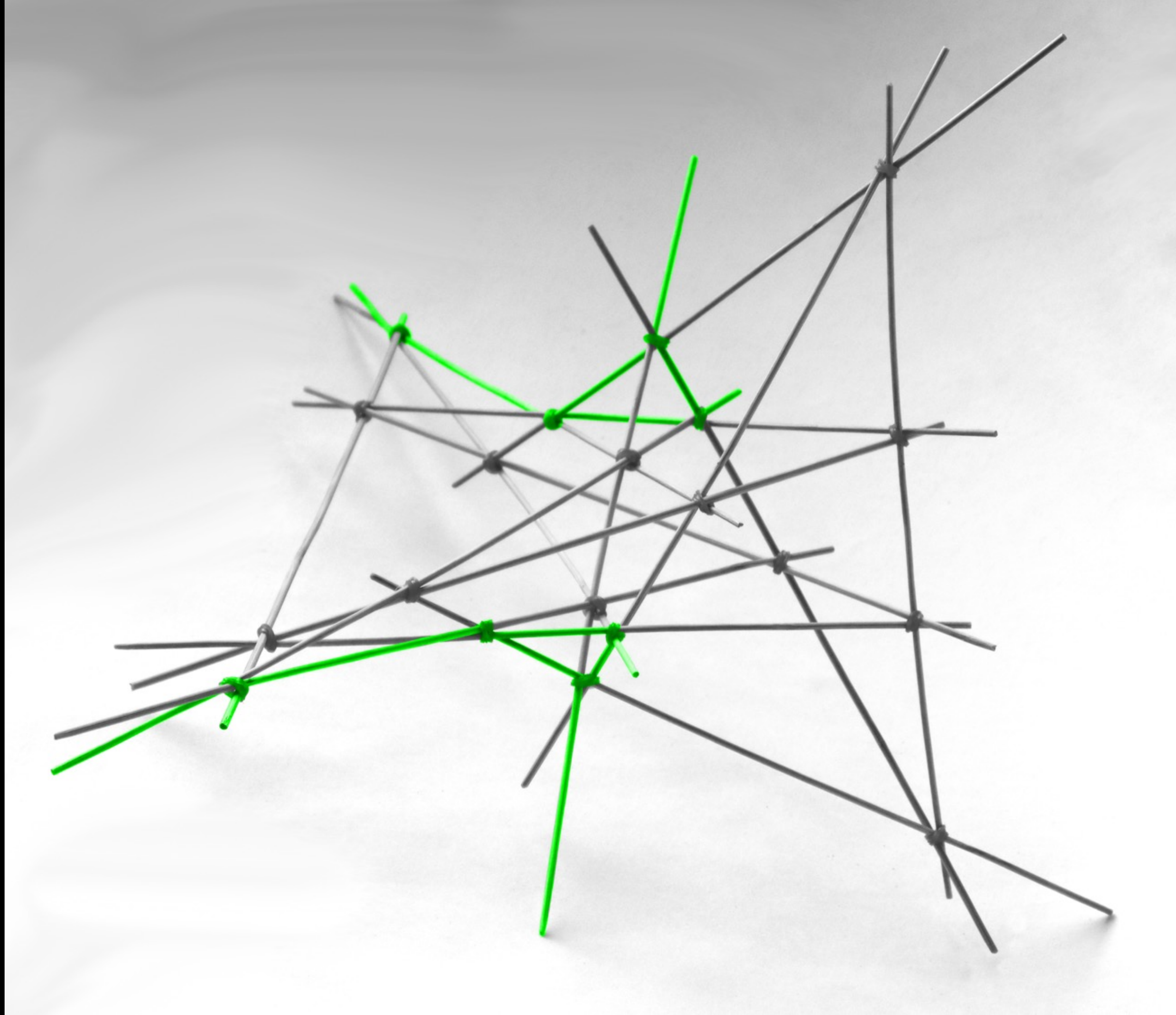




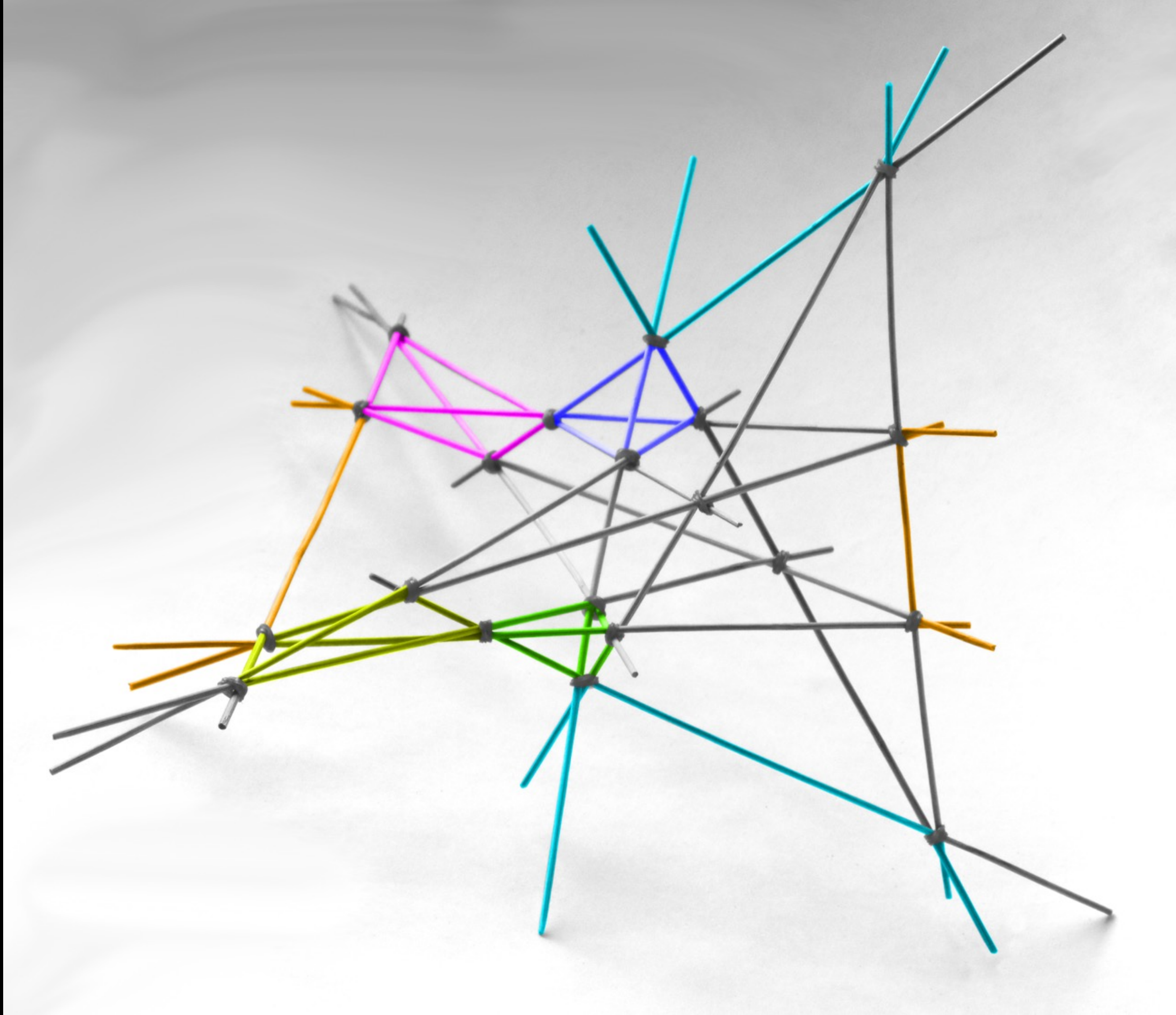








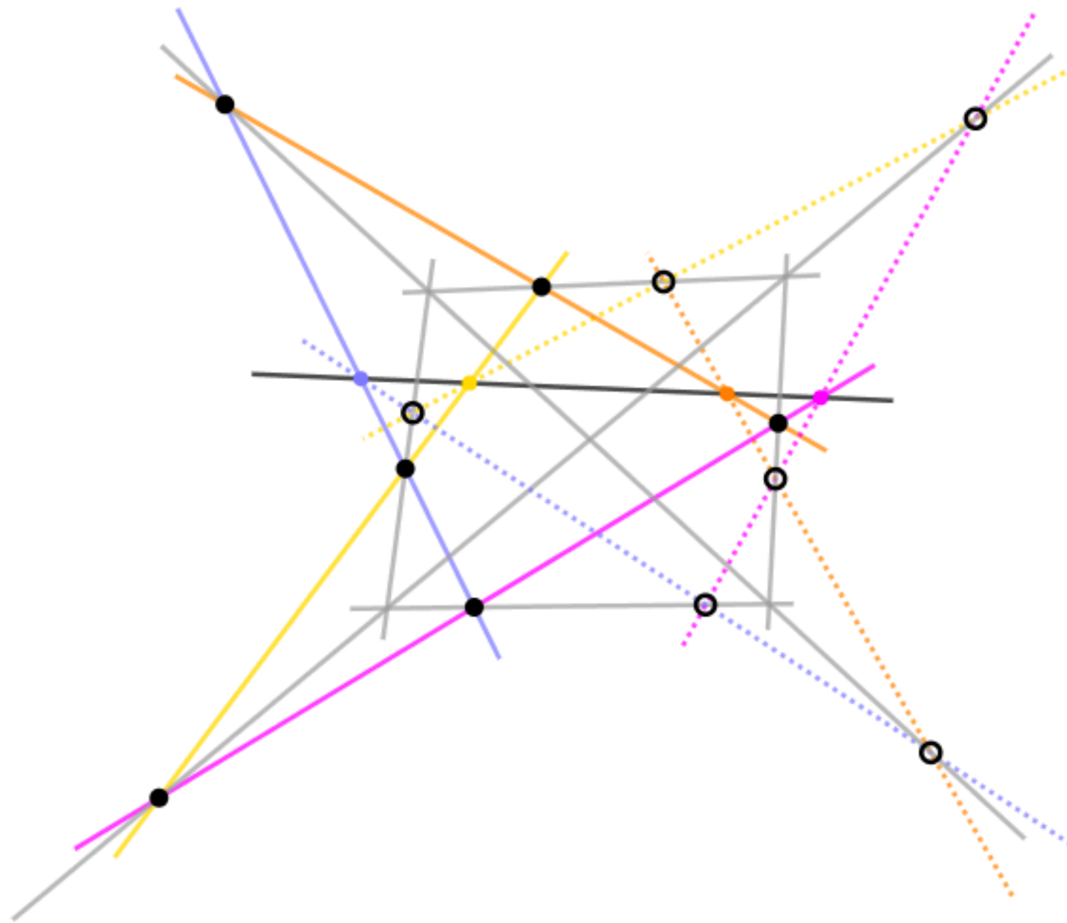
6 x TETRAHEDRAL CELLS



PERSPECTIVITY BETWEEN TWO QUADRILATERALS:

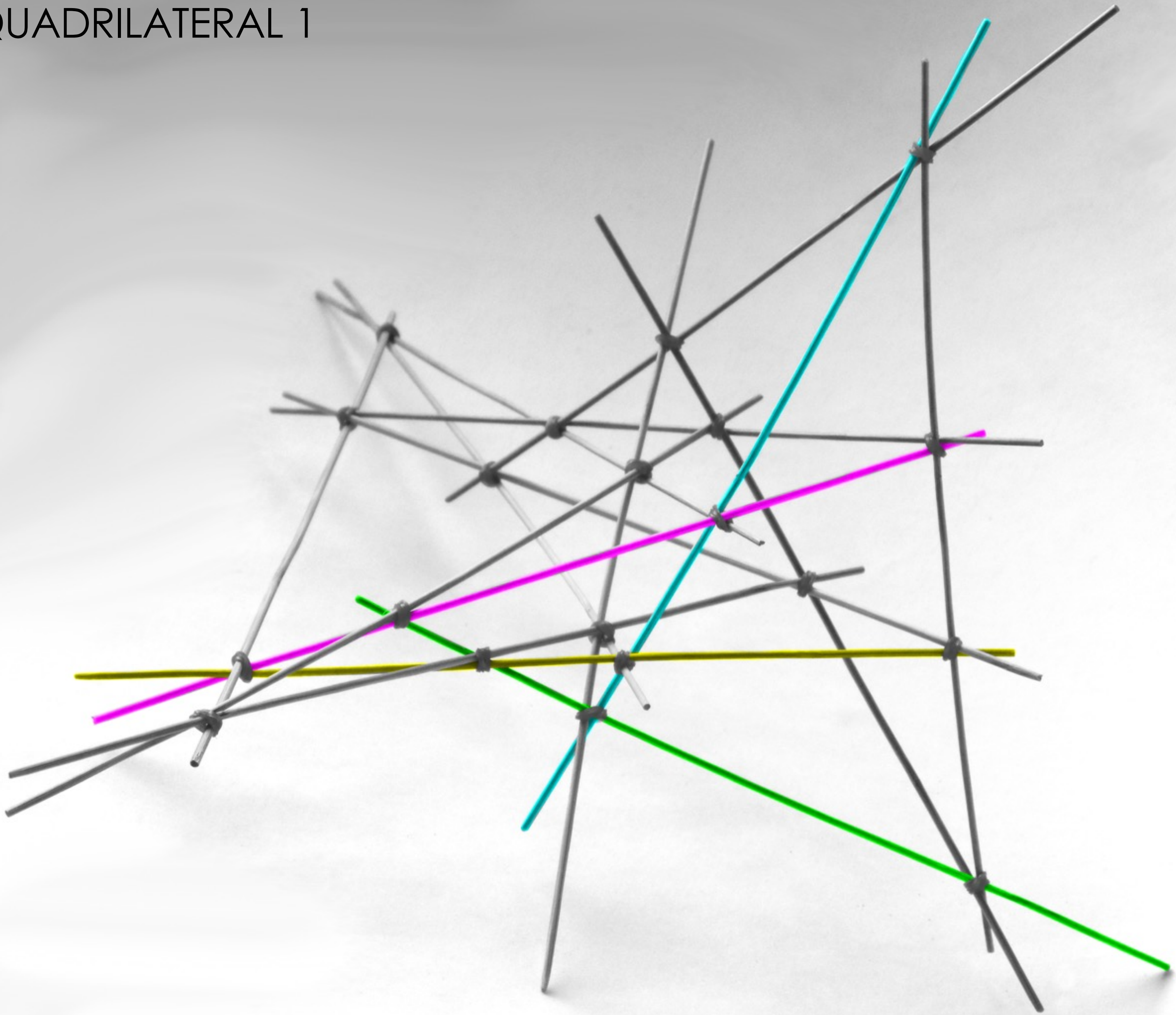
As a planar configuration, the complete hexahedron embodies a theorem about two perspective quadrilaterals – a natural consequence of Desargues' theorem. It states:

> Two quadrilaterals are in perspective from a line
>
> (i.e., the intersection points of the corresponding sides are collinear)
>
> if and only if
>
> the lines determined by their corresponding vertices form a quadrangle.

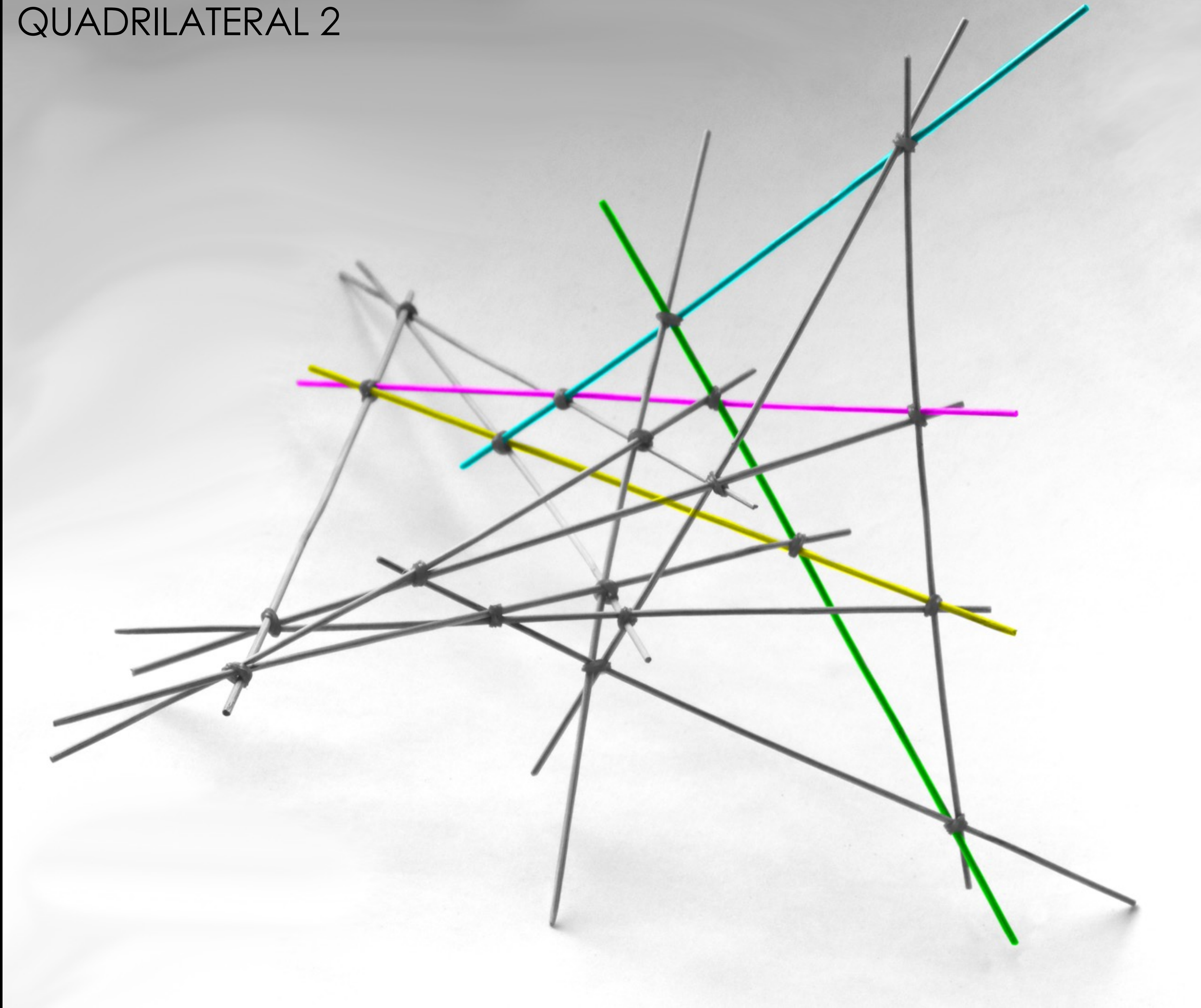


Just like with Desargues theorem, we can gain better insight by observing the situation in three-dimensional setting…

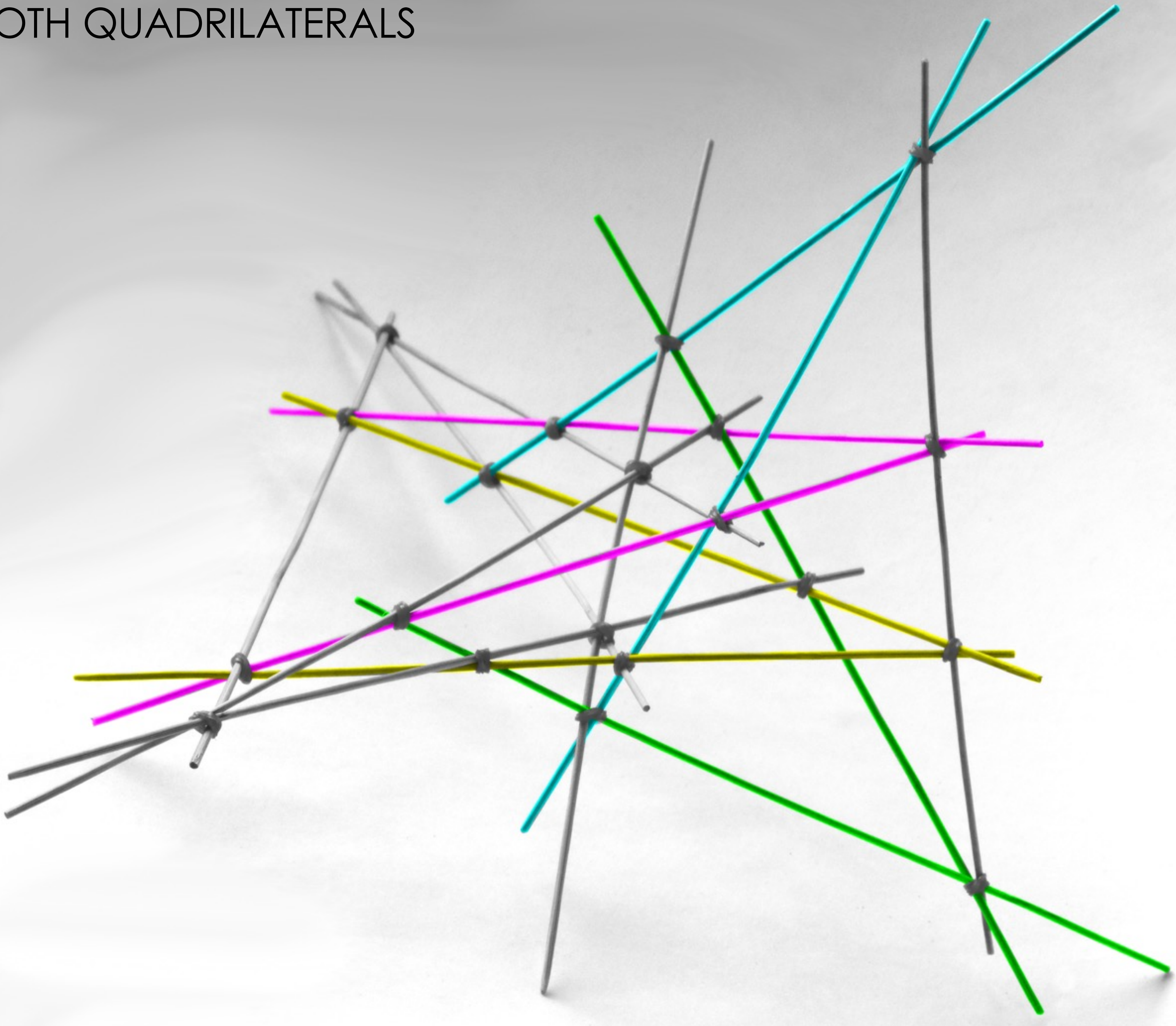
QUADRILATERAL 1

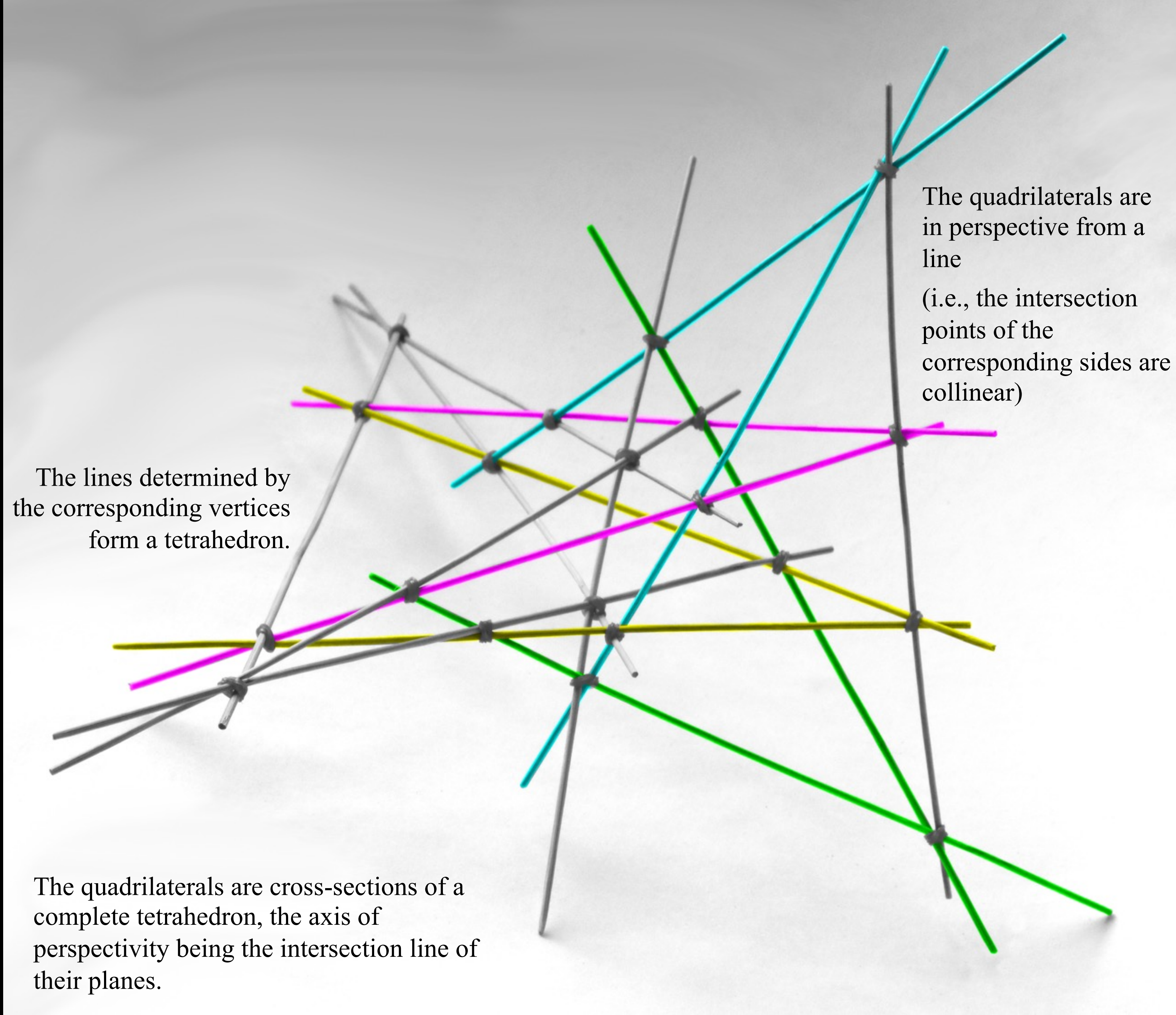


QUADRILATERAL 2



BOTH QUADRILATERALS




The quadrilaterals are in perspective from a line
(i.e., the intersection points of the corresponding sides are collinear)
The lines determined by the corresponding vertices form a tetrahedron.
The quadrilaterals are cross-sections of a complete tetrahedron, the axis of perspectivity being the intersection line of their planes.

PERSPECTIVITY BETWEEN TWO QUADRANGLES:

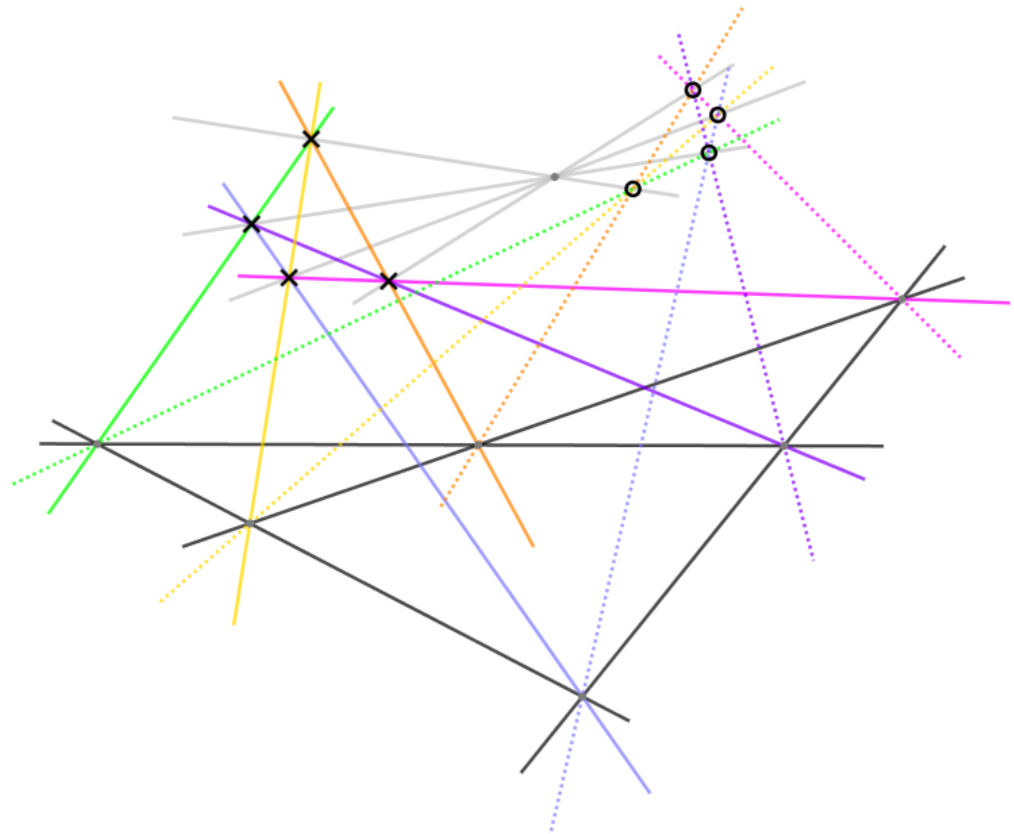
As mentioned previously, there exists a duality – a perfect correspondence between points and lines in the projective plane that extends also to any statements involving them:

A plane-dual of the quadrilateral theorem above can thus be formulated as:

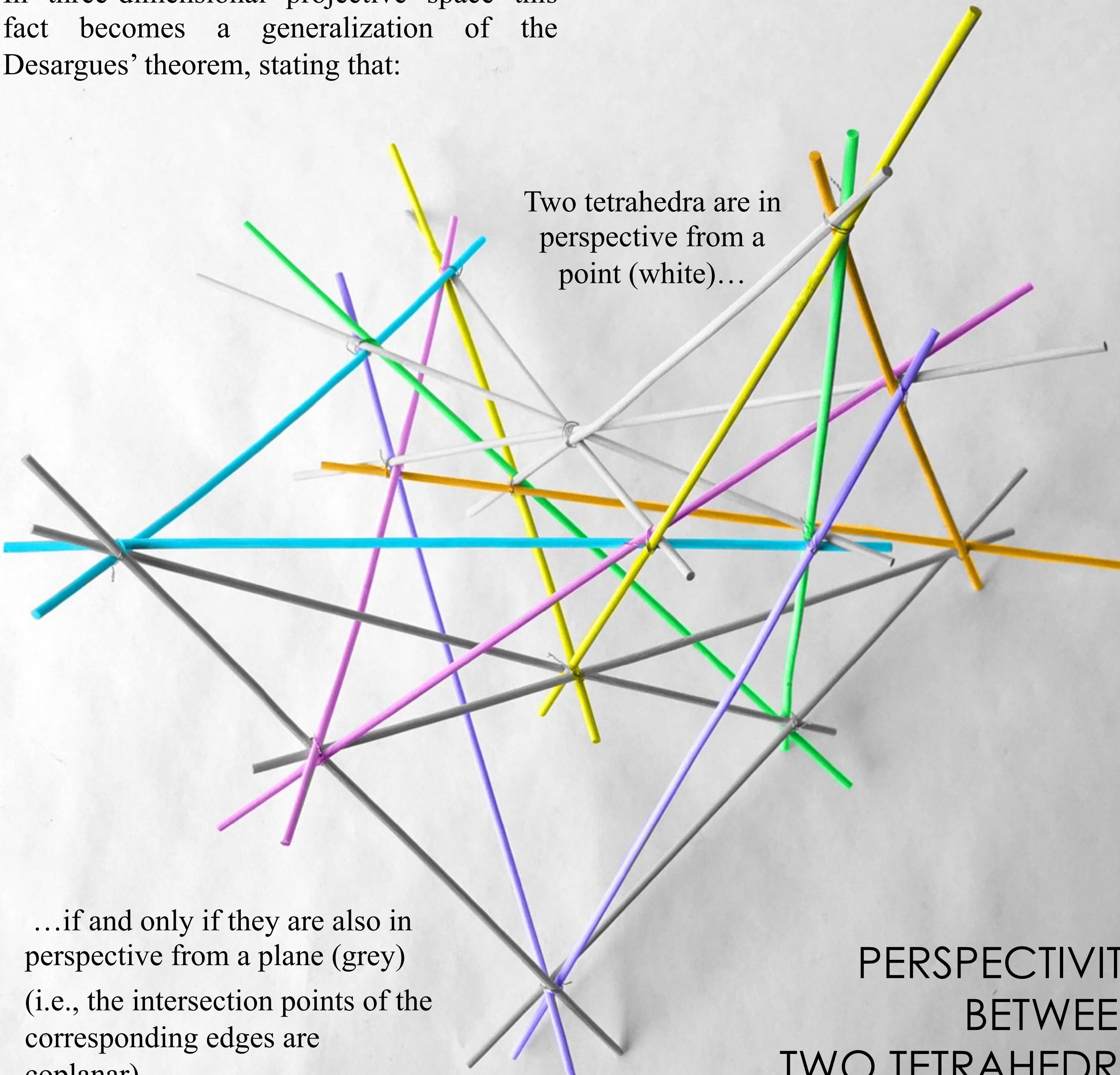
two quadrangles are in perspective from a point,

if and only if,

the meets of their corresponding edges are the vertices of a quadrilateral.



In three-dimensional projective space this fact becomes a generalization of the Desargues' theorem, stating that:



Two tetrahedra are in perspective from a point (white)…

…if and only if they are also in perspective from a plane (grey)

(i.e., the intersection points of the corresponding edges are coplanar).

PERSPECTIVITY BETWEEN TWO TETRAHEDRA

Stick model for
the 'complete hexachoron' configuration