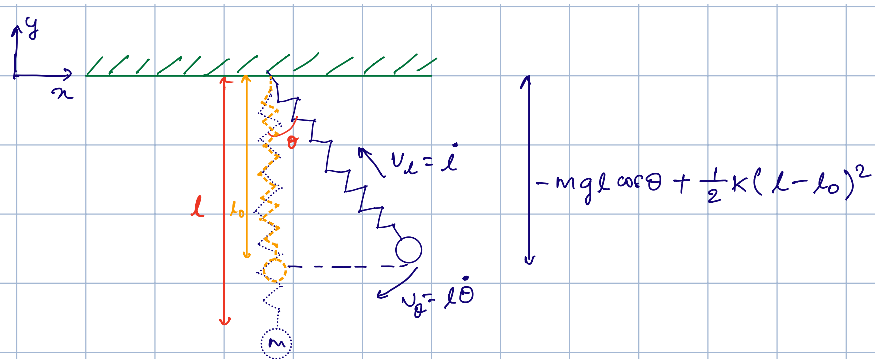
Problem set 4

1. Lagrangian mechanics: Consider a pendulum with an elastic supporting wire. Its rest length is l_0 and spring constant k . Use the length of the wire l and the swinging angle θ as generalized coordinates. The Lagrangian of such a system can be written as,

$$L = \underbrace{\frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)}_{T} - \underbrace{\left(-mgl\cos\theta + \frac{1}{2}k(l-l_0)^2\right)}_{V} \tag{1}$$

where the kinetic energy $T = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)$ and the potential energy $V = -mgl\cos\theta + \frac{1}{2}k(l-l_0)^2$. Convince yourself why the Lagrangian has this form.

Derive the equation of motions (Euler-Lagrange) for generalized co-ordinates l, θ in the limit of a small angle θ.



Generalized co-ordinates: $(q_1, q_2) = (l, \theta)$
$(\dot{q}_1, \dot{q}_2) = (\dot{l}, \dot{\theta})$

$\left\{ \text{2D phase space: } 2N = 2\times 2 = 4 \text{ G.C.} \right\}$

Use Euler-Lagrange equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

where, $q_i \in \{l, \theta\}$ and $\dot{q}_i \in \{\dot{l}, \dot{\theta}\}$ $\rightarrow$ 2 E.O.M.

Small angle approximation: $\cos\theta \approx 1$

2. Hamiltonian mechanics: Find the Hamiltonian of the system in (1) using Legendre transformation and specify the equation of motions.

1. Find generalized momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$ for $q_i = \{\dot{\ell}, \dot{\theta}\}$

2. Use Legendre transformation: $H = \sum_{i=1}^{N} \dot{q}_i p_i - L$

3. Use Hamilton's equation to specify the equation of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$\langle n|n\rangle$

3. Show that if the state $|n\rangle$ is normalized, i.e. $\langle n|n\rangle = 1$, then in order for the state $|n+1\rangle$ to also be normalized, the operator $\hat{a}^\dagger$ must have the property

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{2}$$

(*Hint:* $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$, $\hat{a}^\dagger |0\rangle = |1\rangle$, $\hat{a}|0\rangle = 0$.)

1°. Use the definition: $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$ and use the fact

$|n+1\rangle = \frac{(\hat{a}^+)^{n+1}}{\sqrt{(n+1)!}}|0\rangle$.

4. Show that the operator $\hat{a}^\dagger$ can be represented as

$$\hat{a}^\dagger = \sum_{m=0}^{\infty} \sqrt{m+1}\,|m+1\rangle\langle m|\,. \tag{3}$$

1°. Use the complete basis: $\hat{I} = \sum_n |n\rangle\langle n|$ and the fact that

$\hat{a}^+ = \hat{a}^+ \hat{I}$

5. Transmon qubit: A charge qubit provides an excellent anharmonicity to the energy levels, however the charge dispersion, i.e., the dependence of the energy on the gate charge, introduces a drastic charge noise. Thus, charge noise is the main source of decoherence in the charge qubit.

By adding an additional large shunt capacitance in parallel with the Josephson junction, we suppress the charge noise quite dramatically. This type of qubit is called transmon qubit. The Hamiltonian of the transmon qubit is of the same form as of the charge qubit,

$$H = 4E_{\mathrm{C}}(n - n_{\mathrm{g}})^2 - E_{\mathrm{J}} \cos\varphi. \tag{4}$$

However, for the transmon qubit the energy ratio is in the range $40 < \frac{E_{\mathrm{J}}}{E_{\mathrm{C}}} < 100$. The Josepshon coupling energy dominates the charging energy, thus suppressing the charge noise.

In this assignment, we will quantize the transmon qubit. We begin by setting the offset gate charge $n_{\mathrm{g}} = 0$, since it does not matter how we bias the transmon with the gate charge. Then follow these steps:

a Consider small angle approximation for cosine potential. Approximate (Taylor expansion) the cosine function to the fourth order term.

b Combine the quadratic terms together and express it in terms of harmonic oscillator equation (Eqn. (46) from the lecture material).

c Promote the classical variables n, φ to quantum operators and express them in terms of annihilation $\hat{b}$ and creation $\hat{b}^\dagger$ operators.

d Diagonalize the Hamiltonian by plugging the operators defined in step (c) to the Hamiltonian you have obtained from step (a).

e In the resulting fourth order term, invoke rotating wave approximation i.e., only keep the terms with equal number of $\hat{b}$ and $\hat{b}^\dagger$.

a. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - O(6)$

linear LC oscillator : $\frac{n^2}{2C'} + \frac{\Phi^2}{2L'}$

b, c. $H = \boxed{4E_c n^2 + E_J \frac{\theta^2}{2!}} - E_J \frac{\theta^4}{4!}$

$\hat{n} \propto (\hat{b}^\dagger - \hat{b})$

$\hat{\phi} \propto (\hat{b}^\dagger + \hat{b})$

d, e. $H = \hbar\omega_{ge} \hat{b}^\dagger \hat{b} - \frac{E_c}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}$

$\frac{\hbar\omega_{ge}}{2} \hat{\sigma}_z$

$\omega_{ge} = \sqrt{1/L'C'}$

$\phi^4 \propto (\hat{b}^\dagger + \hat{b})^4$

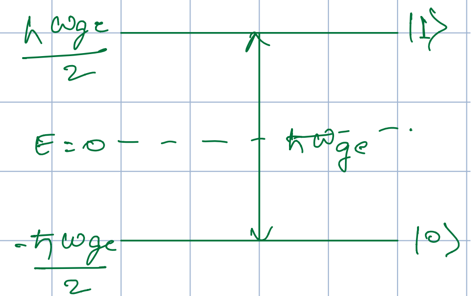
$\propto (\hat{b}^\dagger)^2 (\hat{b})^2$

5. write qubit Hamiltonian in eigenbasis of the qubit.

$$\hat{I}_Q = \sum_k |k\rangle\langle k|$$

$$\hat{H}_Q = \hbar\sum_{k=0}^{1}\omega_k|k\rangle\langle k| = \hbar\omega_0|0\rangle\langle 0| + \hbar\omega_1|1\rangle\langle 1|$$

$$= \frac{\hbar\omega_e}{2}\hat{\sigma}_z$$



$$= -\frac{\hbar\omega_{ge}}{2}|0\rangle\langle 0| + \frac{\hbar\omega_{ge}}{2}|1\rangle\langle 1|$$

$$= \frac{\hbar\omega_{ge}}{2}\Big(\underbrace{|1\rangle\langle 1| - |0\rangle\langle 0|}_{=\hat{\sigma}_z}\Big)$$

$$\hat{H}_Q = -\frac{\hbar\omega_{ge}}{2}\hat{\sigma}_z = \frac{\hbar\omega_{ge}}{2}\hat{\sigma}_z \quad [\pm 1 \text{ eigenvalue}]$$

6. Jaynes-Cummings Hamiltonian: In this exercise, we study a coupled interaction between the resonator and the transmon qubit.

a. Form the total Hamiltonian of the transmon–resonator system, which contains the uncoupled diagonalized Hamiltonians of the resonator and the qubit and the interaction Hamiltonian. The interaction between the resonator and the qubit is modeled by the interation Hamiltonian,

$$\hat{H}_{\text{int}} = -C_{\text{G}}\hat{V}_{\text{Q}} \otimes \hat{V}_{\text{R}}. \quad (5)$$

$\hat{V}_Q = \frac{-2e\hat{n}}{C_\Sigma}(\hat{b}-\hat{b}^\dagger)$

$\hat{V}_R = \hat{q}/C_r = V^0_{rms}(\hat{a}-\hat{a}^\dagger)$

where, the voltage operators take the form $\hat{V}_{\text{Q}} = \frac{-2e}{C_\Sigma}\hat{n}$ and $\hat{V}_{\text{R}} = \frac{\hat{q}}{C_{\text{r}}}$ for the qubit and the resonator respectively. Here, $C_\Sigma = C_{\text{G}} + C_{\text{J}} + C_{\text{S}}$ is the sum capacitance.

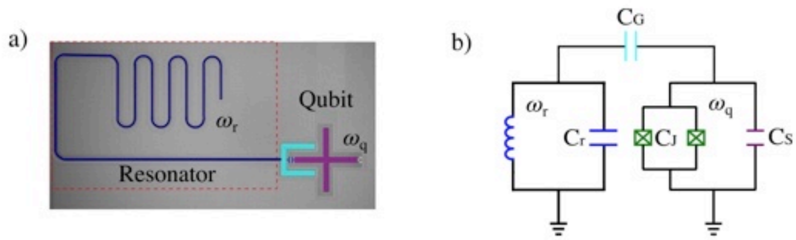


Figure 1: Coupled resonator and transmon qubit. a) Experimental realization, and b) Lumped-circuit model.

b. Invoke rotating wave approximation, and obtain the Jaynes-Cummings Hamiltonian

$$\hat{H}_{\text{JC}} = \hbar\omega_{\text{r}}\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_{\text{q}}}{2}\hat{\sigma}_z + \hbar g(\hat{\sigma}_- \otimes \hat{a}^\dagger + \hat{\sigma}_+ \otimes \hat{a}), \quad (6)$$

where $\hat{\sigma}_- = |0\rangle\langle 1|$ and $\hat{\sigma}_+ = \boxed{|1\rangle\langle 0|}$.

$\Downarrow$ $\omega_q = \omega_{01} = \omega_{ge}$

c. Solve the Jaynes-Cummings Hamiltonian for the case i) $\Delta = \omega_{\text{r}} - \omega_{\text{q}} = 0$, ii) $\Delta = \omega_{\text{r}} - \omega_{\text{q}} = 0.5\,\text{GHz}$, and iii) $\Delta = \omega_{\text{r}} - \omega_{\text{q}} = 1\,\text{GHz}$. Initially, there are $n+1$ photons in the resonator and the qubit is in the ground state. Plot the probability density of the transmon being in the excited states for n = 1, 10 and 100 photons.

1°. $\hat{H}_T = \hat{H}_R + \hat{H}_Q + \hat{H}_{int}$

$\hat{H}_R = \hbar\omega_r(\hat{a}^\dagger\hat{a} + \cancel{\tfrac{1}{2}})$

$|1\rangle\langle 0| = \sigma_+$

$\sigma_- = |0\rangle\langle 1|$

2° $\hat{H}_{int} = -\hbar g(\hat{b}-\hat{b}^\dagger)(\hat{a}-\hat{a}^\dagger)$

3°. $\hat{H}_T = \hbar\omega_r\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z - \hbar g(\hat{b}-\hat{b}^\dagger)(\hat{a}-\hat{a}^\dagger)$

4° RWA, $\hat{H}_T = \underbrace{\hbar\omega_r\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)}_{H_{int}}$ JC

1°. Schrödinger picture: $|\psi(t)\rangle$ evolve (determined by H) & $\langle \hat{O} \rangle$ constant

2° Heisenberg picture: $|\psi\rangle$ const. & $\langle O(t) \rangle$ evolve (determined by H)

3° Dirac (interaction) picture: $|\psi^I\rangle$ evolution determined by H^I_{int}
and $\langle \hat{O}(t) \rangle$ determined by H_0

Interaction Hamiltonian in interaction picture

$$\hat{H}_I^{in}(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{H}_I\,e^{-\frac{i}{\hbar}H_0 t} \qquad \left[\hat{a} \propto e^{-i\omega_r t};\ \hat{\sigma}_- \propto e^{-i\omega_{ge} t}\right]$$

$$|\psi^{in}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}\,|\psi(t)\rangle$$: State ket evolution in interaction picture.

$$\hat{H}_I^{in}(t) = \hbar g\left(\hat{\sigma}_-\hat{a}^{+}e^{i\Delta t} + \hat{\sigma}_+\hat{a}\,e^{-i\Delta t}\right) \quad ; \ \Delta = \omega_r - \omega_{ge}$$

Time evolution

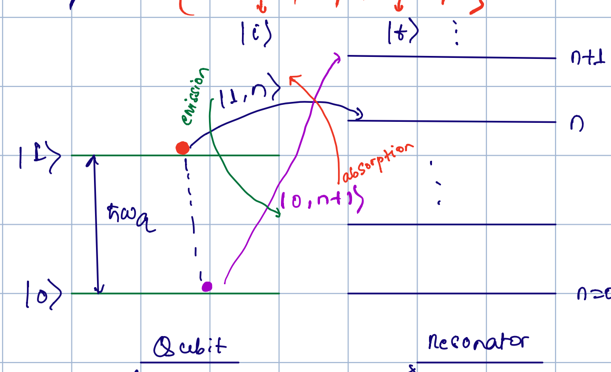
$$i\hbar\frac{\partial}{\partial t}|\psi^{in}(t)\rangle = \hat{H}_I^{in}(t)\,|\psi^{in}(t)\rangle$$

Complete Basis set

$$\{\underset{|i\rangle}{|0,n+1\rangle},\ \underset{|f\rangle}{|1,n\rangle}\}$$

where, we take the Ansatz

$$|\psi^{in}\rangle = c_i|i\rangle + c_f|f\rangle$$



In general,

$$|k\rangle \otimes |n\rangle = |k,n\rangle$$ Tensor product

$$i\hbar\left(\dot{c}_i(t)|i\rangle + \dot{c}_f(t)|f\rangle\right) =$$
$$\hbar g\left(\hat{\sigma}_-\hat{a}^{+}e^{i\Delta t} + \hat{\sigma}_+\hat{a}\,e^{-i\Delta t}\right)\left(c_i|i\rangle + c_f|f\rangle\right)$$

$\vdots$

$$\Rightarrow \begin{aligned} \dot{c}_i(t)|i\rangle &= -ig\sqrt{n+1}\, e^{i\Delta t} c_f |i\rangle \\ \dot{c}_f(t)|f\rangle &= -ig\sqrt{n+1}\, e^{-i\Delta t} c_i |f\rangle \end{aligned} \Rightarrow \begin{bmatrix} \dot{c}_i \\ \dot{c}_f \end{bmatrix} = \begin{bmatrix} 0 & -ig\sqrt{n+1}e^{i\Delta t} \\ -ig\sqrt{n+1}e^{-i\Delta t} & 0 \end{bmatrix} \begin{bmatrix} c_i \\ c_f \end{bmatrix}$$

You can choose to solve this numerically or analytically. I would suggest to solve it analytically to get a feel for it.

(i) For case $\Delta = 0$;

$$\begin{cases} \dot{c}_i = -ig\sqrt{n+1}\, c_f \\ \dot{c}_f = -ig\sqrt{n+1}\, c_i \cdot \left| \frac{d}{dt} \right. \end{cases}$$

~~take~~ d/dt of $\dot{c}_f$ and plug it in $\dot{c}_i$

then solve 2^{nd} order ode.

(ii) For case $\Delta \neq 0$;

$$\begin{cases} \dot{c}_i = -ig\sqrt{n+1}\, e^{i\Delta t} c_f & \rightarrow (a) \\ \dot{c}_f = -ig\sqrt{n+1}\, e^{-i\Delta t} c_i \end{cases}$$

Assume the solution of the form: $c_i = e^{i\Delta t} \xi(t)$ then take the derivative $\frac{d}{dt} c_i = \frac{d}{dt}\left(e^{i\Delta t}\xi(t)\right)$! product rule.

plug $\dot{c}_i$ in (a) and solve.

In the end plot: $P_f(t) = |c_f|^2$ for $\Delta = 0$ & $\Delta \neq 0$ and $n = 1, 10, \& 100$.

Observe the nuances in the results and write it down.

I have given a lot here. So, try to understand the steps thoroughly.