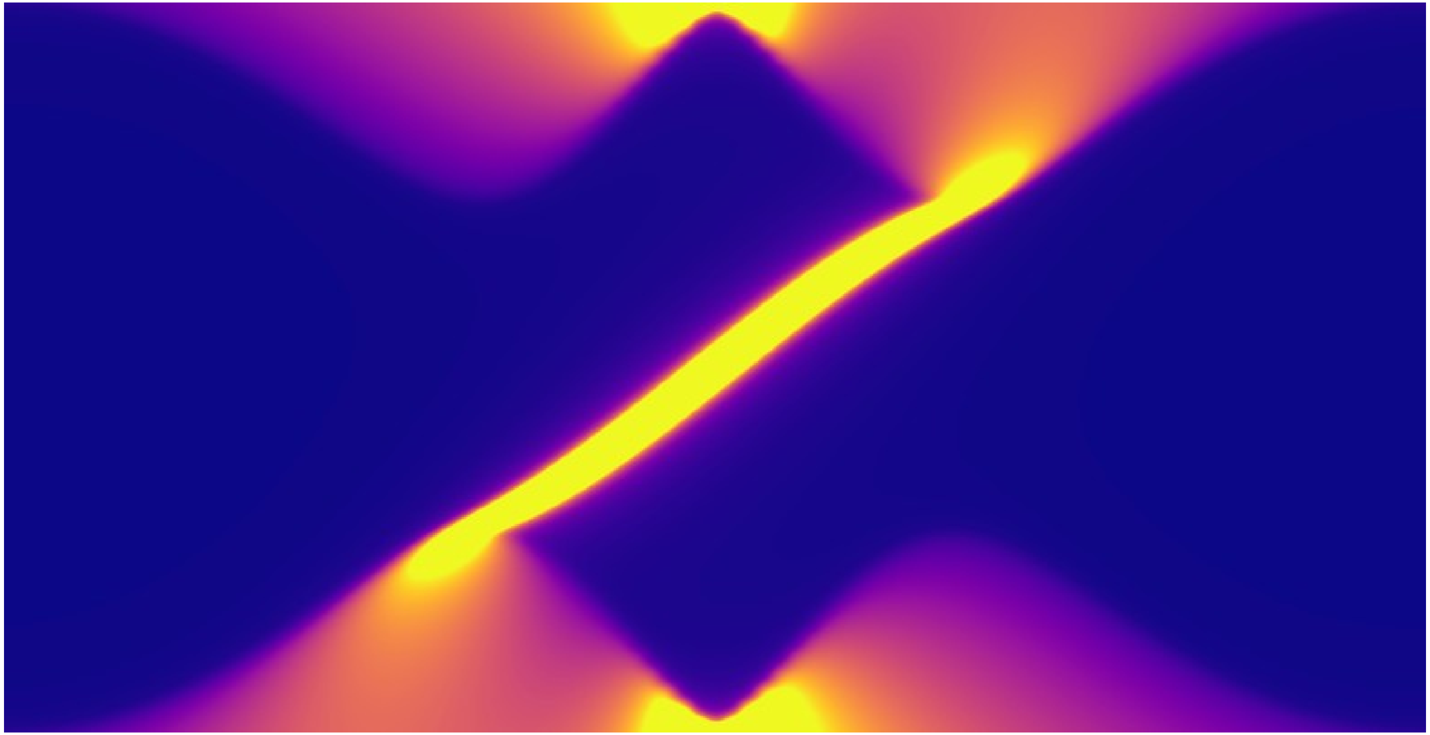
Topological band structure theory



March 29th 2021

Today’s learning outcomes

- Topological states of matter show quantum phenomena resilient to perturbations
- Non-trivial topological invariants give rise to gapless surface excitations

Today’s plan

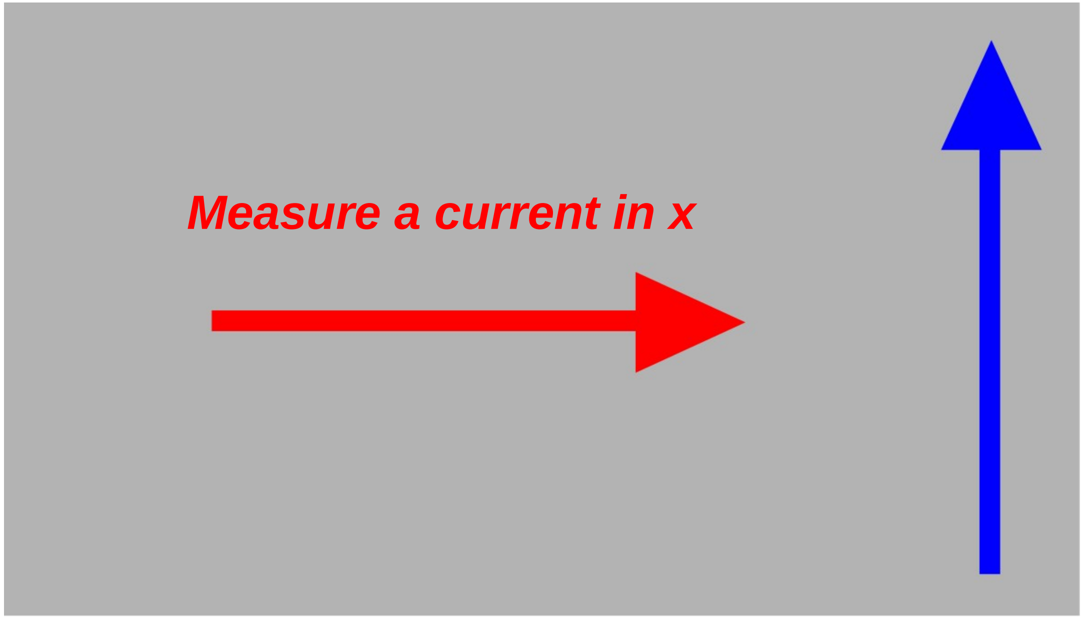
- The quantum Hall effect and its edge states
- The concept of topology in physics
- A minimal model for a topological insulator
- Topology beyond electrons

If you want to know more, there are nice resources online

https://topocondmat.org/

A reminder from session #4: The transverse conductivity

Take a two-dimensional material



Apply a voltage in y

$$J_x = \sigma_{xy} V_y$$

Full Hamiltonian

$$H = H_0 + \lambda V$$

Perturbation

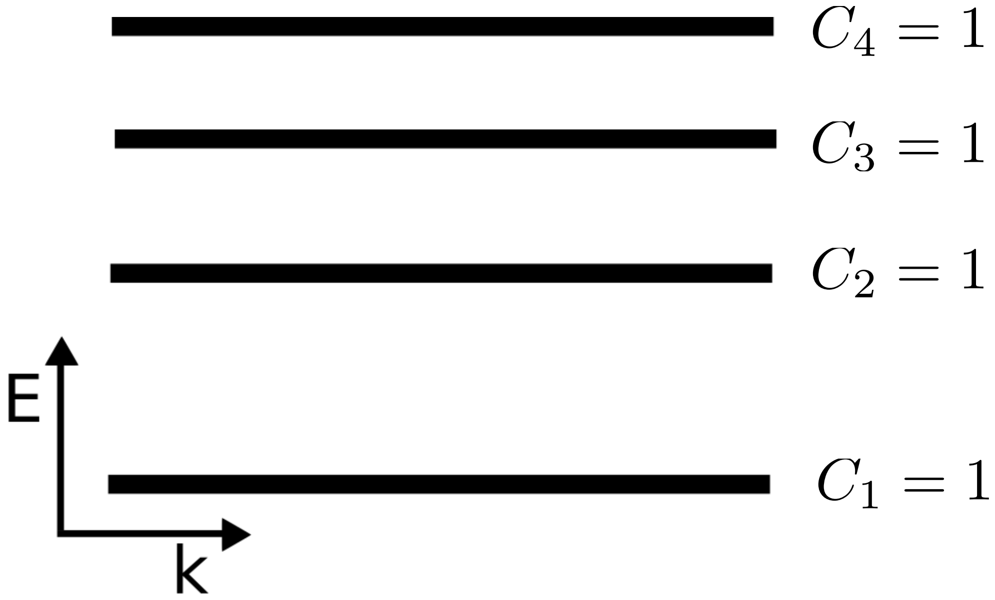
$$V \sim y \sim i\partial_{k_y}$$

Measure

$$J_x \sim \langle \partial H / \partial k_x \rangle$$

A reminder from session #4: the quantum Hall state

Band-structure in the quantum Hall state



Hall conductivity

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

$$\Omega_\alpha = \partial_{k_x} A_y^\alpha - \partial_{k_y} A_x^\alpha$$

$$A_\mu^\alpha = i \langle \partial_{k_\mu} \Psi_\alpha | \Psi_\alpha \rangle$$

Each band (a.k.a Landau level), contributes with Chern number +1

A reminder from session #4: the quantum Hall state

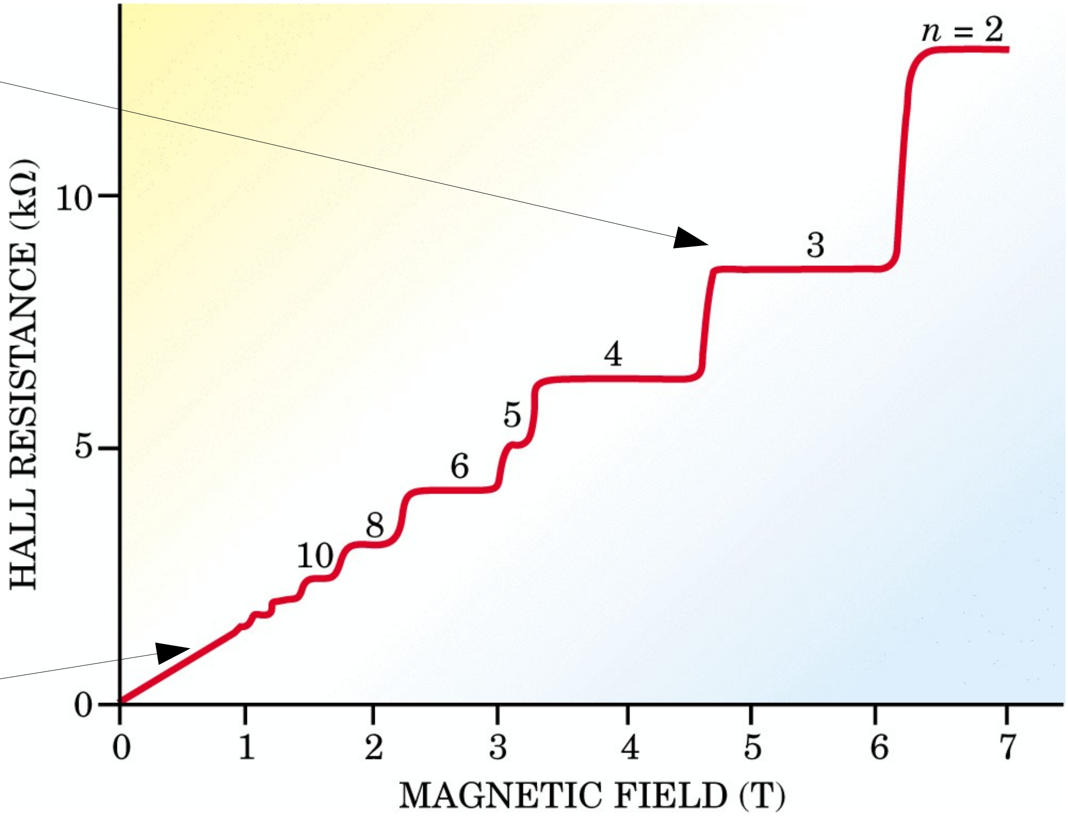
Quantum regime

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

In the quantum Hall regime

$$\rho_{xy} \sim \sigma_{xy}^{-1}$$

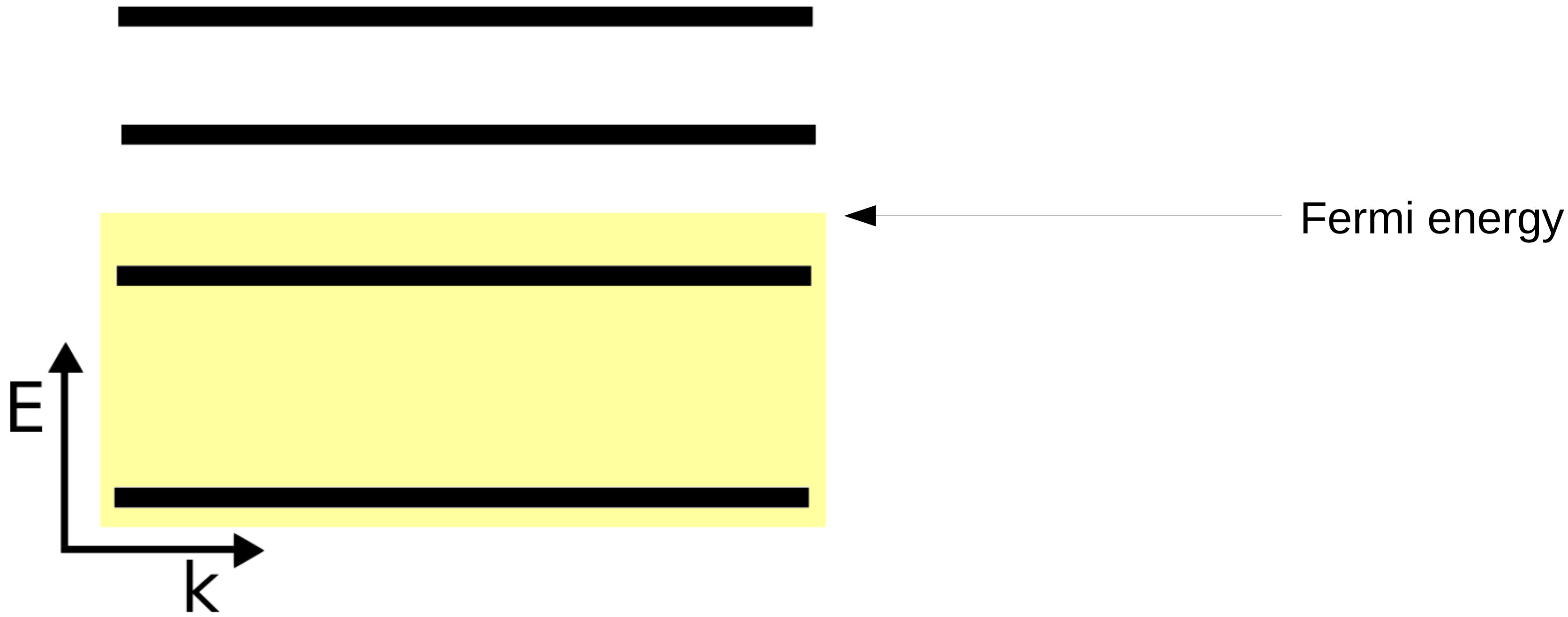
Classical regime



The puzzling quantum Hall effect

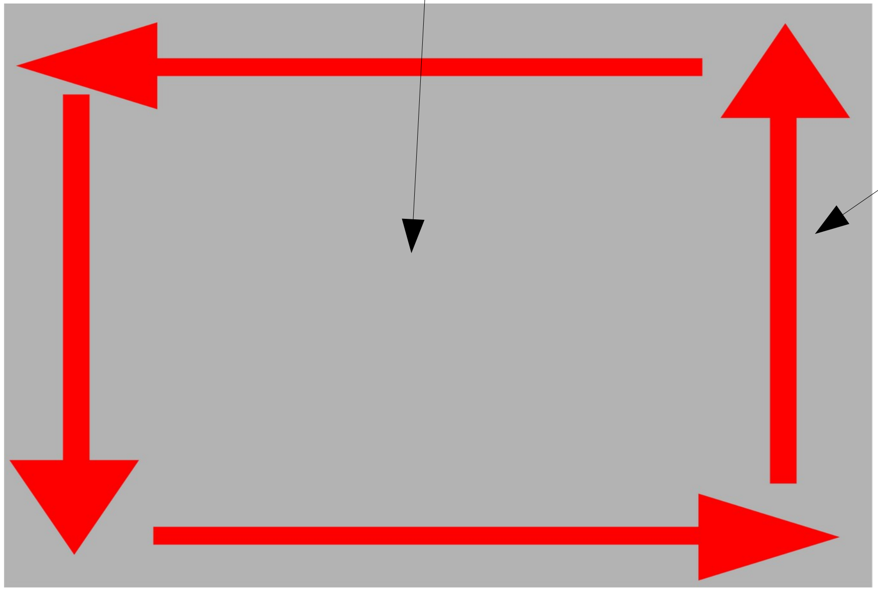
How can an insulator have conductivity?

Bulk band-structure



The puzzling quantum Hall effect

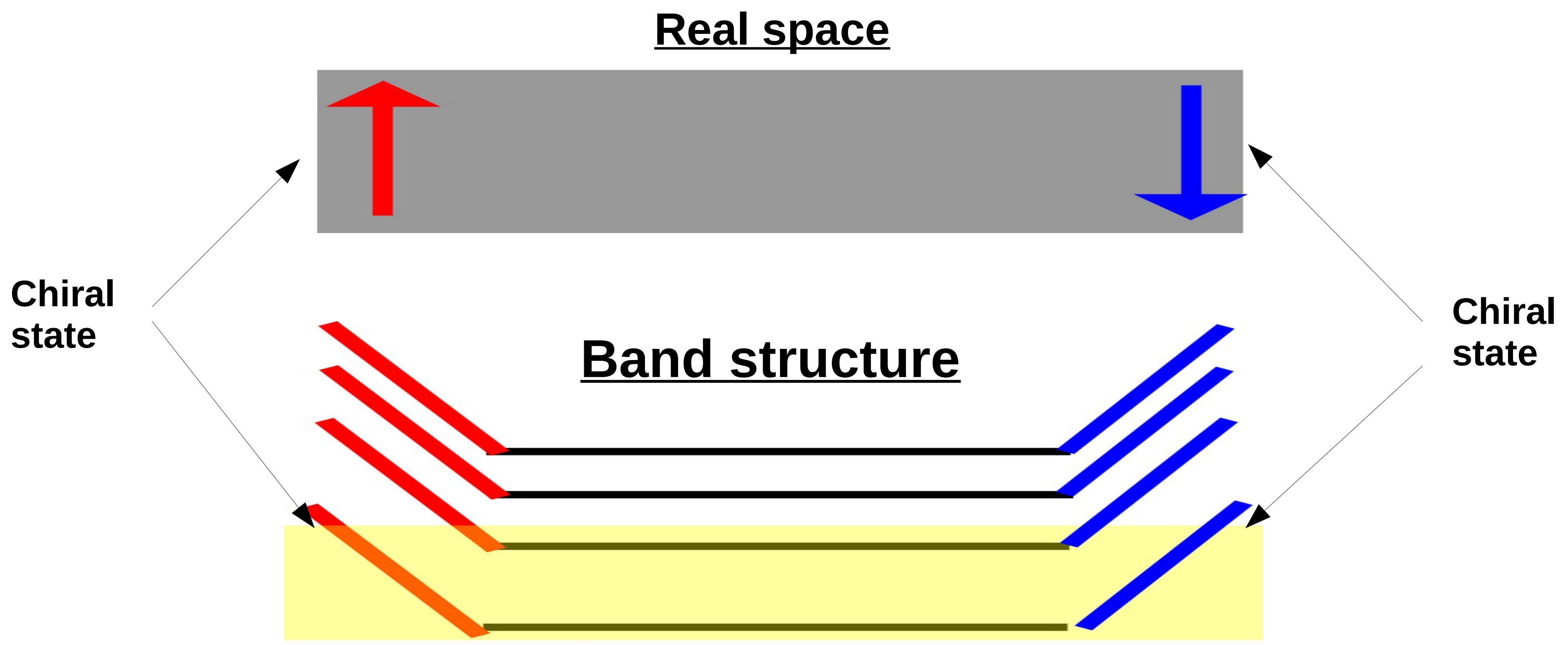
The bulk of a quantum Hall state is insulating



The edge has chiral states

Chiral: propagating
only in one direction

The quantum Hall effect



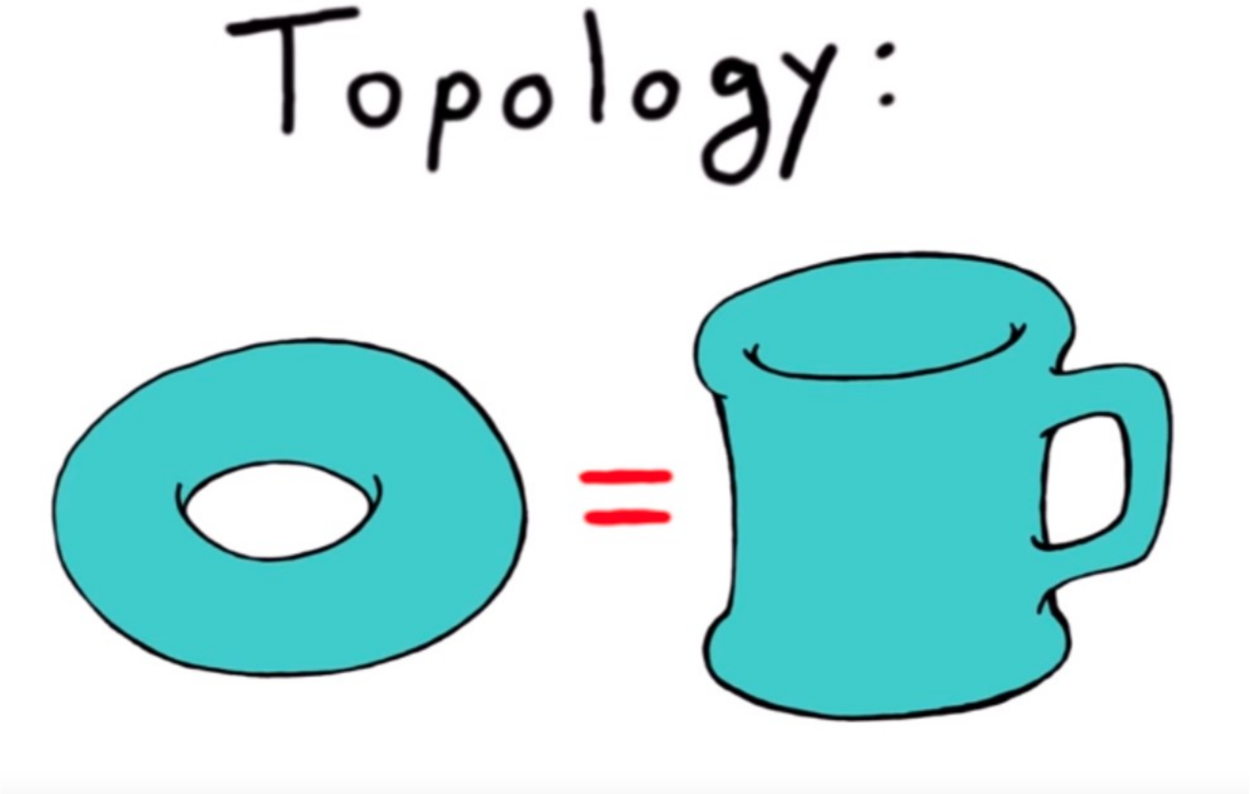
Topology in electronic systems

Topology, doughnuts and knots

Topology classifies object that cannot smoothly deformed into one another

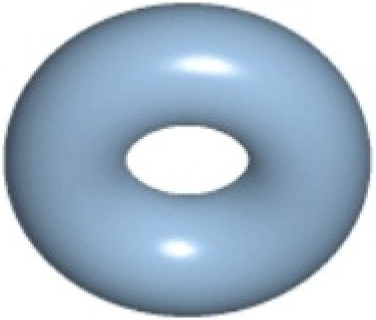
Topology, doughnuts and knots

https://www.youtube.com/watch?v=C-eJW0gEm5w

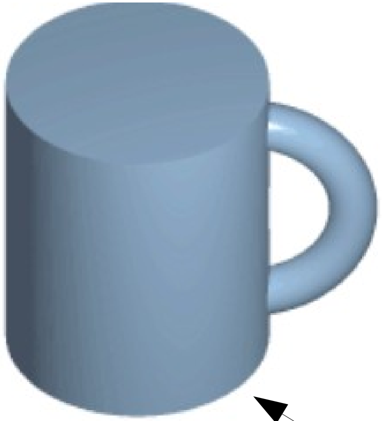


Topology and holes

One hole



One hole



Zero holes

“Topologically” equivalent

“Topologically” different

Topological invariant in a Hamiltonian

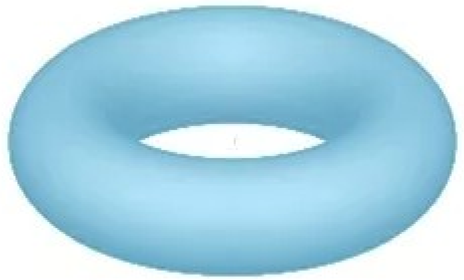
We can classify Hamiltonians according to topological invariants

$$\begin{array}{ccc} \textbf{Hamiltonian} & & \textbf{Wavefunction} \\ H(\mathbf{k}) & \longrightarrow & |\Psi(\mathbf{k})\rangle \\ & & \downarrow \\ C = \int_{BZ} \Omega d^2\mathbf{k} & \longleftarrow & \Omega = \nabla \times \mathbf{A}\big|_z \\ \textbf{Topological invariant} & & \textbf{Metric} \\ \textbf{(Chern number)} & & \textbf{(Berry curvature)} \end{array}$$

The role of a topological invariant

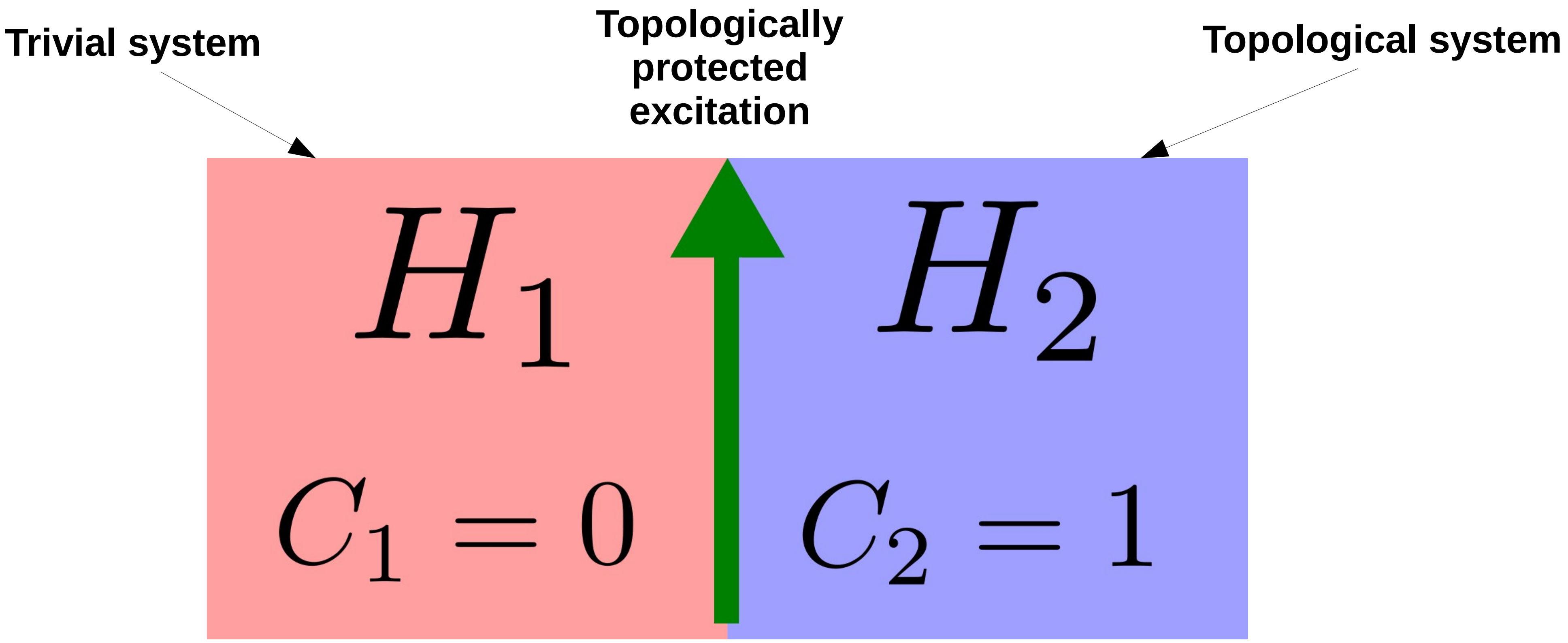
Hamiltonians with different topological invariants can not be deformed one to another

$$C = 1$$



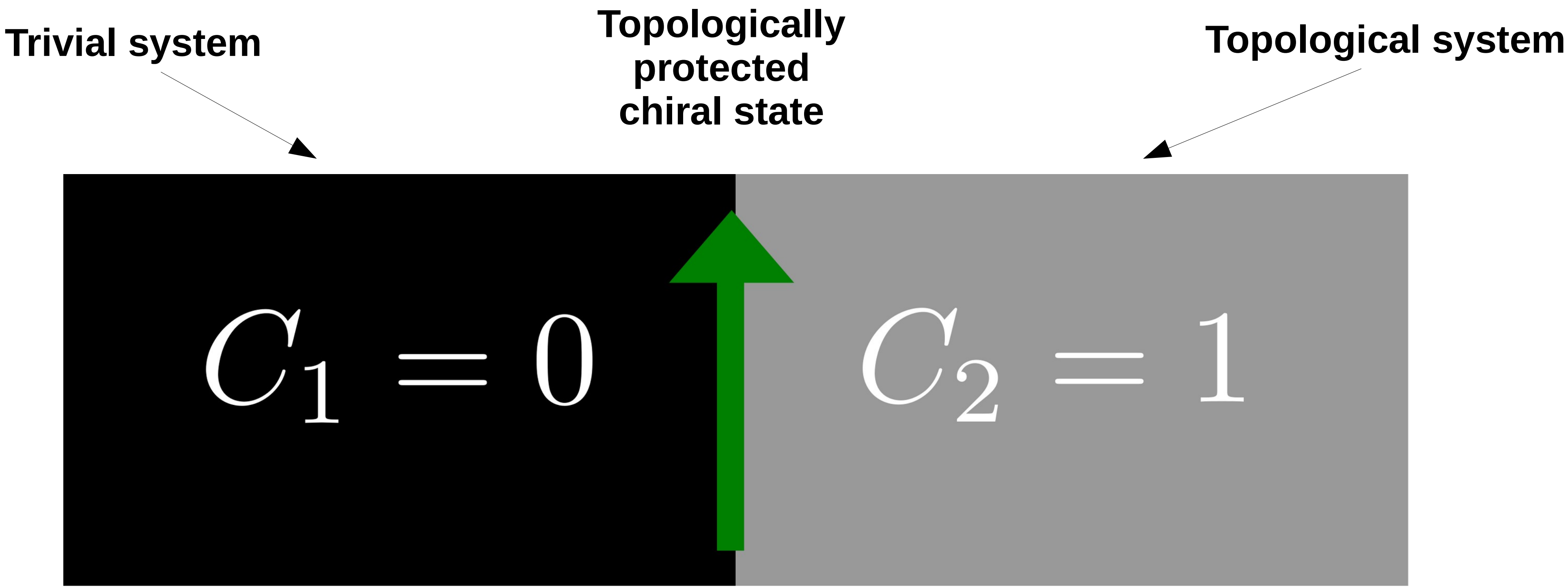
$$C = 2$$

The consequence of different topological invariants



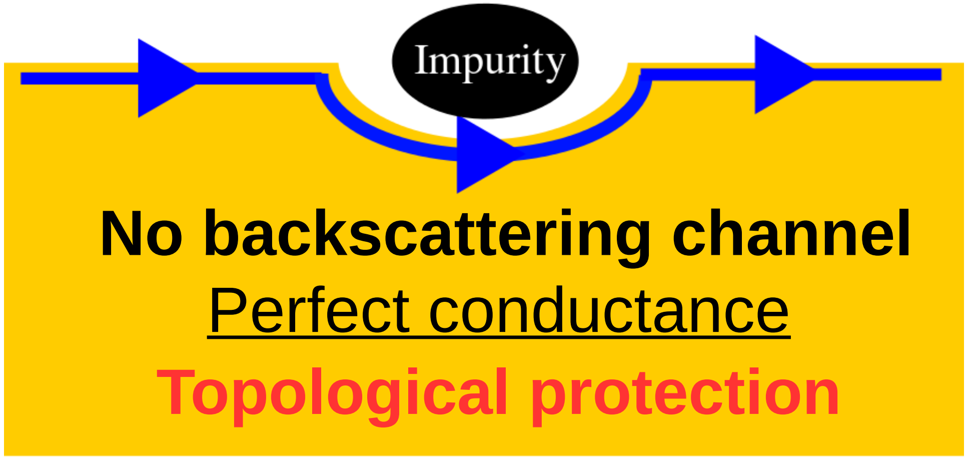
Topological excitations appear between topologically different systems

The edge states of the quantum Hall effect



The edge states of the quantum Hall effect are topological excitations

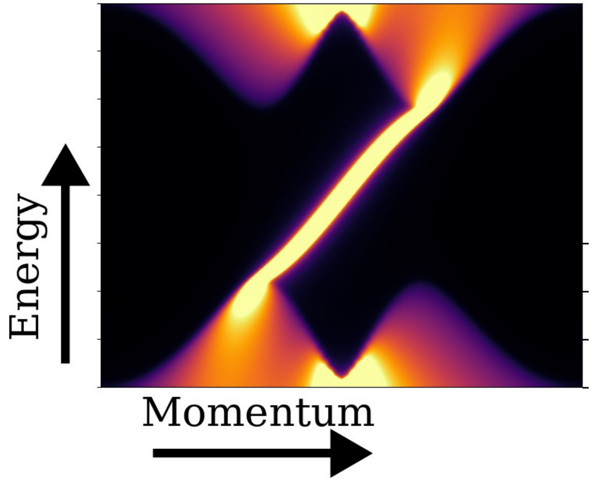
The edge states of the quantum Hall effect



The edge states of the quantum Hall effect are topologically protected

Three important topological materials

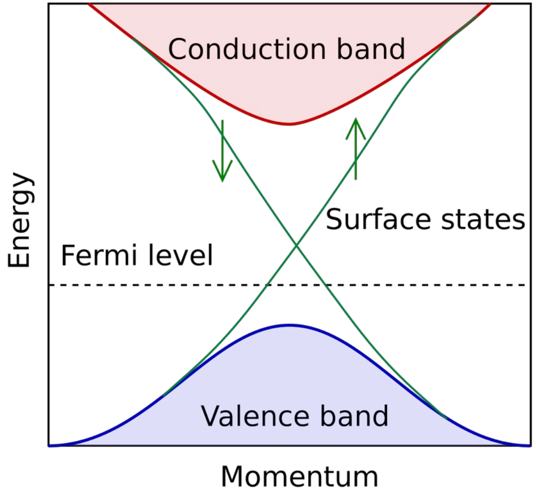
Chern insulators



Chiral states

Electronics

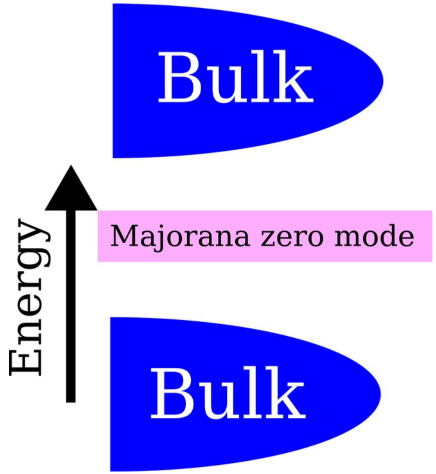
Quantum spin Hall insulators



Helical states

Spintronics

Topological superconductors

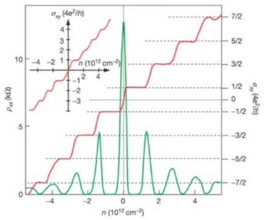


Majorana excitations

Topological quantum computing

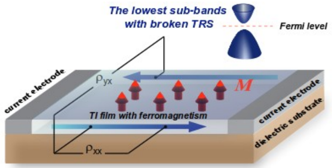
Materials and topological states of matter

Quantum Hall effect



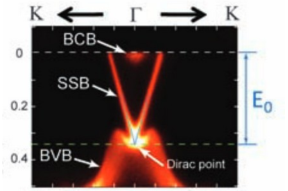
graphene

Quantum Anomalous Hall effect



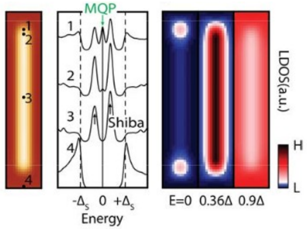
Cr-doped $(Bi,Sb)_2Te_3$

Quantum Spin Hall effect



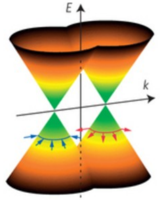
Bi_2Se_3

Topological superconductor



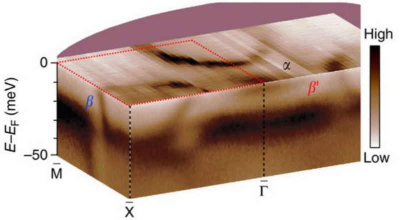
Fe@Pb

Weyl semimetal



TaAs

Topological Kondo insulator

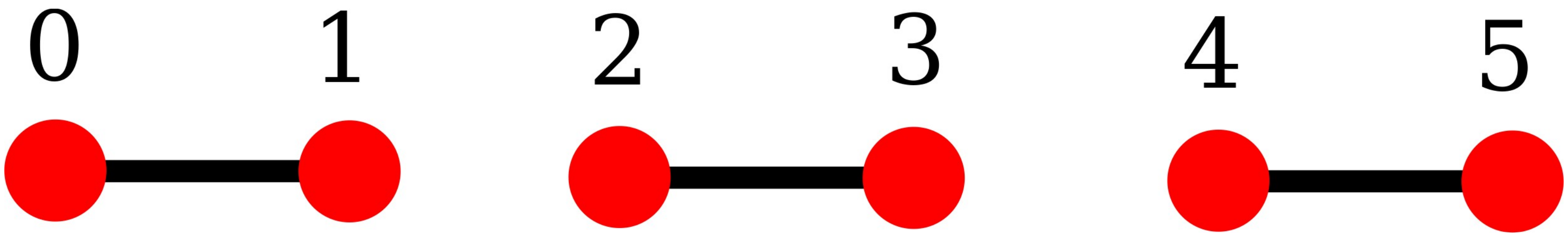


SmB_6

Many different topological states in nature

A toy model for a topological insulator: the SSH model

The SSH model

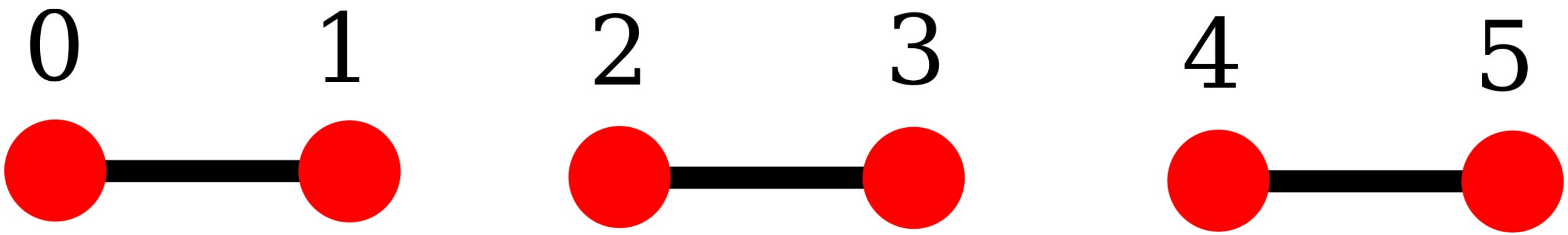


$$H = \sum_{n=0}^{4} [1 + (-1)^n]/2 c_n^\dagger c_{n+1} + h.c.$$

$$H = c_0^\dagger c_1 + c_2^\dagger c_3 + c_4^\dagger c_5 + h.c.$$

Let us consider a finite dimerized chain

The SSH model



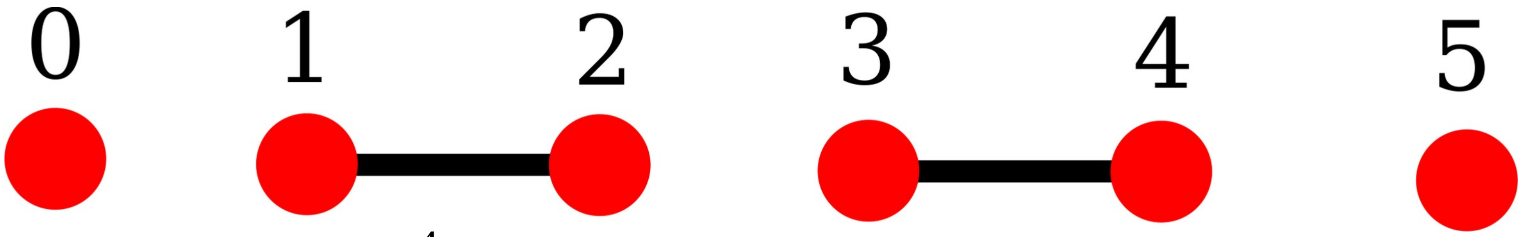
$$H = \sum_{n=0}^{4} [1 + (-1)^n]/2 c_n^\dagger c_{n+1} + h.c.$$

What is the spectra of this Hamiltonian?

$$\epsilon_n = \pm 1$$

$$\epsilon_n = 0, \pm 1$$

The SSH model

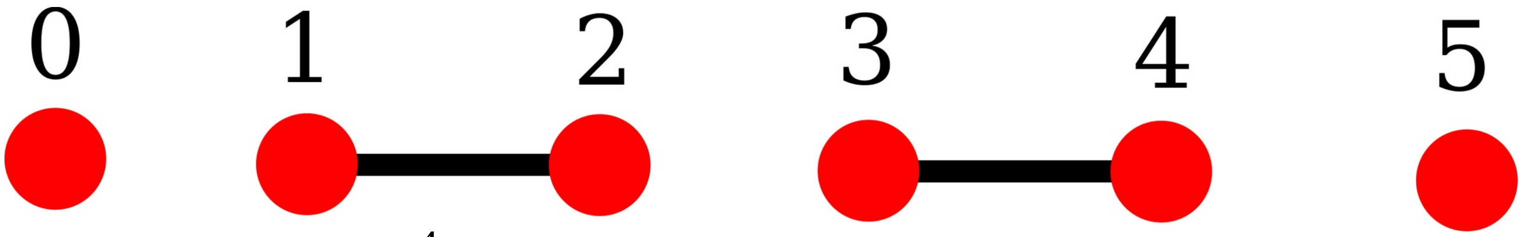


$$H = \sum_{n=0}^{4} [1 - (-1)^n]/2 c_n^\dagger c_{n+1} + h.c.$$

$$H = c_1^\dagger c_2 + c_3^\dagger c_4 + h.c.$$

Let us consider now the other dimerization (with dangling sites)

The SSH model



$$H = \sum_{n=0}^{4} [1-(-1)^n]/2 c_n^\dagger c_{n+1} + h.c.$$

What is the spectra of this Hamiltonian?

$$\epsilon_n = \pm 1$$

$$\epsilon_n = 0, \pm 1$$

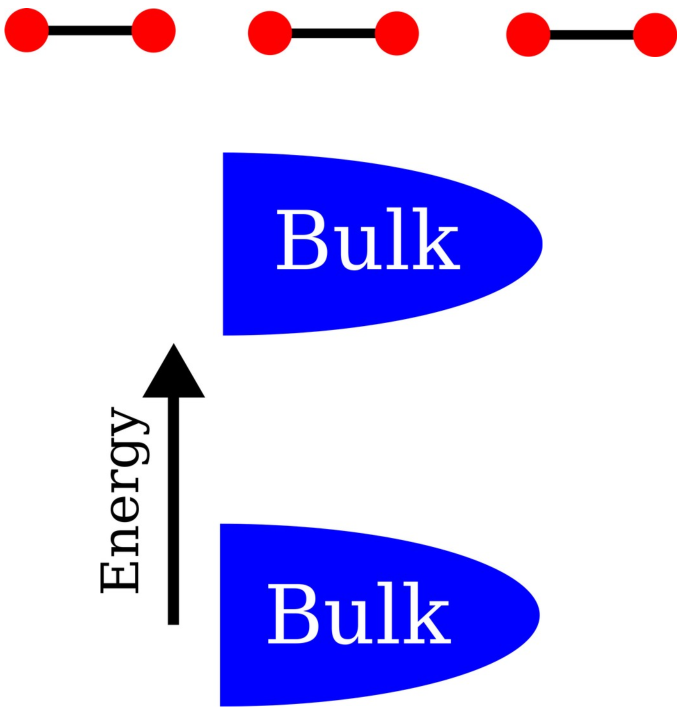
The two phases of the SSH model

“Trivial” phase (gaped everywhere)

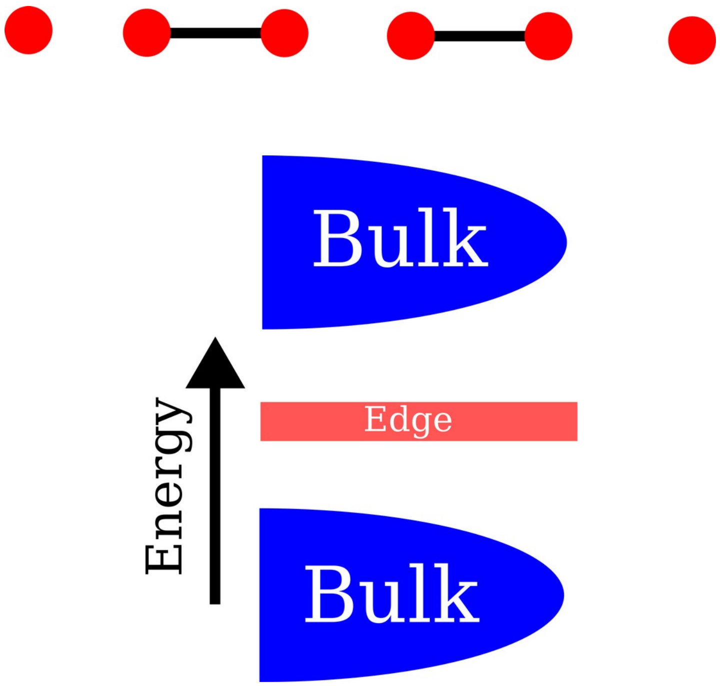
“Topological” phase (gapless zero modes)

The two phases of the SSH model

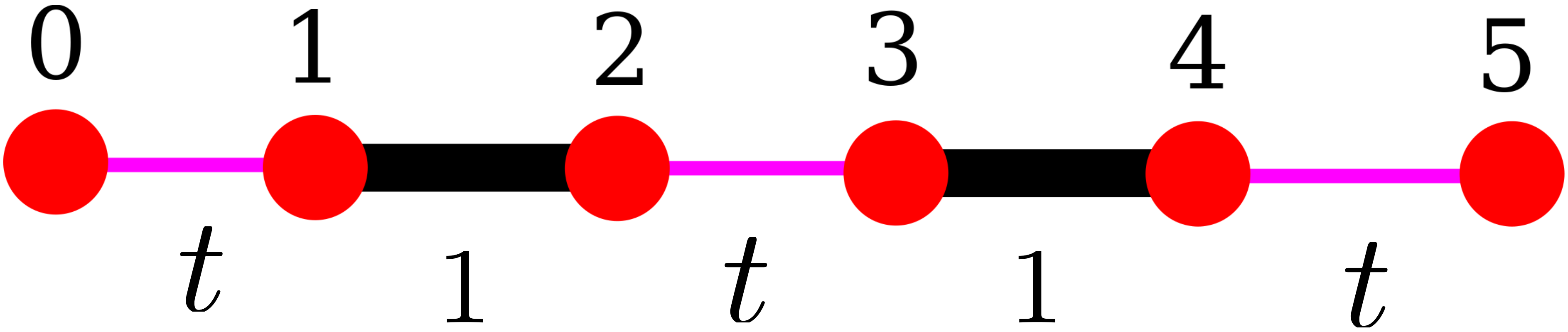
Trivial



Topological



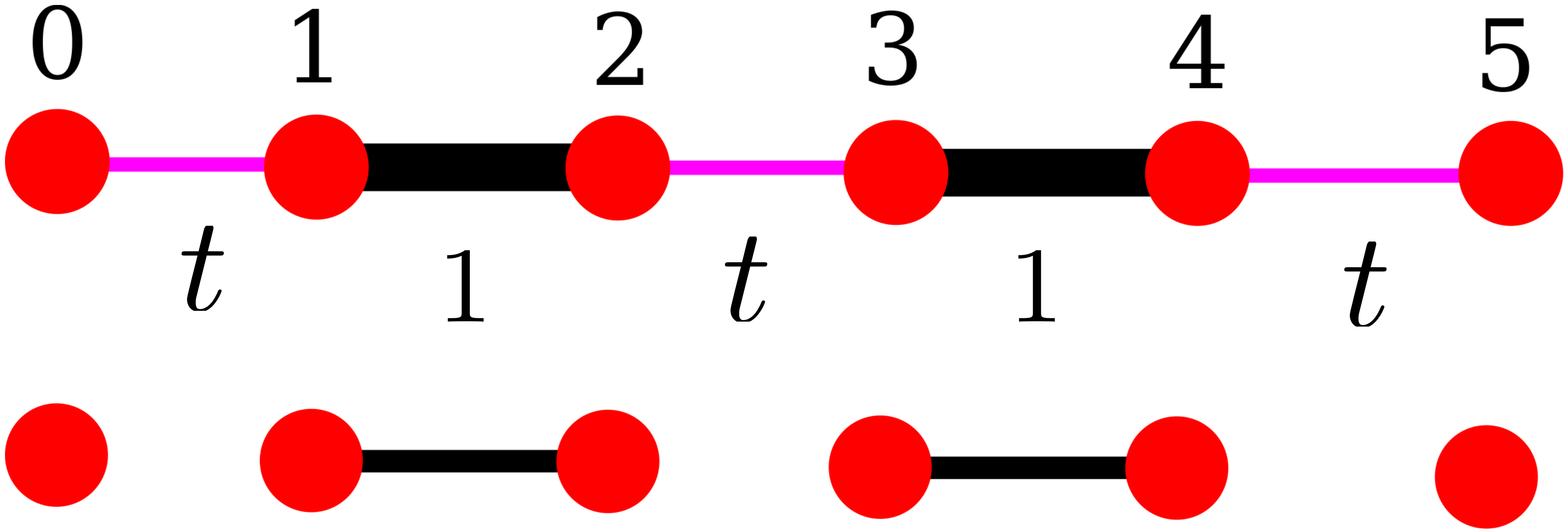
Coupling the dimers



$$H = tc_0^\dagger c_1 + c_1^\dagger c_2 + tc_2^\dagger c_3 + c_3^\dagger c_4 + h.c.$$

Does this Hamiltonian have a surface zero mode?

Coupling the dimers

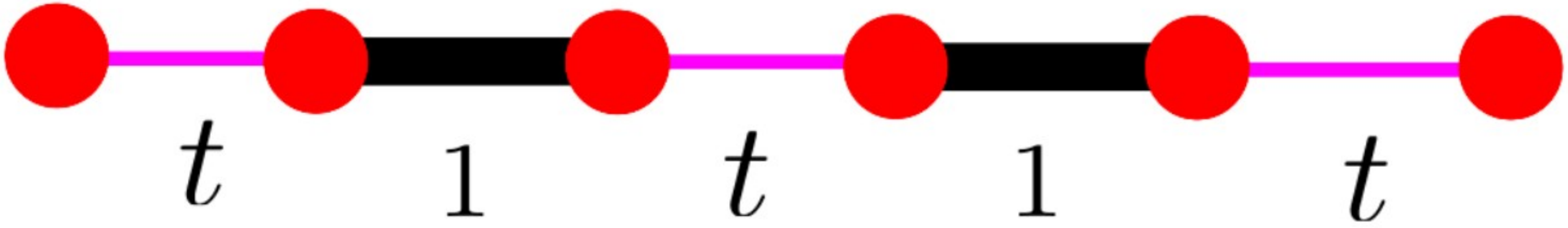


For t<1, both Hamiltonians are topologically equivalent

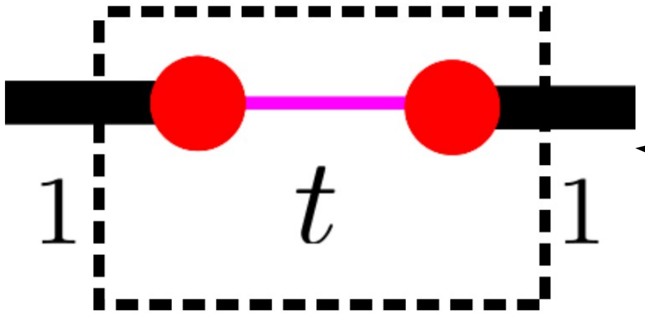
They can be deformed into one another without closing the bulk gap

The bulk Hamiltonian in the SSH model

For a finite system of this form



The unit cell is

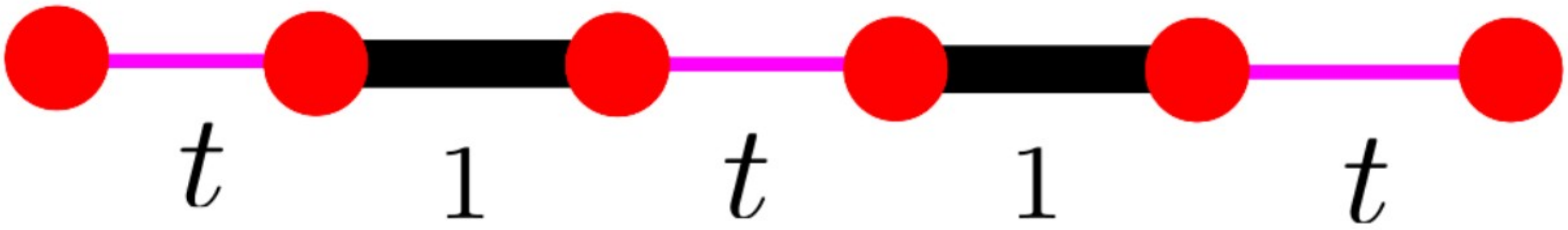


What is the Bloch Hamiltonian for this unit cell?

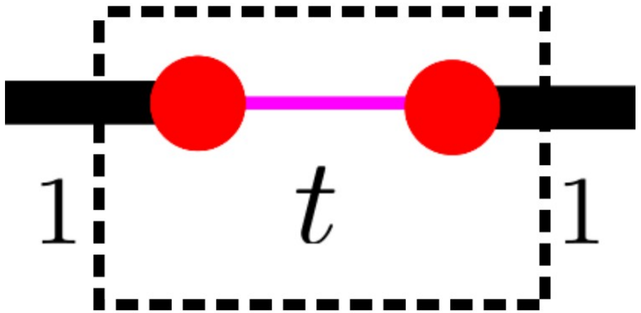
Hint: the Hamiltonian is a 2x2 matrix

The bulk Hamiltonian in the SSH model

For a finite system of this form



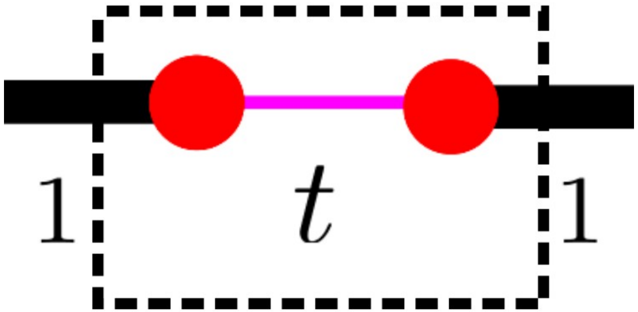
The unit cell is



The Hamiltonian is

$$H = \begin{pmatrix} 0 & t + e^{ik} \\ t + e^{-ik} & 0 \end{pmatrix}$$

The bulk invariant in the SSH model



The Hamiltonian is

$$H = \begin{pmatrix} 0 & t + e^{ik} \\ t + e^{-ik} & 0 \end{pmatrix}$$

$|\Psi(k)\rangle$ ← Lowest energy wavefunction

The topological invariant for this system is the Zak phase

$$\phi = \int_{BZ} A dk$$

$$A = i\langle \Psi(k) | \partial_k | \Psi(k) \rangle$$

Zak phase

The bulk invariant in the SSH model

Hamiltonian

$$H = \begin{pmatrix} 0 & t + e^{ik} \\ t + e^{-ik} & 0 \end{pmatrix}$$

Zak phase

$$\phi = \int_{BZ} A dk$$

$$A = i\langle \Psi(k) | \partial_k | \Psi(k) \rangle$$

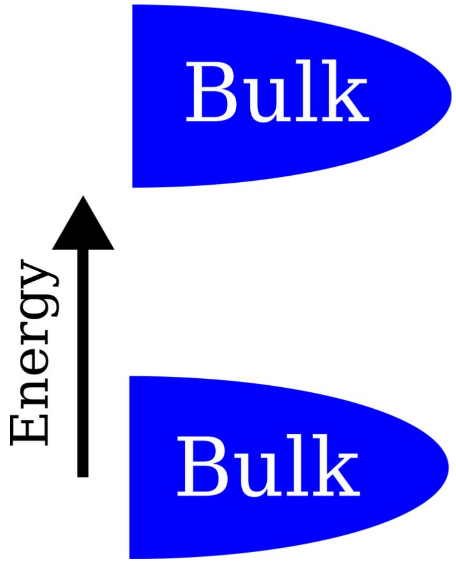
Two different possible values for the Zak phase

$\phi = 0$	$t > 1$	**Trivial insulator**
$\phi = \pm\pi$	$t < 1$	**Topological insulator**

The bulk-boundary correspondence in the SSH model

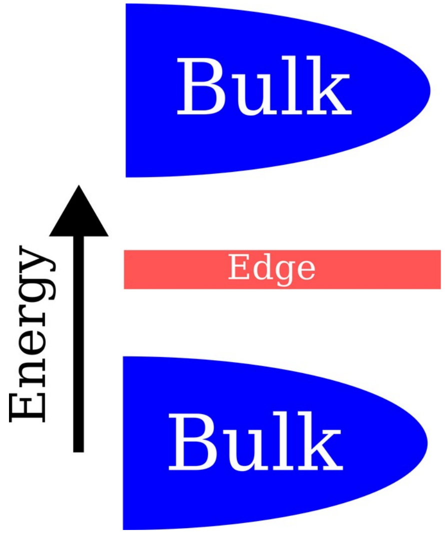
Trivial

$$\phi = 0$$



Topological

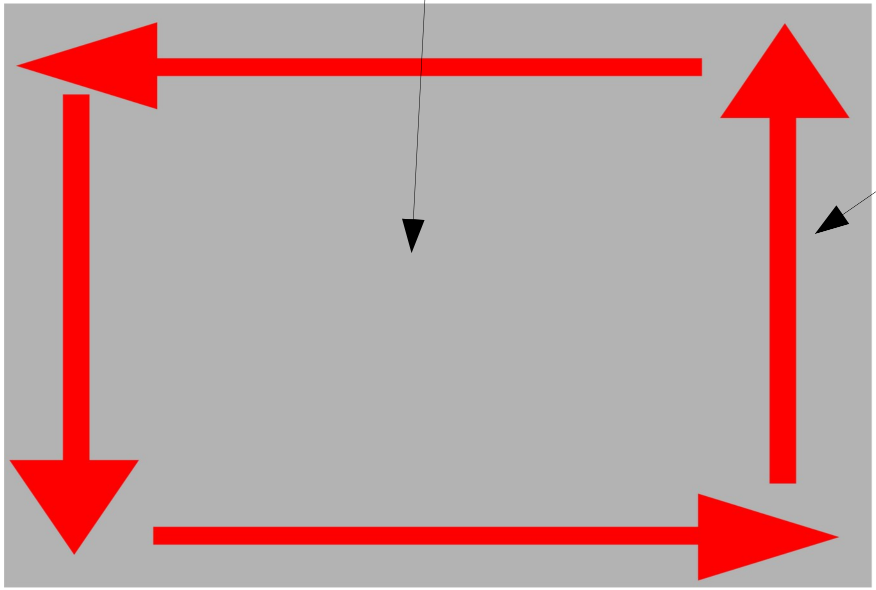
$$\phi = \pm\pi$$



Two-dimensional topological insulators

Chern insulators

The bulk of a quantum Hall state is insulating



The edge has chiral states

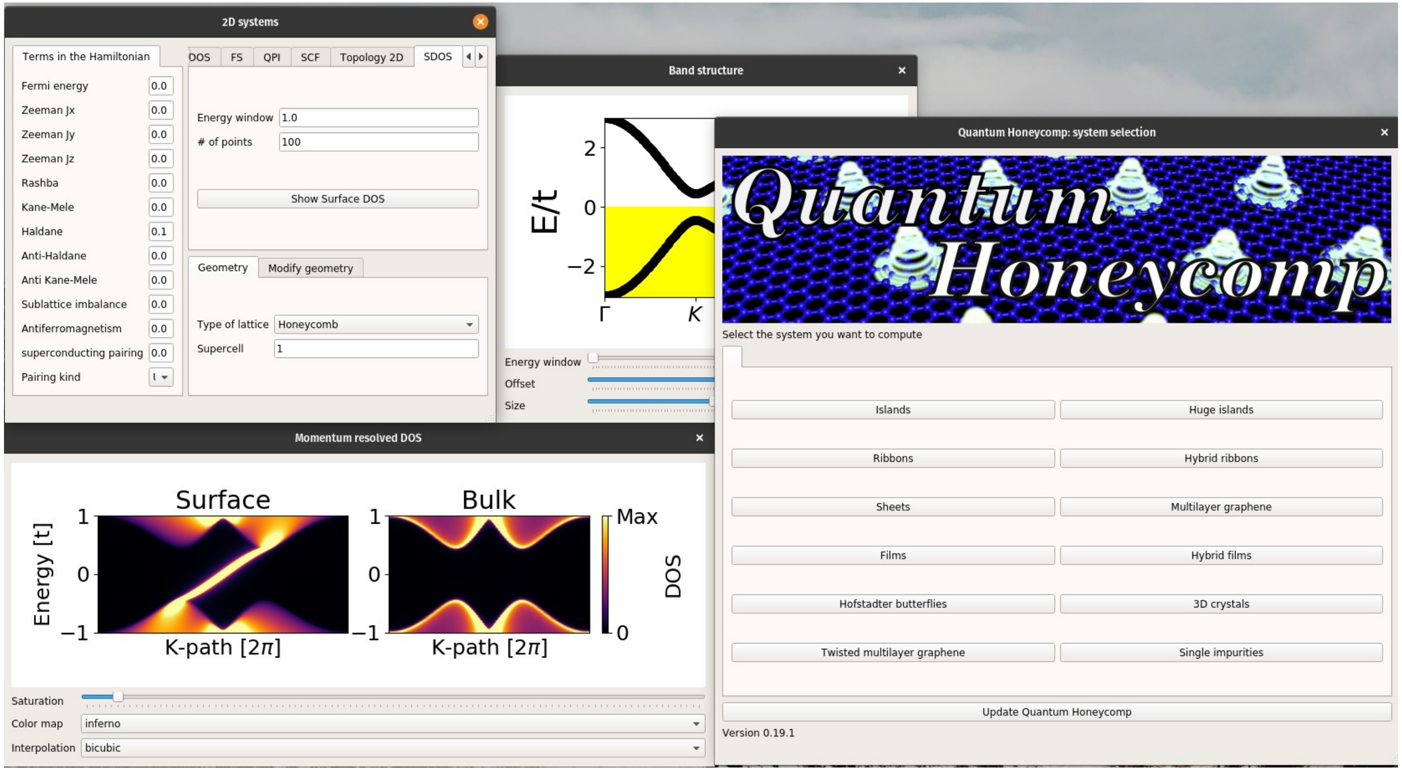
Hall conductivity (Chern number)

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

$$\Omega_\alpha = \partial_{k_x} A_y^\alpha - \partial_{k_y} A_x^\alpha$$

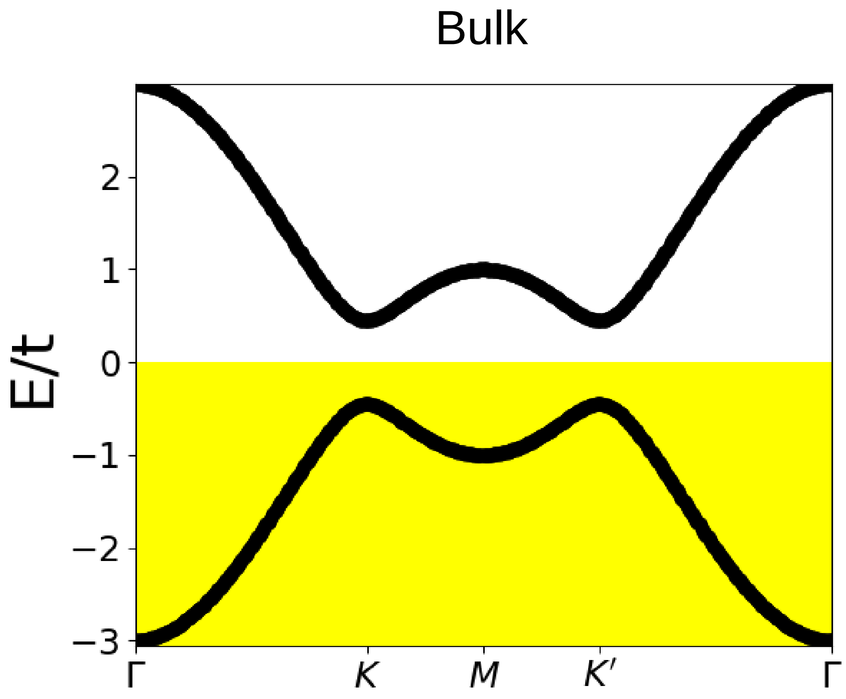
$$A_\mu^\alpha = i\langle \partial_{k_\mu} \Psi_\alpha | \Psi_\alpha \rangle$$

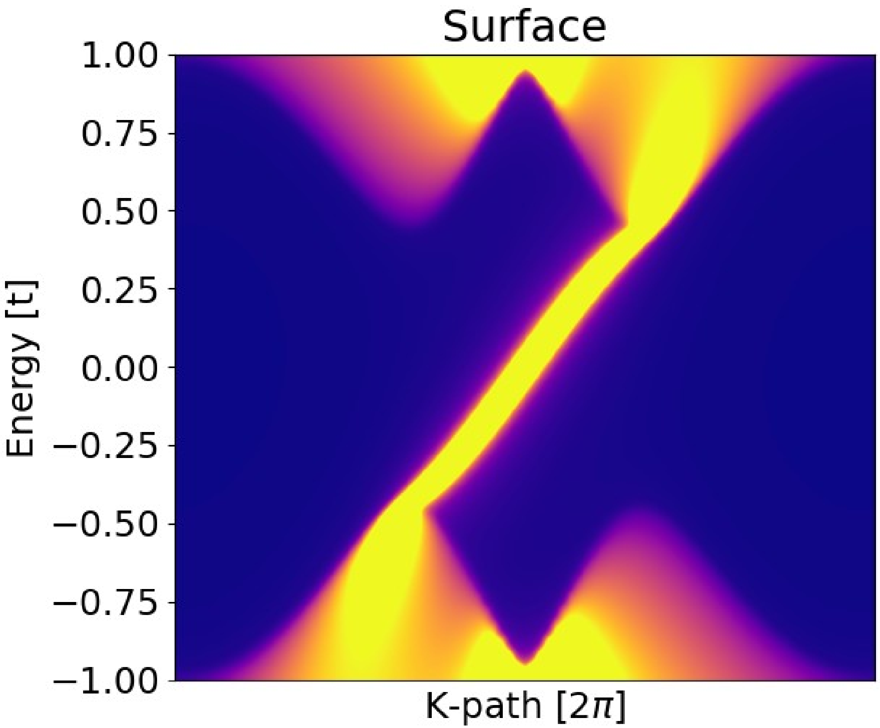
Chern insulators (interactively)



https://github.com/joselado/quantum-honeycomp

Bulk boundary correspondence





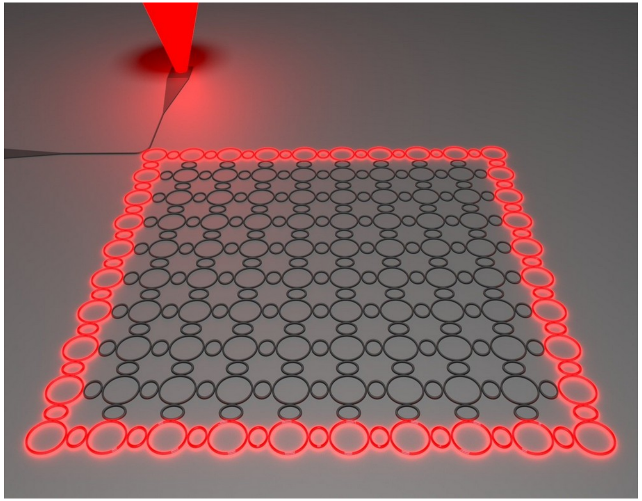
Topology beyond electrons

Topological modes beyond electrons

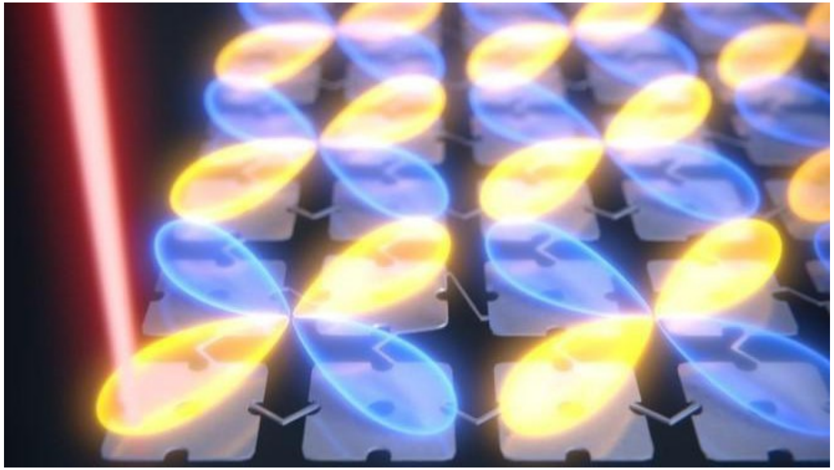
Topological edge modes are a signature of systems that can be described with matrices

Other systems can be described with mathematically analogous tools

Photonic systems

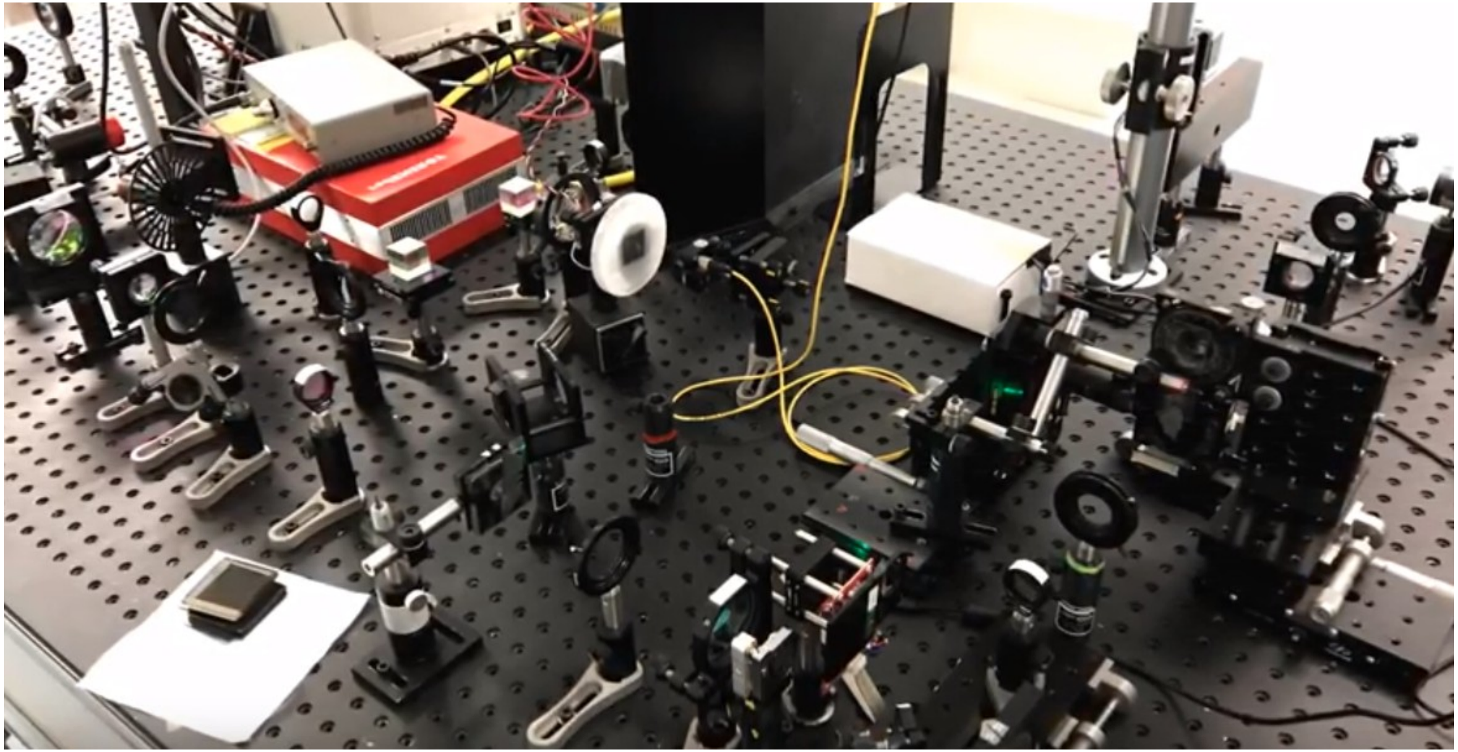


Classical (mechanical) systems



Topological lasers

https://www.youtube.com/watch?v=qIg6PVbs1BI



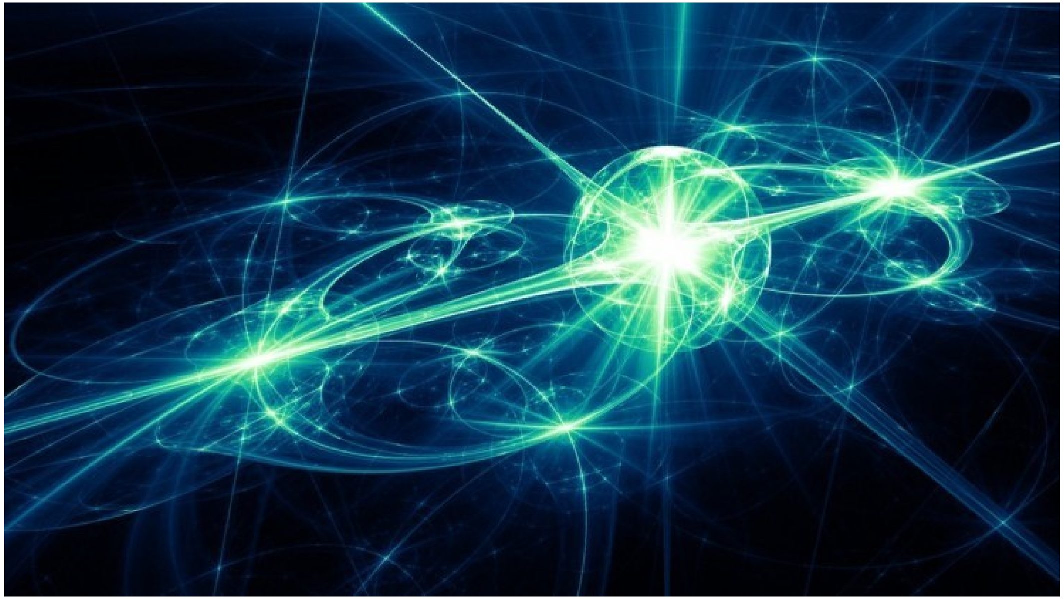
Take home

- Topological systems have protected edge modes
- The existence of edge modes is associated with a non-trivial topological invariant
- Reading material:

  Bernevig & Hughes 15-25

In the next session

Quantum Hall effect



A starting point toward fractional excitations