MEC-E8003 Beam, plate and shell models, week 13/2021

1. Derive the components of the elastic isotropic Kirchhoff plate constitutive equation for bending. Consider the Cartesian (x,y,n) coordinate system and use the definitions $\vec{M}=\vec{\vec{B}}:\vec{\kappa}$ and $\vec{\kappa}=-\nabla_0\nabla_0 w$. Cartesian coordinate system representation of the elasticity tensor is

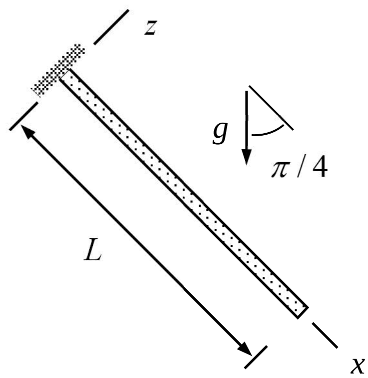
$$\vec{\vec{B}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}\text{ in which }[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}.$$

Answer $M_{xx}=-D(\dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2})$, $M_{yy}=-D(\nu\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2})$, $M_{xy}=-D(1-\nu)\dfrac{\partial^2 w}{\partial x\partial y}$.

2. Show that the transverse displacement of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2\nabla_0^2 w=b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix}\dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n\\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x\\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y\end{Bmatrix}=0,\quad \begin{Bmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{Bmatrix}=\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\dfrac{\partial\theta}{\partial x}\\ -\dfrac{\partial\phi}{\partial y}\\ \dfrac{\partial\theta}{\partial y}-\dfrac{\partial\phi}{\partial x}\end{Bmatrix},\quad \begin{Bmatrix}Q_x\\ Q_y\end{Bmatrix}=Gt\begin{Bmatrix}\dfrac{\partial w}{\partial x}+\theta\\ \dfrac{\partial w}{\partial y}-\phi\end{Bmatrix}.$$

3. Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx}=N(x)$, $Q_x=Q(x)$, $M_{xx}=M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.

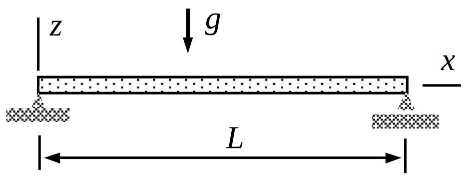


Answer

$$\frac{dN}{dx}+\frac{\rho g t}{\sqrt{2}}=0,\ \frac{dQ}{dx}-\frac{\rho g t}{\sqrt{2}}=0,\ \frac{dM}{dx}-Q=0\ \text{ in } (0,L),\qquad N=0,\ M=0,\ Q=0\ \text{ at } x=L.$$

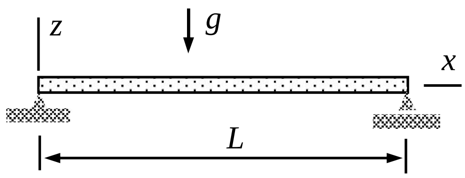
$$N=\frac{tE}{1-\nu^2}\frac{du}{dx},\ Q=Gt(\frac{dw}{dx}+\theta),\ M=D\frac{d\theta}{dx}\ \text{ in } (0,L),\qquad u=0,\ w=0,\ \theta=0\ \text{ at } x=0.$$

4. Consider a plate strip of the figure loaded by its own weight. Determine deflection w and rotation θ of the plate according to the Kirchhoff model. Thickness, and length of the plate are t and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Assume that stress resultants, displacements, and rotations depend on x only (consider a plate of width H).



Answer $w = -\frac{\rho g t L^4}{24D}[\frac{x}{L} - 2(\frac{x}{L})^3 + (\frac{x}{L})^4]$, $\theta = \frac{\rho g t L^3}{24D}[1 - 6(\frac{x}{L})^2 + 4(\frac{x}{L})^3]$

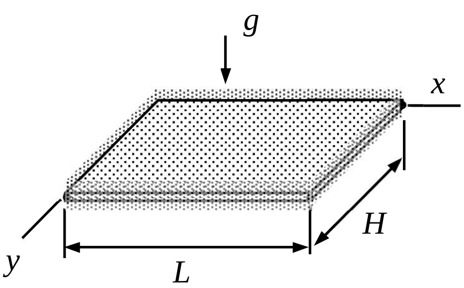
5. Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = \mathrm{a}_0(1 - x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \begin{Bmatrix} \frac{\partial^2 \delta w}{\partial x^2} \\ \frac{\partial^2 \delta w}{\partial y^2} \\ 2\frac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Answer $w = -\frac{\rho g t L^4}{24D}(1 - \frac{x}{L})\frac{x}{L}$

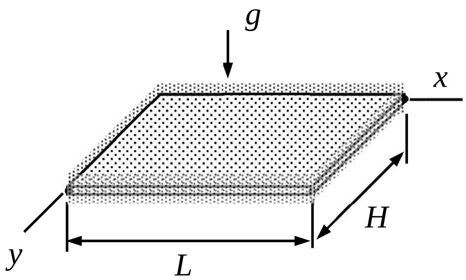
6. A simply supported rectangular plate of size $L \times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Determine the displacement at the center point with $w = \mathrm{a}_0(xy/LH)(1 - x/L)(1 - y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0$ $\forall \delta \mathrm{a}_0 \in \mathbb{R}$. Assume that the solution does not explicitly depend on ν. Virtual work expression



$$\delta W=-\int_{\Omega}\begin{Bmatrix}\frac{\partial^2\delta w}{\partial x^2}\\ \frac{\partial^2\delta w}{\partial y^2}\\ 2\frac{\partial^2\delta w}{\partial x\partial y}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}dA+\int_{\Omega}\delta w b_n dA.$$

Answer $w=\frac{5}{8}\frac{\rho g t}{D}\frac{H^4L^4}{3H^4+5H^2L^2+3L^4}\frac{x}{L}(1-\frac{x}{L})\frac{y}{H}(1-\frac{y}{H})$

7. A simply supported rectangular plate of size $L\times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Find an approximation to the transverse displacement by using $w=\mathrm{a}_0\sin(\pi x/L)\sin(\pi y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W=0\ \ \forall\delta\mathrm{a}_0\in\mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W=-\int_{\Omega}\begin{Bmatrix}\frac{\partial^2\delta w}{\partial x^2}\\ \frac{\partial^2\delta w}{\partial y^2}\\ 2\frac{\partial^2\delta w}{\partial x\partial y}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}dA+\int_{\Omega}\delta w b_n dA.$$

Answer $w=\frac{16}{\pi^6}\frac{\rho g t}{D}\frac{H^4L^4}{(H^2+L^2)^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})$

8. Find the representation $\nabla_0\cdot(\nabla_0\cdot\vec{\vec{M}})+b_n=0$ of the Kirchhoff plate bending equation in terms of components M_{rr}, $M_{r\phi}$ and $M_{\phi\phi}$ of the polar coordinate system. Assume rotation symmetry so that the moment components depend on r only.

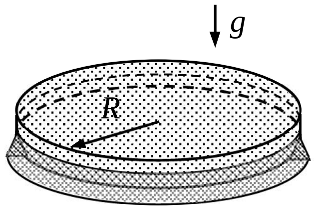
Answer $\frac{1}{r}\frac{d}{dr}[r(\frac{dM_{rr}}{dr}-\frac{1}{r}M_{\phi\phi})]+b_n=0$

9. Derive the components of the elastic isotropic Kirchhoff plate constitutive equation for bending in polar (r,ϕ,n) coordinate system. Use definitions $\vec{\vec{M}}=\vec{\vec{\vec{\vec{B}}}}:\vec{\vec{\kappa}}$, $\vec{\vec{\kappa}}=-\nabla_0\nabla_0 w$ and assume rotation symmetry $\partial w/\partial\phi=0$. The polar coordinate system representation of the bending elasticity tensor of plate model is

$$\vec{\vec{B}} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ in which } D = \frac{Et^3}{12(1-\nu^2)}.$$

Answer $M_{rr} = -D(\frac{d^2w}{dr^2} + \nu \frac{1}{r}\frac{dw}{dr})$, $M_{\phi\phi} = -D(\nu \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr})$, $M_{r\phi} = M_{\phi r} = 0$.

10. A simply supported circular plate of radius R is loaded by its own weight as shown in the figure. Determine the displacement of the plate at the midpoint by using the Kirchhoff plate model in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that the solution depends on the radial coordinate only.



Answer $w(0) = -\frac{3}{16}\frac{\rho g R^4}{Et^2}(5+\nu)(1-\nu)$

Derive the components of the elastic isotropic Kirchhoff plate constitutive equations for bending. Consider the Cartesian (x,y,n) coordinate system and use the definitions $\vec{M}=\ddot{\vec{B}}:\vec{\kappa}$ and $\vec{\kappa}=-\nabla_0\nabla_0 w$. Cartesian coordinate system representation of the elasticity tensor is

$$\ddot{\vec{B}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix} \text{ in which } D=\frac{Et^3}{12(1-\nu^2)}.$$

Solution

The (mid-plane) gradient operator of the Cartesian (x,y,n) coordinate system coordinate system gives

$$\vec{\kappa}=-\nabla_0\nabla_0 w=-(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})w=-\vec{i}\vec{i}\frac{\partial^2 w}{\partial x^2}-(\vec{i}\vec{j}+\vec{j}\vec{i})\frac{\partial^2 w}{\partial x\partial y}-\vec{j}\vec{j}\frac{\partial^2 w}{\partial y^2} \ .$$

By using the constitutive equation and elasticity tensor

$$\vec{M}=\ddot{\vec{B}}:\vec{\kappa}=-\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}.$$

Therefore, the Cartesian coordinate system components of the bending moment constitutive equation are

$$M_{xx}=-D(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}),\ M_{yy}=-D(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}),\ M_{xy}=M_{yx}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}. \quad \leftarrow$$

Show that the vertical displacement $w(x,y)$ of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2\nabla_0^2 w = b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial\theta}{\partial x} \\ -\dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\theta}{\partial y}-\dfrac{\partial\phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

Solution

Kirchhoff constraints are first used to write the constitutive equations in terms of the transverse displacement

$$\theta=-\frac{\partial w}{\partial x} \text{ and } \phi=\frac{\partial w}{\partial y} \quad\Rightarrow\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=-D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix}=-D\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2} \\ (1-\nu)\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix}.$$

In the Kirchhoff model, shear forces Q_x and Q_y are in the role of constraint forces to be solved from the moment equations. Eliminating the shear forces from the equilibrium equation in the transverse direction by using the moment equations gives

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0 \quad\Rightarrow\quad \frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+b_n=0\,.$$

The biharmonic equation for the transverse displacement follows from the equilibrium equation above, when the constitutive equations for the moments are substituted there

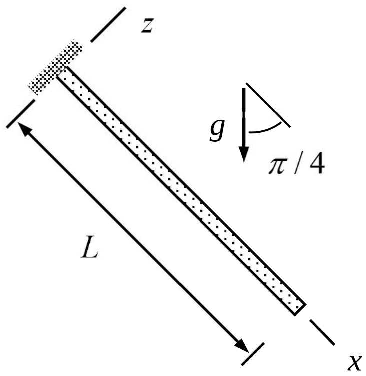
$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+b_n=0 \quad\Leftrightarrow$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2})+2(1-\nu)\frac{\partial^2}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2})-\frac{b_n}{D}=0 \quad\Leftrightarrow$$

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-\frac{b_n}{D}=0 \quad\text{or}\quad \nabla_0^2\nabla_0^2 w=\frac{b_n}{D}\,. \;\Leftarrow$$

The last invariant form holds also, e.g., in the polar coordinate system.

Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx}=N(x)$, $Q_x=Q(x)$, $M_{xx}=M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



Solution

Equilibrium and constitutive equations of the thin-slab and bending modes are

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x}+\dfrac{\partial N_{xy}}{\partial y}+b_x \\ \dfrac{\partial N_{yy}}{\partial y}+\dfrac{\partial N_{xy}}{\partial x}+b_y \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

Derivatives with respect to y vanish, $b_x=\rho g t/\sqrt{2}$, and $b_n=-\rho g t/\sqrt{2}$. The Reissner-Mindlin plate equations of the planar problem simplify to

$$\frac{dN}{dx}+\frac{\rho g t}{\sqrt{2}}=0, \quad \frac{dQ}{dx}-\frac{\rho g t}{\sqrt{2}}=0, \quad \frac{dM}{dx}-Q=0 \quad \text{in} \quad (0,L),$$

$$N=\frac{tE}{1-\nu^2}\frac{du}{dx}, \quad Q=Gt\left(\frac{dw}{dx}+\theta\right), \quad M=D\frac{d\theta}{dx} \quad \text{in} \quad (0,L),$$

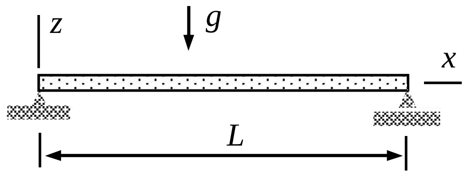
Boundary conditions can be deduced from the figure:

$$u=0, \quad w=0, \quad \theta=0 \quad \text{at} \quad x=0,$$

$$N=0, \quad M=0, \quad Q=0 \quad \text{at} \quad x=L.$$

Solution to equations can be obtained by considering the equilibrium equations and the boundary conditions at the free end first. After that, solutions to the displacement components follow from the constitutive equations and the boundary conditions at the clamped edge.

Consider a plate strip of the figure loaded by its own weight. Determine deflection w and rotation θ of the plate according to the Kirchhoff model. Thickness, and length of the plate are and L, respectively. Density ρ, You$t$ng’s modulus E, and Poisson’s ratio ν are constants. Assume that stress-resultants, displacements, and rotations depend on x only (consider a plate of width H).



Solution

Assuming that all derivatives with respect to y vanish, the plate equations of the formulae collection (just the equations associated with the bending modes) are

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}=0 \quad \text{in} \quad \Omega\,,$$

$$Q_n-\underline{Q}+\frac{\partial}{\partial s}(M_{ns}-\underline{M}_s)=0 \quad \text{or} \quad w-\underline{w}=0 \quad \text{on} \quad \partial\Omega\,,$$

$$M_{nn}-\underline{M}_n=0 \quad \text{or} \quad \frac{\partial w}{\partial n}+\underline{\theta}_s=0 \quad \text{on} \quad \partial\Omega\,.$$

Taking into account only the equations needed and using notation $M_{xx}\equiv M$ and $Q_x\equiv Q$

$$\frac{dQ}{dx}+b_n=0\,,\ \frac{dM}{dx}-Q=0\,,\ M_{xx}=D\frac{d\theta}{dx}\,,\ \text{and}\ \frac{dw}{dx}+\theta=0 \ \text{in}\ (0,L)\,.$$

Boundary conditions specify either displacement or shear force and bending moment or rotation. From the figure

$$M=0 \quad \text{and} \quad w=0 \quad \text{on}\ \{0,L\}\,.$$

After elimination of the shear force and the bending moment, the boundary value problem for the deflection w becomes (the equation system can also be solved one equation at a time in its original form)

$$-D\frac{d^4w}{dx^4}+b_n=0 \ \text{in}\ (0,L) \quad \text{and} \quad w=\frac{d^2w}{dx^2}=0 \ \text{on}\ (0,L)\,. \ \leftarrow$$

Generic solution to the differential equation can be obtained by repeated integrations

$$\frac{d^4w}{dx^4}=\frac{b_n}{D} \quad \Rightarrow \quad w(x)=\frac{b_n}{D}\frac{x^4}{24}+ax^3+bx^2+cx+d \,.$$

Boundary conditions imply that (the equations can be solved starting from the first, then using the already obtained solution in the second etc.)

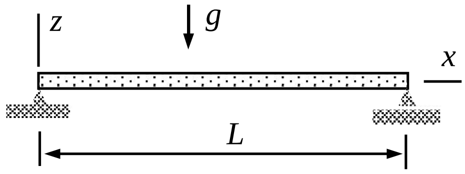
$$\begin{Bmatrix} w(0) \\ \dfrac{d^2w}{dx^2}(0) \\ \dfrac{d^2w}{dx^2}(L) \\ w(L) \end{Bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 6L & 2 & 0 & 0 \\ L^3 & L^2 & L & 1 \end{bmatrix}\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}+\frac{b_nL^2}{2D}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ L^2/12 \end{Bmatrix}=0 \quad \Rightarrow$$

$$d=b=0 \;\Rightarrow\; a=-\frac{b_nL}{12D} \;\Rightarrow\; c=\frac{b_nL^3}{24D}\,.$$

Therefore, using the expressions of the coefficient in the displacement solution, expression of the distributed force $b_n=-\rho g t$, and the constraint $\theta=-dw/dx$ of the Kirchhoff model

$$w=-\frac{\rho g t L^4}{24D}[\frac{x}{L}-2(\frac{x}{L})^3+(\frac{x}{L})^4] \quad \Rightarrow \quad \theta=-\frac{dw}{dx}=\frac{\rho g t L^3}{24D}[1-6(\frac{x}{L})^2+4(\frac{x}{L})^3]\,.$$

Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = \mathrm{a}_0(1-x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \delta \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} dA + \int_\Omega b_n \delta w dA .$$

Solution

Principle of virtual work gives a straightforward way to find approximate/series solutions to beam, plate etc. problem. First, approximation of the "right" form is substituted into the virtual work expression. The approximation is a sum of terms having multipliers to be determined. Second, principle of virtual work and the fundamental lemma of variation calculus are applied with respect to the multipliers. Finite element method, sine series solutions, etc. can be taken just particular cases of the method. Virtual work expression of the Kirchhoff plate model bending mode

$$\delta W = -\int_\Omega \delta \vec{\kappa}_\mathrm{c} : \vec{M} dA + \int_\Omega b_n \delta w dA$$

in terms of the transverse displacement follow when the constitutive equation $\vec{M} = \vec{\vec{B}} : \vec{\kappa}$, strain definition $\vec{\kappa} = -\nabla_0 \nabla_0 w$, and the Cartesian coordinate system representation of the elasticity tensor

$$\vec{B} = D \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix} .$$

are substituted there. Approximation to the transverse displacement (notice that the polynomial shape is known and variation of displacement is through the multiplier)

$$w = \mathrm{a}_0 (1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = -\mathrm{a}_0 \frac{2}{L^2} \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y^2} = 0 \quad \Rightarrow$$

$$\delta w = \delta \mathrm{a}_0 (1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \delta \frac{\partial^2 w}{\partial x^2} = -\delta \mathrm{a}_0 \frac{2}{L^2} \quad \text{and} \quad \delta \frac{\partial^2 w}{\partial x \partial y} = \delta \frac{\partial^2 w}{\partial y^2} = 0 .$$

When the approximation is substituted there, virtual work expression simplifies to

$$\delta W = -\int_0^H \int_0^L D(-\delta a_0 \frac{2}{L^2})(-a_0 \frac{2}{L^2}) dxdy + \int_0^H \int_0^L b_n \delta a_0 (1-\frac{x}{L})(\frac{x}{L}) dxdy \quad \Rightarrow$$

$$\delta W = -\delta a_0 H (\frac{4}{L^3} D a_0 + b_n \frac{1}{6}) .$$

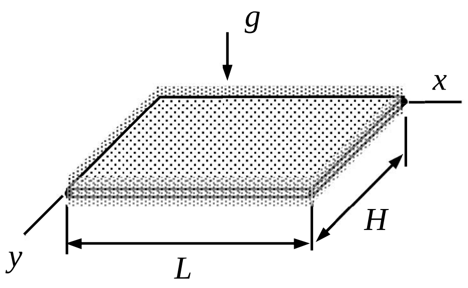
Principle of virtual work $\delta W = 0 \ \forall \delta a_0$ and the fundamental lemma of variation calculus give with $b_n = -\rho g t$

$$\delta a_0 HL(-\frac{4}{L^4} D a_0 + b_n \frac{1}{6}) = 0 \ \ \forall \delta a_0 \quad \Leftrightarrow \quad -\frac{4}{L^4} D a_0 + b_n \frac{1}{6} = 0 \quad \Rightarrow \quad a_0 = \frac{L^4 b_n}{24D} = -\frac{L^4 \rho g t}{24D} .$$

Therefore, an approximation to the transverse displacement is given by

$$w = -\frac{\rho g t L^4}{24D}(1-\frac{x}{L})\frac{x}{L} .$$

A simply supported rectangular plate of size $L \times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Determine the displacement at the center point with $w = \mathrm{a}_0 (xy / LH)(1 - x / L)(1 - y / H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \;\; \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Solution

As the solution does not depend on the Poisson's ratio ν (additional information), one may consider D and ν as the two independent material parameters of a linearly elastic material and choose a convenient value like $\nu = 1$ to simplify the calculations. Then, the virtual work expression simplifies to

$$\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = -\int_\Omega D \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \end{Bmatrix} dA + \int_\Omega b_n \delta w dA .$$

Approximation to the transverse displacement is chosen to be (the polynomial shape is known and variation of displacement is through the multiplier)

$$w = \mathrm{a}_0 [\frac{x}{L} - (\frac{x}{L})^2][\frac{y}{H} - (\frac{y}{H})^2] \quad \Rightarrow \quad \delta w = \delta \mathrm{a}_0 [\frac{x}{L} - (\frac{x}{L})^2][\frac{y}{H} - (\frac{y}{H})^2] ,$$

$$\frac{\partial^2 w}{\partial x^2} = -\mathrm{a}_0 \frac{2}{L^2}[\frac{y}{H} - (\frac{y}{H})^2] \quad \Rightarrow \quad \delta \frac{\partial^2 w}{\partial x^2} = -\delta \mathrm{a}_0 \frac{2}{L^2}[\frac{y}{H} - (\frac{y}{H})^2] ,$$

$$\frac{\partial^2 w}{\partial y^2} = -\mathrm{a}_0 \frac{2}{H^2}[\frac{x}{L} - (\frac{x}{L})^2] \quad \Rightarrow \quad \frac{\partial^2 w}{\partial y^2} = -\delta \mathrm{a}_0 \frac{2}{H^2}[\frac{x}{L} - (\frac{x}{L})^2].$$

When the approximation is substituted there, virtual work expression takes the form

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 \int_0^L \int_0^H D \begin{Bmatrix} \frac{2}{L^2}[\frac{y}{H}-(\frac{y}{H})^2] \\ \frac{2}{H^2}[\frac{x}{L}-(\frac{x}{L})^2] \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \frac{2}{L^2}[\frac{y}{H}-(\frac{y}{H})^2] \\ \frac{2}{H^2}[\frac{x}{L}-(\frac{x}{L})^2] \end{Bmatrix} dxdy \mathrm{a}_0 \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 4D(\frac{1}{L^4}\frac{HL}{30} + \frac{2}{H^2 L^2}\frac{HL}{36} + \frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 ,$$

$$\delta W^{\text{ext}} = \delta \mathrm{a}_0 \int_0^L \int_0^H b_n [\frac{x}{L}-(\frac{x}{L})^2][\frac{y}{H}-(\frac{y}{H})^2] dxdy = \delta \mathrm{a}_0 b_n \frac{HL}{36} .$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta \mathrm{a}_0 [4D(\frac{1}{L^4}\frac{HL}{30} + \frac{2}{H^2 L^2}\frac{HL}{36} + \frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 - b_n \frac{HL}{36}] .$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}_0$ and the fundamental lemma of variation calculus imply that (here $b_n = \rho g t$)

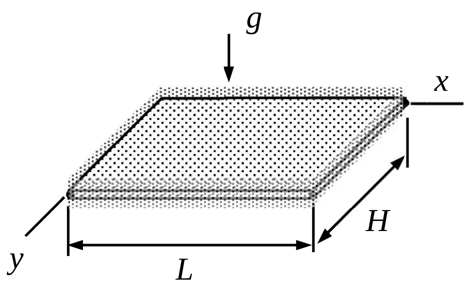
$$4D(\frac{1}{L^4}\frac{HL}{30} + \frac{2}{H^2 L^2}\frac{HL}{36} + \frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 - b_n \frac{HL}{36} = 0 \quad \Leftrightarrow \quad \mathrm{a}_0 = \frac{\rho g t}{8D}\frac{5H^4 L^4}{3H^4 + 5H^2 L^2 + 3L^4} .$$

Therefore, approximation to the transverse displacement becomes

$$w = \frac{5\rho g t}{8D}\frac{H^4 L^4}{3H^4 + 5H^2 L^2 + 3L^4}\frac{x}{L}(1-\frac{x}{L})\frac{y}{H}(1-\frac{y}{H}) .$$

Notice! The double sine series solution with 100 terms in both directions gives in case of the square plate displacement $w = 0.0041 b_n L^4 / D$ at the center point. Displacement given by the one parameter approximation is $w = 0.0036 b_n L^4 / D$.

A simply supported rectangular plate of size $L\times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Find an approximation to transverse displacement by using $w=\mathrm{a}_0\sin(\pi x/L)\sin(\pi y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W=0\ \ \forall\delta\mathrm{a}_0\in\mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W=-\int_\Omega \begin{Bmatrix} \dfrac{\partial^2\delta w}{\partial x^2} \\ \dfrac{\partial^2\delta w}{\partial y^2} \\ 2\dfrac{\partial^2\delta w}{\partial x\partial y} \end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix} dA+\int_\Omega \delta w b_n dA\,.$$

Solution

As the solution does not depend on the Poisson's ratio ν (additional information), one may consider D and ν as the two independent material parameters of a linearly elastic material and choose a convenient value like $\nu=1$ to simplify the calculations. Then, the virtual work expression simplifies to

$$\delta W=\delta W^{\mathrm{int}}+\delta W^{\mathrm{ext}}=-\int_\Omega D\begin{Bmatrix} \dfrac{\partial^2\delta w}{\partial x^2} \\ \dfrac{\partial^2\delta w}{\partial y^2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \end{Bmatrix} dA+\int_\Omega b_n\delta w dA\,.$$

Principle of virtual work gives a straightforward way to find approximate/series solutions to beam, plate etc. problem. First, approximation of the "right" form is substituted into the virtual work expression. Second, principle of virtual work and the fundamental lemma of variation calculus are applied with respect to the multipliers. Approximation to the transverse displacement is chosen to be (variation of displacement is through the multiplier)

$$w=\mathrm{a}_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \text{ and } \quad \delta w=\delta\mathrm{a}_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\,,$$

$$\frac{\partial^2 w}{\partial x^2}=-\mathrm{a}_0\frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \text{ and } \delta\frac{\partial^2 w}{\partial x^2}=-\delta\mathrm{a}_0\frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\,,$$

$$\frac{\partial^2 w}{\partial y^2}=-\mathrm{a}_0\frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \text{ and } \delta\frac{\partial^2 w}{\partial y^2}=-\delta\mathrm{a}_0\frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\,.$$

Orthogonality of sines and cosines and known integrals like

$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx=\delta_{ij}\frac{L}{2},\ \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx=\delta_{ij}\frac{L}{2},$$

$$\int_0^L \sin(i\pi\frac{x}{L})dx=\frac{L}{\pi}\frac{1-(-1)^i}{i}$$

simplify the calculations with sinusoidal shape functions. When the approximation is substituted there, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}}=-\delta \mathrm{a}_0\int_0^L\int_0^H D\begin{Bmatrix}\frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\\ \frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\\ \frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H})\end{Bmatrix}dxdy\mathrm{a}_0 \quad\Rightarrow$$

$$\delta W^{\text{int}}=-\delta \mathrm{a}_0\int_0^L\int_0^H D\sin^2(\pi\frac{x}{L})\sin^2(\pi\frac{y}{H})\begin{Bmatrix}\frac{\pi^2}{L^2}\\ \frac{\pi^2}{H^2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\frac{\pi^2}{L^2}\\ \frac{\pi^2}{H^2}\end{Bmatrix}dxdy\mathrm{a}_0 \quad\Rightarrow$$

$$\delta W^{\text{int}}=-\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^4+2(\frac{\pi}{L})^2(\frac{\pi}{H})^2+(\frac{\pi}{H})^4]\mathrm{a}_0=-\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^2+(\frac{\pi}{H})^2]^2\mathrm{a}_0,$$

$$\delta W^{\text{ext}}=\delta \mathrm{a}_0 b_n\int_0^L \sin(\pi\frac{x}{L})dx\int_0^H \sin(\pi\frac{y}{H})dy=\delta \mathrm{a}_0 b_n\frac{4LH}{\pi^2}.$$

Virtual work expression

$$\delta W=-\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^2+(\frac{\pi}{H})^2]^2\mathrm{a}_0+\delta \mathrm{a}_0\frac{4LH}{\pi^2}b_n.$$

Principle of virtual work $\delta W=0\ \forall\delta \mathrm{a}_0$ and the fundamental lemma of variation calculus imply that (here $b_n=\rho g t$)

$$\mathrm{a}_0=\frac{4LH}{\pi^2}\frac{b_n}{D}/D\frac{LH}{4}[(\frac{\pi}{L})^2+(\frac{\pi}{H})^2]^2=\frac{16}{\pi^6}\frac{H^4L^4}{(L^2+H^2)^2}\frac{b_n}{D}.$$

Therefore, approximation to the transverse displacement becomes

$$w=\frac{16}{\pi^6}\frac{H^4L^4}{(L^2+H^2)^2}\frac{b_n}{D}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}).$$

Derive the component form of the Kirchhoff plate equation (bending)

$$\nabla_0 \cdot (\nabla_0 \cdot \vec{\vec{M}}) + b_n = 0$$

in terms of components M_{rr}, $M_{r\phi}$ and $M_{\phi\phi}$ of the polar coordinate system. Assume rotation symmetry so that the moment components depend on r only.

Solution

Assuming symmetry $\vec{\vec{M}} = \vec{\vec{M}}_{\mathrm{c}}$, the component representations of the planar gradient operator and moment tensor in the (r,ϕ,n) coordinate system are

$$\nabla_0 = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \phi} \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi},$$

$$\vec{\vec{M}} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} M_{rr} & M_{r\phi} \\ M_{r\phi} & M_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \vec{e}_r\vec{e}_r M_{rr} + \vec{e}_r\vec{e}_\phi M_{r\phi} + \vec{e}_\phi\vec{e}_r M_{r\phi} + \vec{e}_\phi\vec{e}_\phi M_{\phi\phi}.$$

First divergence of the moment tensor by considering the four terms of $\vec{\vec{M}}$ one by one (each term of $\vec{\vec{M}}$ may give rise to 6 derivative terms). The derivatives of the basis vectors are $\partial\vec{e}_r / \partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi / \partial\phi = -\vec{e}_r$ and $\partial M_{rr} / \partial\phi = 0$, $\partial M_{\phi\phi} / \partial\phi = 0$, and $\partial M_{r\phi} / \partial\phi = 0$ by assumption.

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r\vec{e}_r M_{rr}) = \vec{e}_r \frac{dM_{rr}}{dr} + \vec{e}_r \frac{1}{r} M_{rr} = \vec{e}_r \frac{1}{r}\frac{d}{dr}(rM_{rr}),$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r\vec{e}_\phi M_{r\phi}) = \vec{e}_\phi \frac{dM_{r\phi}}{dr} + \vec{e}_\phi \frac{1}{r} M_{r\phi} = \vec{e}_\phi \frac{1}{r}\frac{d}{dr}(rM_{r\phi}),$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_\phi\vec{e}_r M_{r\phi}) = \vec{e}_\phi \frac{1}{r} M_{r\phi},$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_\phi\vec{e}_\phi M_{\phi\phi}) = -\frac{1}{r}\vec{e}_r M_{\phi\phi}.$$

Therefore, the divergence of the moment simplifies to

$$\nabla_0 \cdot \vec{\vec{M}} = \vec{e}_r \frac{1}{r}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + \vec{e}_\phi \frac{1}{r}[\frac{d}{dr}(rM_{r\phi}) + M_{r\phi}].$$

Application of the divergence operator again gives

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot \vec{e}_r[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}] = \frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}]) + \frac{1}{r}[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}],$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot \vec{e}_\phi [\frac{1}{r}\frac{d}{dr}(rM_{r\phi}) + \frac{1}{r}M_{r\phi}] = 0\,.$$

Finally, combining the terms

$$\nabla_0 \cdot (\nabla_0 \cdot \overset{\leftrightarrow}{M}) + b_n = \frac{1}{r}\frac{d}{dr}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + b_n = 0\,.$$

Derive the components of the elastic isotropic Kirchhoff plate constitutive equations for bending in polar (r,ϕ,n) coordinate system. Use definitions $\vec{M} = \vec{\vec{B}}:\vec{\kappa}$, $\vec{\kappa} = -\nabla_0\nabla_0 w$ and assume rotation symmetry $\partial w/\partial\phi = 0$. The polar coordinate system representation of the bending elasticity tensor of plate model is

$$\vec{\vec{B}} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ in which } D = \frac{Et^3}{12(1-\nu^2)}.$$

Solution

The (mid-plane) gradient operator of the polar (r,ϕ,n) coordinate system gives

$$\vec{\kappa} = -\nabla_0\nabla_0 w = -(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi})(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi})w \quad \Rightarrow$$

$$\vec{\kappa} = -\nabla_0\nabla_0 w = -(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi})(\vec{e}_r\frac{\partial w}{\partial r}) = -\vec{e}_r\vec{e}_r\frac{d^2w}{dr^2} - \vec{e}_\phi\vec{e}_\phi\frac{1}{r}\frac{dw}{dr} \ .$$

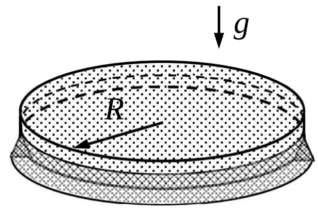
By using the constitutive equation and the elasticity tensor

$$\vec{M} = \vec{\vec{B}}:\vec{\kappa} = -\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{d^2w}{dr^2} \\ \dfrac{1}{r}\dfrac{dw}{dr} \\ 0 \end{Bmatrix}.$$

Therefore, the polar coordinate system components of the bending moment constitutive equation are

$$M_{rr} = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}),\ M_{\phi\phi} = -D(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}),\ \text{and } M_{r\phi} = M_{\phi r} = 0.$$

A simply supported circular plate of radius R is loaded by its own weight as shown in the figure. Determine the displacement of the plate at the midpoint by using the Kirchhoff plate model in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that the solution depends on the radial coordinate only.



Solution

Under the rotation symmetry assumption, the equilibrium equation and the two constitutive equations of Kirchhoff plate bending

$$\nabla_0 \cdot (\nabla_0 \cdot \vec{M}) + b_n = \frac{1}{r}\frac{d}{dr}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + b_n = 0,$$

$$M_{rr} = -D(\frac{d^2 w}{dr^2} + \nu \frac{1}{r}\frac{dw}{dr}), \text{ and } M_{\phi\phi} = -D(\nu \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr})$$

Give, after elimination of the moment resultants, the equilibrium equation

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D}.$$

The boundary value problem for a simply supported circular plate of the problem

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D} \text{ in } (0,R) \quad \text{and} \quad w = M_{rr} = 0 \text{ at } r = R.$$

Repeated integrations in the equilibrium equation give

$$\frac{d}{dr}(r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w) = \frac{b_n}{D}r \quad \Rightarrow \quad r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^2}{2} + a \quad \Rightarrow$$

$$\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w) = \frac{b_n}{D}\frac{r}{2} + \frac{a}{r} \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^2}{4} + a\ln r + b \quad \Rightarrow$$

$$\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^3}{4} + ar\ln r + br \quad \Rightarrow \quad r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^4}{16} + a(-\frac{r^2}{4} + \frac{1}{2}r^2\ln r) + b\frac{r^2}{2} + c \quad \Rightarrow$$

$$\frac{d}{dr}w = \frac{b_n}{D}\frac{r^3}{16} + a(-\frac{r}{4} + \frac{1}{2}r\ln r) + b\frac{r}{2} + c\frac{1}{r} \quad \Rightarrow \quad w = \frac{b_n}{D}\frac{r^4}{64} + a\frac{1}{4}r^2(\ln r - 1) + b\frac{r^2}{4} + c\ln r + d$$

or by redefining the coefficients to get a more compact solution

$$w = \frac{b_n}{D}\frac{r^4}{64} + a + br^2 + cr^2(1 - \log r) + d\log r.$$

The generic solution contains parameters a, b, c, and d to be determined from the boundary conditions. As origin belongs to the solution domain and only the distributed load is acting on the plate, derivatives should be bounded at the origin which implies that $c = d = 0$. Boundary condition on the outer edge

$$M_{rr}(R) = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr})(R) = -2bD(1+\nu) - \frac{1}{16}b_n R^2(3+\nu) = 0 \quad \Rightarrow \quad b = -\frac{1}{32}\frac{b_n R^2}{D}\frac{3+\nu}{1+\nu},$$

$$w(R) = a + bR^2 + \frac{b_n R^4}{64D} = 0 \quad \Rightarrow \quad a = \frac{b_n}{D}\frac{R^4}{64}\frac{5+\nu}{1+\nu}.$$

Displacement at the center point when $b_n = -\rho g t$

$$w(0) = \frac{b_n}{D}\frac{R^4}{64}\frac{5+\nu}{1+\nu} = -\frac{3}{16}\frac{\rho g R^4}{Et^2}(5+\nu)(1-\nu) \quad (D = \frac{t^3 E}{12(1-\nu^2)}). \leftarrow$$